21-301 Combinatorics
Section B

Assignment 2

Due Wednesday 11th February (at the beginning of the lecture)

Please write up and hand in solutions to all of the starred questions below (note the last question is starred!). The un-starred questions are for you to practise and do not need to be handed in. Questions may ask for a closed form of a summation. An example of this would be

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

1.* Find a closed form for

$$\sum_{n=0}^{\infty} \binom{n+k}{k} x^n.$$

2.* Find a_n if

$$a_n = 2a_{n-1} + 3a_{n-2}$$

and $a_0 = -1$ and $a_1 = -15$.

3.*

(a) How many strings of length n consisting of 0's and 1's have no two consecutive 1's.

(b) How many strings of length n consisting of 0's and 1's have no three consecutive 1's and no three consecutive 0's?

4.* Find a closed form for

$$\sum_{n=0}^{\infty} n^2 x^n.$$

5. Find a closed form for

$$\sum_{n=0}^{\infty} n^3 x^n.$$

6. The number of ways of partitioning n into k parts, i.e. finding a set of positive integers $\{a_1, a_2, \ldots a_k\}$ (repetitions allowed) such that

$$a_1 + a_2 + \cdots + a_k = n$$

is equal to the number of ways of partitioning n into any number of parts, the largest of which is k. For example, if $n = 4$ and $k = 2$ then

$$2 + 2 = 3 + 1 = 4$$

and also

$$2 + 2 = 2 + 1 + 1 = 4.$$

Find a combinatorial proof of this for general n and k.

7.* A composition of n into k parts is the same as a partition of n (as above) but we care about the order of the sum. For example the compositions of 4 are 4, $1 + 3$, $3 + 1$, $2 + 2$, $1 + 1 + 2$, $1 + 2 + 1$, $2 + 1 + 1$, $1 + 1 + 1 + 1$.

(a) How many compositions of n (a positive integer) are there?

(b) How many compositions of n into k parts are there for general positive integers $k \leq n$?

Provide combinatorial proofs of your solutions.