21-301 Combinatorics
Section B
Assignment 1 - Solutions

1.* Give a combinatorial proof of Vandermonde's identity, namely that

$$\sum_{r=0}^{k} \binom{m}{r}\binom{n}{k-r} = \binom{m+n}{k}.$$

Solution Consider a set of size $m+n$ and the number of ways of choosing a k element subset from this set. This is equal to $\binom{m+n}{k}$. If we colour m elements of this set red and the remaining n elements blue, then for any subset of size k there will be some number, say r, for $0 \le r \le k$, that are red, and the remaining $k-r$ will be blue. For each k and r there are $\binom{m}{r}$ and $\binom{n}{k-r}$ respectively ways of choosing these subsets and so there are $\binom{m}{r}\binom{n}{k-r}$ ways of choosing a subset of size k with r red elements. Summing over $r=0$ to k counts all such possible sets and this gives the identity as required. $\square$

2.* In class we proved using differentiation that

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$

Provide a combinatorial proof.

Solution The equation $\sum_{k=0}^{n}\binom{n}{k}$ is the number of subsets of a set of size n and we saw in lectures that this was equal to 2^n. The equation $\sum_{k=0}^{n} k\binom{n}{k}$ is the sum of the sizes of all subsets of a set of size n, since for each k there are $\binom{n}{k}$ subsets of size k. For each subset of size k we can pair it up with it's complement which is of size $n-k$. This tells us that the average size of a subset is $n/2$. The sum of the sizes of the subsets is equal to the average size multiplied by the number of subsets and this is equal to $n/2 \times 2^n = n2^{n-1}$ as required.

3.* Suppose you have an $n \times m$ grid. A monotone path is one that is made up of segments $(x,y) \to (x+1,y)$ or $(x,y) \to (x,y+1)$. How many monotone paths are there from $(0,0)$ to (n,m)?

Solution A monotone path is made up of path segments that go to the right $((x,y) \to (x+1,y))$ or up $((x,y) \to (x,y+1))$. Once the path has reached (n,m) it must have travelled right n times and up m times. The number of such paths is the number of ways in which these $n+m$ events can be ordered. The path can be defined by writing a string of length $m+n$ with letters R and U where R appears n times and U appears m times. Each path uniquely defines such a string by writing down R whenever you go right and U whenever you go up. Equally, each string uniquely defines a path and so this is a bijection and the number of each is equal. Each string is uniquely defined by where you put the n U letters in the string and there $m+n$ places to choose from and so the number of ways of choosing n locations for the U letters is $\binom{n+m}{n}$ which we note is also equal to $\binom{n+m}{m}$.

4.* Suppose you have a 1 by n chess style board (i.e. alternating white and black squares). Let T_n be the number of ways of covering the 1 by n board using 1 by 1 tiles and 1 by 2 dominoes and let $T_0 = 1$.

Give and justify a recurrence relation for T_n.

Solution The first square can either be covered by a 1×1 tile or a 2×2 domino that covers the first and second squares. If it is covered by a 1×1 tile, there are T_{n-1} ways of covering the remaining $n-1$ squares. If it is covered by a 2×2 domino then there are $n-2$ squares remaining and there are T_{n-2} ways of covering the remaining squares. So in total there are $T_{n-1} + T_{n-2}$ ways of covering the whole board. This gives us that $T_n = T_{n-1} + T_{n-2}$, we have that $T_0 = 1$ and clearly $T_1 = 1$ also. (As an aside, this is the Fibonacci sequence).

5.* In lectures we proved the following;

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}.$$

Use this identity to find a closed form for

$$\sum_{i=1}^{n} i^2,$$

providing a combinatorial proof.

Solution For each $i \in [n]$, i^2 is the number of ordered pairs we can chose from $[i] \times [i]$ (as we have i choices for the first and then i for the second). Either these two elements are equal in which case there are $i = \binom{i}{1}$ choices or they are different in which case there are $i(i-1) = 2\binom{i}{2}$ choices. Therefore we have that

$$i^2 = \binom{i}{1} + 2\binom{i}{2}.$$

Summing over all i gives us

$$\sum_{i=1}^{n} i^2 = \sum_{i=1}^{n} \binom{i}{1} + 2\binom{i}{2}.$$

This is equal to

$$\binom{1}{1} + \binom{2}{1} + \binom{3}{1} + \cdots + \binom{n}{1} + 2\left(\binom{1}{2} + \binom{2}{2} + \binom{3}{2} + \cdots + \binom{n}{2}\right).$$

Noting that $\binom{1}{2}$ is 0 we see that these sums match the form from the given identity with $k = 1$ and 2 respectively. Therefore we have

$$\sum_{i=1}^{n} i^2 = \binom{n+1}{2} + 2\binom{n+1}{3} = \frac{n(n+1)(2n+1)}{6}.$$

6. Suppose n students hand in their homework without writing their names on the papers they submit. Aside from the fact that they won't receive credit for the work they've done, if the examiner hands back the homework sheets at random, what is the probability that no student receives their own work back?

Solution Label the students with numbers 1 to n. If we consider the mapping $f : [n] \to [n]$ such that $f(i) = j$ if the homework of student i is given to student j. This is a one to mapping on $[n]$ to itself and so is a permutation. We saw in class that there are $n!$ such permutations. If no student receives their own work, then we have a permutation $f : [n] \to [n]$ such that $f(i) \neq i$ for all $i \in [n]$ and this is precisely the definition of a derangement. We saw that the number of such derangements is

$$n! \sum_{k=0}^{n} (-1)^k \frac{1}{k!}$$

and so the probability that no student receives their own homework is equal to the probability that our random permutation is a derangement which is

$$n! \sum_{k=0}^{n} (-1)^k \frac{1}{k!} / n! = \sum_{k=0}^{n} (-1)^k \frac{1}{k!}.$$

7. How many permutations of the numbers $1, 2, \dots 10$ exist that map no even numbers to themselves?

Solution The bad permutations are those that map an even number to themselves. For $i \in E = \{2, 4, 6, 8, 10\}$ define $A_i = \{\pi : [n] \to [n] : \pi(i) = i\}$. We want the permutations that do not lie in any of the A_i. By inclusion-exclusion we have

$$\bigcap_{i \in E} \overline{A}_i = \sum_{S \subset E} (-1)^{|S|} |A_S|.$$

We now need to calculate the size of A_S for a given S. This is the set of permutations that map each element of S to itself and then permute the remaining elements in any way. Once the $|S|$ elements are fixed there $(10 - |S|)!$ ways of permuting the remaining $10 - |S|$ elements so $|A_S| = (10 - |S|)!$. This gives us,

$$\bigcap_{i \in E} \overline{A}_i = \sum_{S \subset E} (-1)^{|S|} (10 - |S|)!,$$

and reordering to sum over the sizes of the $|S|$ and noting that there are $\binom{5}{k}$ ways of choosing a set S of size k, we get

$$\bigcap_{i \in E} \overline{A}_i = \sum_{k=0}^{5} (-1)^k (10-k)! \binom{5}{k} = 10! - 9! \times 5 + 8! \times 10 - 7! \times 10 + 6! \times 5 - 5! \times 1,$$

which is equal to 2170680.