21-301 Combinatorics
Section B

Assignment 1

Due Wednesday 4th February (at the beginning of the lecture)

Please write up and hand in solutions to all of the starred questions below. You may restate results from lectures without proof as long as you state the result you are using. If a question asks for a combinatorial proof, then no credit will be given for solutions using purely non-combinatorial proofs such as induction or direct algebraic manipulation of equations. You must provide working and justification for your answers. No marks will be awarded for an answer provided without proof.

1.* Give a combinatorial proof of Vandermonde's identity, namely that

$$\sum_{r=0}^{k} \binom{m}{r}\binom{n}{k-r} = \binom{m+n}{k}.$$

2.* In class we proved using differentiation that

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$

Provide a combinatorial proof.

3.* Suppose you have an $n \times m$ grid. A monotone path is one that is made up of segments $(x, y) \to (x+1, y)$ or $(x, y) \to (x, y+1)$. How many monotone paths are there from $(0, 0)$ to (n, m)?

4.* Suppose you have a 1 by n chess style board (i.e. alternating white and black squares). Let T_n be the number of ways of covering the 1 by n board using 1 by 1 tiles and 1 by 2 dominoes and let $T_0 = 1$.

Give and justify a recurrence relation for T_n.

5.* In lectures we proved the following;

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}.$$

Use this identity to find a closed form for

$$\sum_{i=1}^{n} i^2,$$

providing a combinatorial proof.

6. Suppose n students hand in their homework without writing their names on the papers they submit. Aside from the fact that they won't receive credit for the work they've done, if the examiner hands back the homework sheets at random, what is the probability that no student receives their own work back?

7. How many permutations of the numbers $1, 2, \ldots 10$ exist that map no even numbers to themselves?