

A UNIFIED CREDIT RISK MODEL

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Abstract

In recent years a lot of work has been done to try to bridge the gap between the two main approaches in credit risk modelling: structural and reduced form models. Many papers tries to obtain this using special assumptions about the problem. For instance, Duffie and Lando(1999)[16] makes the connection using incomplete accounting information. Whereas we propose a unified approach that attains this in full generality. A model where the credit default event is defined as the minimum of the two default times, one from the structural default and the other from the exogenous intensity.

In particular, we look at the effect of having stochastic volatility in the structural approach. We study the effects of time scales on the credit spread yield curves both for the stochastic volatility and the stochastic intensity. In this framework we use perturbation analysis to derive closed-form approximations for the credit spreads that would ease the work of calibration of parameters.

The main quantities of interest are not only include fixed-income market data such as bond spreads but also the equity market data such as variance and leverage ratio. We test for the calibration of the model and stability of the model parameters for particular names of different credit ratings. We observe and analyze the differences of the behaviors of credit spreads between highly and lowly rated names. We analyze these differences. Results emphasize the importance of equitiy market data, such as variance and leverage ratio and also the stochastic default intensity.

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Chapter 1

Credit Risk Modeling

Credit risk is the distribution of financial losses due to unexpected changes in the credit quality of a counter-party in a financial agreement. Examples range from agency downgrades to failure to service debt liquidation. Credit risk pervades virtually all financial transactions.

The distribution of credit losses is complex. At its center is the probability of default, by which we mean any type of failure to honor a financial agreement and the time of the default, by which we mean the time that it is known to everybody that there is a failure. To estimate these two main quantities, we need to specify the models of

- i. investor uncertainty;
- ii. the available information and its evolution over time;
- iii. the definition of the default event.

However, default probabilities and default times alone are not sufficient to price credit sensitive securities. We need to, also, model

- i. the risk-free interest rate
- ii. recovery upon default
- iii. the premium investors required as compensation for bearing systematic credit risk

The credit premium maps actual default probabilities to market implied probabilities that are embedded in market prices. To price securities that are sensitive to the credit risk of multiple issuers and to measure the aggregated portfolio credit risk, we also need to specify a model that links defaults of several entities.

Our main goal is modeling credit risk for measuring credit risk and for pricing defaultable bonds, credit derivatives and other securities that are exposed to credit risk. We first present critical assessments of the existing theoretical approaches for pricing the instruments sensitive to credit risk, pointing out the advantages and disadvantages of the current frameworks. In particular, we review two broad classes of models: *Reduced-form* models that assume an exogenous process for the migration of default probabilities, that are calibrated to historical market data, and *Structural* models that are based directly on the issuer's ability or willingness to pay its liabilities. Structural models are in general based around a stochastic model of variation in asset liability ratio. Both classes have its own advantages and disadvantages, e.g. although reduced form models have a lot of room for calibrating to historical data, they lack the financial ingredient for the model parameters. On the other hand structural models have a nice explanation in financial terms and rather intuitive, they lack measuring in particular the short-term credit risk and much harder to apply when there is more than one name involved. Hence, a framework that would combine the two general classes of models in a way that none of the above criticisms do not apply would be the ideal framework to model credit risk. And yet the model still should be parsimonious enough so that it could calibrated to historical data easily which is hard to find most of the time even for the single name products. Also the framework should be open to correlation to other market variables such as interest rates, equities or in fact more importantly the credit risk of other names.

Rest of this chapter is organized as follows. In section 1.1 we summarize the basics of zero coupon bond pricing with or without default and explain the concept of credit yield spreads. Section 1.2 and 1.3 we describe the two main schools of credit risk modelling and review the already existing models in the literature for single name

credit derivatives. We state the advantages and disadvantages of each model, especially the ones that relates to our work. In section 1.4 we present the basic results of the thesis and finally in section 1.5 we mention some recent related work and future research directions that we would like to pursue.

1.1 Defaultable Bond Pricing

1.1.1 Classical Bond Pricing

In the most classical sense bonds are priced using models of the term structures of interest rates that are used for the pricing and hedging of fixed-income securities. We fix a d -dimensional Standard Brownian Motion $W = (W_1, W_2, \dots, W_d)$ in $\mathbb{R}^d$, for some dimension $d \geq 1$, restricted to some time interval $[0, T]$, on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We also fix the standard filtration $\mathcal{F} = \{\mathcal{F}_t : 0 \leq t \leq T\}$ of W .

We take as given an adapted short-rate process r_t with $\int_0^T |r_t| dt$. Conceptually, r_t is the continually compounding rate of interest on a riskless securities at time t . This is formalized by supposing that, for any time t , one can invest one unit of account and achieve a market value at any future time s of $\exp(\int_t^s r_u du)$ units of account. This may be viewed as the proceeds of continual reinvestment at the short rate r_t .

Consider a zero-coupon bond maturing at some future time $s > t$. By definition, the bond pays no dividends before time s , and offers a fixed payment at time s that we can take without loss of generality to be 1 unit of account. Although it is not always the case essentially to do so, we assume that such a bond exists for each maturity date s . Main objective is to characterize the price $B(t, s)$ at time t of the s -maturity bond, and behavior over time. Absence of arbitrage assumption, under some technical conditions, implies the existence of an equivalent martingale measure. Such a probability measure Q has the property that any security whose dividend is in the form of a payment of X at some time s has a price, at any time $t < s$, of

$$\mathbb{E}_t^Q \left\{ \exp\left(-\int_t^s r_u du\right) X \right\} \quad (1.1)$$

where E_t^Q denotes $\mathcal{F}_t$ conditional expectation under the probability measure Q . Here, X is some $\mathcal{F}_s$ measurable random variable such that the expectation (1.1) is well defined. In particular, taking $X = 1$ in the expression (1.1), the price at time t of the zero-coupon bond maturing at s is

$$B(t, s) \equiv \mathbb{E}_t^Q \left\{ \exp \left(- \int_t^s r_u du \right) \right\}$$

The doubly indexed process is known as the *term structure of interest rates*. And the term structure is often expressed in terms of the yield curve. The continuously compounding yield $y(t, s)$, on a zero-coupon bond maturing at time s is defined by

$$y(t, s) \equiv - \frac{\log(B(t, s))}{s - t}$$

And mostly in the literature the short rate, r_t , is modeled by a single or multi-factor model given by an SDE of the form

$$dr_t = \mu(r_t, t)dt + \sigma(r_t, t)dW_t^Q$$

with various choices of μ and σ .

1.1.2 Bond Pricing with Defaults

Pricing defaultable zero-coupon bonds is more complicated than pricing riskless zero-coupon bonds. Consider the same setting of section (1.1.1) but with an additional filtration $\mathcal{G} = \{\mathcal{G}_t : 0 \leq t \leq T\}$ where $\mathcal{G}_t \supseteq \mathcal{F}_t$. The default time of the bond, τ , is a stopping time with respect to this possibly larger filtration $\mathcal{G}$.

Consider a *defaultable* zero-coupon bond maturing at some time $s > t$. The payoff of the bond is a lumpsum amount of X at time s and no dividends before time s if there is no default until time s , i.e. $\tau > s$. Again without loss of generality assuming that $X = 1$, this time as opposed to the equation (1.1) we have

$$P(t, s) \equiv \mathbb{E}_t^Q \left\{ \exp \left(- \int_t^s r_u du \right) 1_{\tau > s} \right\} \quad (1.2)$$

The concept *credit spread* is the difference between the yield on a defaultable bond and the yield on an otherwise equivalent default-free zero coupon bond. It gives the excess return demanded by the bond investors to carry the potential default losses. Hence the formula for credit spread, $S(t, s)$, at time t is

$$S(t, s) \equiv -\frac{1}{s-t} \log\left(\frac{P(t, s)}{B(t, s)}\right)$$

where $B(t, s)$ is the risk-free bond maturing at time s as defined in equation (1.1). The *term structure of credit spreads* is the function $S(t, \cdot)$, i.e. the schedule of $S(t, s)$ against s holding t fixed.

Term structures of credit spreads can also be modelled using an HJM[39] framework. Although we do not work all the details since we do not use it, but see Appendix B for a brief overview.

1.2 Structural Models

This class of models defines default as a contingent claim by describing precisely when the default occurs and then prices the defaultable security using the methods of derivative security pricing. We shall consider three models in this class, first the classical Merton Model,[59], second as an extension of the Merton Model, first passage time models,[6] and the Longstaff-Schwartz Model,[52]. Models in this class are differentiated by the way that in which the default event is defined and particular definitions may well be better suited to describing particular defaults. All these models relate the default to the process for the firm's asset backing and define the default event in terms of a boundary condition on this process. A major deficiency of these models, as noted in Madan and Unal,[55], is that they treat the value of the asset backing as a primary asset of the economy when in fact it may be a derivative asset in its own right with exposure to other more primitive state variables of the economy. Prices should be reduced to exogenous state variables of the economy and it is unclear that a firm's asset value is such a variable.

1.2.1 The Merton Model

For a simple start, we outline the classic Black-Scholes-Merton,[7],[59], model of corporate debt and equity valuation. We suppose that the firms future cash flows have a total market value at time t given by S_t , where the process S is a Geometric Brownian Motion, satisfying

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

for constants μ and σ , and where we have taken as the dimension of the underlying Brownian Motion W to be 1. One sometimes refers to S_t as the total asset level of the firm. For simplicity, assuming that the firm produces no cash flows before a given time T . We take it that the original owners of the firm have chosen a capital structure consisting of pure equity and of debt in the form of a single zero-coupon bond maturing at time T , of face value K . In the event, that the total value S_T is less than the contractual payment K due on the debt., the firm defaults at time T giving its future cash flows, worth S_T to debtholders. That is, debtholders receive $\min(S_T, K)$ at T . Equityholders receive the residual $\max(S_T - K, 0)$. We suppose for simplicity that there are no other distributions (such as dividends) to debt or equity. We will shortly confirm the natural conjecture that the market value of equity is given by the Black-Scholes,[7], option pricing formula, treating the firm's asset value as the price of the underlying security. Bond and the equity investors have already paid the original owners of the firm for their respective securities. The absence of well-behaved arbitrage implies that at any time $t < T$, the total of the market values E_t of equity and Y_t of debt must be the market value of the S_t of the assets.

$$S_t = E_t + Y_t$$

Markets are complete given riskless borrowing or lending at a constant rate r and access to a self-financing trading strategy whose value process is S_t . This implies that there is at most one equivalent martingale measure. Letting $W_t^Q = W_t + \eta t$ where $\eta = \frac{\mu-r}{\sigma}$, we have

$$dS_t = r S_t dt + \sigma S_t dW_t^Q$$

By Girsanov's theorem,[48] as also stated in section 2.2.1, W^Q defines a Standard Brownian Motion under the equivalent probability measure $\mathbb{Q}$ defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp(-\eta W_T - \frac{\eta^2 T}{2})$$

By Ito's formula, $\{e^{-rt}S_t : t \in [0, T]\}$ is a $\mathbb{Q}$ -martingale. It follows that, after deflation by e^{-rt} , $\mathbb{Q}$ is the equivalent martingale measure. As $\mathbb{Q}$ is unique in this regard, we have the unique price process E_t of equity in the absence of well-behaved arbitrage given by

$$E_t = \mathbb{E}_t^Q\{e^{-r(T-t)} \max(S_T - K, 0)\}$$

Thus, the equity price E_t is computed by the Black-Scholes formula, treating S_t as the underlying asset price, σ as the volatility coefficient, the face value K as the strike price, and $T - t$ as the time remaining to exercise. The market value of the debt at time t is the residual, $S_t - E_t$.

When the original owners of the firm sold the debt with face value K and the equity, they realized a total initial market value of $E_0 + Y_0 = S_0$, which does not depend on the chosen face value K of debt. This is one aspect of the Modigliani-Miller Theorem,[61]. The same irrelevance of capital structure for the total valuation of the firm applies much more generally, has nothing to do with the choice of Geometric Brownian Motion, nor the specific nature of the debt and the equity as noted in [38]. Besides all the nice intuitive definition of default and mathematical attractiveness of Merton's model, there are also several shortcomings e.g. the defaults according to this model are either happening at a fixed maturity T or never which leads to an inconsistency in pricing multiple maturity derivatives. Also, the default naturally could happen any time until the maturity of the derivative of interest. Also as a result of this late defaults, the credit spreads that are produced by the model are 0 in the beginning and very near zero for short maturities which is unrealistic. Also the model does not make the connection between the defaults and other market variables such as interest rates.

Despite all these features of the Merton's model, it still serves as a benchmark model for comparison and provides a useful basic framework to develop more complicated

models based on it. In our calculations in sections 2.5.1, 2.7.1 and 2.8.1 we always start with this setting and improve the definition of default and calculations based on it.

1.2.2 First Passage Model

Black and Cox,[6], introduced the idea that the default would occur at the first time that assets drop to a sufficiently low default boundary, whether or not at the maturity of the date of the debt. It is the most natural extension of the Merton's model described in section 1.2.1, that would allow defaults other than the maturity of the financial product. They assumed a simple time-dependent default boundary, exploiting the fact that there is an explicitly known probability distribution (and in fact also the Laplace transform) for the first time that a Brownian Motion with constant drift and volatility parameters reaches a given level. Many subsequent structural models, including those of Fischer et al.,[26], Leland,[51], Anderson and Sundareson,[1], and Mella-Barral,[57], have considered incentive-based models for the default boundary and the default recovery.

Focusing for the moment on this simple first-passage model, suppose default occurs at the first time at which the log-normally distributed asset level, S_t , reaches a constant default threshold K , which need not to be the face value of the debt, but rather chosen by the firm so as to maximize the market value of the equity. For each time horizon T , the survival probability is then the probability $\mathbb{P}(t, T)$ that the distance to default does not reach between t and T or

$$\mathbb{P}(t, T) = \mathbb{P}(S_s \geq K, t \leq s \leq T | S_t) = H(S_t, T - t)$$

where

$$H(x, s) = N\left(\frac{x - K + ms}{\sqrt{s}}\right) - \exp(-2m(x - K))N\left(\frac{-x + K + ms}{\sqrt{s}}\right)$$

Following the discussion at the end of section 1.2.1, first passage time default models still suffer from the near zero short term spreads, simply due to the fact that a

continuous process needs some time to reach the level of default boundary. The tractability of the model declines rapidly as one enriches the models used for the asset process, S_t , and allows for a time varying default threshold K . Although some extensions, including an allowance of jumps in asset process have been introduced for purposes of bond pricing, for instance by Merton,[58], the pricing formulas are not as nice as in this section. And also there is an issue of calibration about how to interpret the market data for the jumps. As explained in more detail in section 1.4, we use the first passage framework in our hybrid framework and do the pricing in sections 2.5.2, 2.7.2 and 2.8.1 but we enrich the default definition with an outside source so that we do not come across the problem of near zero spreads, see figure 2.1.

1.2.3 Longstaff-Schwartz Model

This model attempts to address a number of shortcomings of the Merton model. By allowing early defaults time consistency is attained in pricing multiple maturity derivatives. Default is defined as occurring at the first time that asset value reaches a threshold level K and at this time all maturities still outstanding default simultaneously. The possibility of early default may also induce higher credit spreads and eliminate the problem of no short maturity credit spreads. Furthermore, by allowing for stochastic interest rates one may better calibrate to existing term structures of interest rates. The model supposes that asset values S_t , follow Geometric Brownian Motion while interest rates r_t follow a Vasicek process,[68], with movements that are correlated with the stock. Specifically it is supposed that

$$\left\{ \begin{array}{lcl} dS_t & = & rS_t dt + \sigma S_t dW_t^S \\ dr_t & = & \kappa(\theta - r_t)dt + \eta dW_t^r \\ dW_t^S dW_t^r & = & \rho dt \end{array} \right.$$

At the first passage time of S to K the default threshold, we have a default in which the recovery level is some constant write down of the face and creditors receive $(1 - l)F$. In pricing the defaultable claim one may account for correlation between interest rates and asset values by following Jamshidian,[44] and Geman, El Karoui

and Rochet,[24] and expressing the dynamic evolution of asset values and interest rates using the price of the risk-free bond, $B(t, T)$, as the numeraire. Hence we let Q^T be the probability under which asset prices discounted by the price of the risk free bond price $B(t, T)$, are martingales. The price of the defaultable bond, $P(t, T)$, may then be written as

$$P(t, T) = B(t, T)[G(t, T) + (1 - l)F(1 - G(t, T))]$$

where $G(t, T)$ is the probability of no default in the interval (t, T) and in this case we receive the unit face, while under default, with probability $(1 - G(t, T))$ we receive $(1 - l)F$ at time T by definition. The claim is priced on determining the probability of no default $G(t, T)$ under the measure Q^T .

The explicit dynamics for asset values and interest rates under Q^T may be determined by an application of Girsanov's theorem as

$$\begin{cases} d \log S_t &= (r - \frac{\sigma^2}{2} - \rho\sigma\eta M(T-t))dt + \sigma dW_t^{*S} \\ dr_t &= (\theta - \kappa r_t - \eta^2 M(T-t))dt + \eta dW_t^{*r} \\ M(T-t) &= \frac{1 - \exp(-\kappa(T-t))}{\kappa} \end{cases}$$

where W_t^{*S}, W_t^{*r} are Brwonian motions under Q^T . To determine G , Longstaff and Schwartz present and solve an integral equation that equates the Gaussian probability of $\log(\frac{S}{K})$ being negative at T to the integral over first passage times to zero of the first passage probability to zero times the Gaussian probability of then ending up below zero, starting now at zero at the first passage time. It has been observed by Robert Goldstein that this integral equation is incorrect as one must also integrate over the level of the random interest rate at the first passage time of $\log(\frac{S}{K})$ to zero. It is simple to show using characteristic functions for example, that with two independent Brownian motions the distribution of the first Brownian motion at the first passage time of the second Brownian motion to zero is a Cauchy random variable with an infinite mean. The Longstaff and Schwartz model is, at this writing, as yet unsolved. Especially for pricing of credit derivatives with longer maturities, the modelling of interest rates are important and having a stochastic process process in interest rates

is favorable feature. It allows for easier calibration and also interaction with the default risk. However, the problem gets too hard if one wants to do it in a structural framework. We introduce a more direct approach in chapter 3, in the context of CDS premium calculation, to model the interaction between interest rates and default probabilities.

1.3 Reduced Form Models

We first introduce the model for a default time as a stopping time τ with a given intensity process λ_t which will be defined in more detail below. From the joint behavior of λ_t and the short rate process r_t , the promised payment of the security, and the model recovery at default, as well as risk premia, one can characterize the stochastic behavior of the term structure of yields on defaultable bonds.

As far as extensions go, default intensities are allowed to depend on observable variables that are linked with the likelihood of default, such as debt-to-equity ratios, asset volatility measures, other accounting measures of indebtedness, market equity prices, bond yield spreads, industry performance measures and macroeconomic variables related to business cycle.

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\{\mathcal{G}_t : t \geq 0\}$ satisfying the usual conditions, which were listed in section 1.1.2. As we depart from the case of purely Brownian information, it is important to make a distinction between an *adapted process* and a *predictable process*. A predictable process is, intuitively speaking, one whose value at any time s depends only on the information in the underlying filtration that is available up to, but not including, time s . As defined in section 2.2 in more details, a non-explosive counting process N_t (for example a Poisson process) has an intensity λ_t if λ_t is a predictable nonnegative process satisfying $\int_0^t \lambda_s ds < \infty$, with the property that the compensated counting process defined by M_t ,

$$M_t = N_t - \int_0^t \lambda_s ds \tag{1.3}$$

is indeed a martingale.

We will say the a stopping time τ has an intensity λ_t if τ is the first jump time of a nonexplosive counting process whose intensity process is λ_t . The intuition is that, at any time t and state w with $t < \tau(w)$, the $\mathcal{G}_t$ -conditional probability that $\tau < t + \delta$ is approximately $\lambda(w, t)\delta$ for small δ .

A stopping time τ is non-trivial if $\mathbb{P}(\tau < \infty) > 0$. If a stopping time is nontrivial and if the filtration $\mathcal{G}_t$ is the standard filtration of some Brownian Motion W_t , then τ could not have an intensity, see Kusuoka (1999), [50], for an example. We know this from the fact that, if $\mathcal{G}_t$ is the standard filtration of W_t , then the associated compensated counting process M_t of (1.3) could be represented as a stochastic integral with respect to W_t , and therefore cannot jump, but M_t must be jump at τ . In order to have an intensity, a stopping time must be totally inaccessible, a property whose definition, [16], suggests arrival as a sudden surprise but there are no such surprises on a Brownian filtration.

As an example, Duffie and Lando [16] models the firms equityholders or managers are equipped with some Brownian filtration for purposes of determining their optimal default time τ but that bondholders have imperfect monitoring, and may view τ as having an intensity with respect to the bondholders own filtration $\mathcal{G}_t$, which contains less information than the Brownian filtration.

We say that τ is doubly stochastic with intensity λ_t if the underlying counting process whose first jump time τ is doubly stochastic with intensity λ_t . The doubly-stochastic property implies that

$$\mathbb{P}(\tau > s | \mathcal{G}_t) = \mathbb{E}_t \left\{ \exp \left(- \int_t^s \lambda_u du \right) \right\}, \quad t < \min(\tau, s) \quad (1.4)$$

where $\mathbb{E}_t$ denotes the $\mathcal{G}_t$ -conditional expectation. This property (1.4) is convenient for calculations, as evaluating the expectation in (1.2) is computationally equivalent to the pricing of a default-free zero-coupon bond, treating λ_t as a short rate and the $\mathbb{E}_t$ as risk-neutral.

As we prove in section 2.2, it would be sufficient for (1.4) that $\lambda_t = \Lambda(X_t, t)$ for

some measurable function Λ , where X_t is a d -dimensional process that solves a stochastic differential equation of the form

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t$$

for some $\mathcal{G}_t$ standard Brownian Motion $W \in \mathbb{R}^d$.

More generally, (1.4) follows from assuming the doubly stochastic counting process N_t whose first jump time is τ is driven by some filtration $\{\mathcal{F}_t : T > 0\}$ where $\mathcal{F}_t \subseteq \mathcal{G}_t$. The idea of the doubly-stochastic assumption is that the intensity is $\mathcal{F}_t$ -predictable and that the conditional on λ_t , N_t is a Poisson process with time varying intensity λ_t . In particular, for any time $s > t$, conditional on the tribe $\mathcal{G}_t \vee \mathcal{F}_s$ generated by the events in $\mathcal{G}_t \cup \mathcal{F}_s$, the number $N_s - N_t$ of arrivals between t and s is distributed as a Poisson random variable with parameter $\int_t^s \lambda_u du$. Thus, letting A be the event that $N_s - N_t = 0$, the law of iterated expectations implies that, for $t < \tau$

$$\begin{aligned} \mathbb{P}(\tau > s | \mathcal{G}_t) &= \mathbb{E}\{1_A | \mathcal{G}_t\} \\ &= \mathbb{E}\{\mathbb{E}[1_A | \mathcal{G}_t \vee \mathcal{F}_s] | \mathcal{G}_t\} \\ &= \mathbb{E}\{\mathbb{P}(N_s - N_t = 0 | \mathcal{G}_t \vee \mathcal{F}_s) | \mathcal{G}_t\} \\ &= \mathbb{E}\{\exp(-\int_t^s \lambda_u du) | \mathcal{G}_t\} \end{aligned}$$

consistent with (1.4). This is only a sketch of the idea, Duffie and Lando [16] offer a proper development.

1.3.1 Intensity Process under Risk-Neutral Measure

For purposes of the market valuation of bonds and other securities whose cash flows are sensitive to default timing, we would want to have an equivalent martingale measure Q and a risk-neutral intensity process, that is, an intensity λ_t^Q for the default time τ that is associated with $(\Omega, \mathcal{F}, \mathbb{P})$ and the given filtration $\{\mathcal{G}_t : t > 0\}$. As usual, there may be more than one equivalent martingale measure. In [16] the ratio $\frac{\lambda_t^Q}{\lambda}$ (for λ strictly positive) is called a multiplicative risk premium for the uncertainty

associated with the timing of default.

Proposition 1 *Suppose a nonexplosive counting process N_t has a $\mathbb{P}$ –intensity process and $\mathbb{Q}$ is any probability measure equivalent to $\mathbb{P}$. Then N_t has a $\mathbb{Q}$ –intensity process.*

A version of Girsanov’s theorem provides conditions suitable for calculating the change of probability measure associated with a change of intensity, by analogy with the change in drift of a Brownian Motion. Suppose N_t is a non-explosive counting process with intensity λ_t , and ϕ is a strictly positive predictable process such that, for some fixed time horizon T , $\int_0^T \phi_s \lambda_s ds < \infty$ almost surely. A martingale is then defined by

$$\xi_t = \exp\left(\int_0^t (1 - \phi_s) \lambda_s ds\right) \prod_{\{i: T(i) \leq t\}} \phi_{T(i)} \quad (1.5)$$

Theorem 1 (Girsanov) *Suppose ξ_t in equation (1.5) is a martingale. Then an equivalent martingale measure $\mathbb{Q}$ is defined by $\frac{d\mathbb{Q}}{d\mathbb{P}} = \xi(T)$. Moreover, restricted to the time interval $[0, T]$, the counting process N_t has $\mathbb{Q}$ –intensity $\phi_t \lambda_t$.*

Care must be taken with assumptions, for the convenient doubly-stochastic property need not be preserved with a change to an equivalent probability measure. Illustrative counter-examples are cited in [16]. A proof of this version of the Girsanov theorem is given in Appendix A, which also gives sufficient conditions for the martingale property of ξ , and for N_t to be doubly stochastic both under $\mathbb{P}$ and $\mathbb{Q}$. Under certain conditions to the filtration $\{\mathcal{G}_t : t \geq 0\}$ outlined in section 2.2, the martingale representation property applies and for any equivalent martingale measure $\mathbb{Q}$, one can obtain the associated $\mathbb{Q}$ –intensity of N_t from the martingale representation of the associated density process.

1.3.2 Defaultable Zero Coupon Bond Pricing

We fix a short rate process r_t and an equivalent martingale measure $\mathbb{Q}$ after deflation by $\exp(-\int_0^t r_u du)$. We consider the valuation of a security that pays $K 1_{\{\tau > s\}}$ at a given time $s > 0$, where K is some $\mathcal{G}_s$ –measurable, bounded random variable. As $1_{\{\tau > s\}}$ is the random variable that is 1 in the event of no default by s and 0 otherwise,

we may view K as the contractually promised payment of the security at time s , with default by s leading to no payment. The case of a defaultable zero-coupon bond is treated by letting $K = 1$. In the next section 1.3.3, we will consider recovery at default.

From the definition of $\mathbb{Q}$ as an equivalent martingale measure, the price $P(t, s)$ of this security at any time $t < s$ is given by

$$P(t, s) = \mathbb{E}_t^{\mathbb{Q}} \{ e^{-\int_t^s r_u du} 1_{\{\tau > s\}} K \} \quad (1.6)$$

where $\mathbb{E}_t^{\mathbb{Q}}$ denotes the $\mathcal{G}_t$ -conditional expectation under $\mathbb{Q}$. From (1.6) and the fact that τ is a stopping time, $P(t, s)$ must be 0 for all $t \geq \tau$. Let us assume under $\mathbb{Q}$, the default time τ has intensity $\lambda_t^{\mathbb{Q}}$.

Theorem 2 *Suppose that K, r_t and $\lambda_t^{\mathbb{Q}}$ are bounded and that τ is doubly stochastic under $\mathbb{Q}$ driven by a filtration $\mathcal{F}_t$ such that r_t is $\mathcal{F}_t$ -adapted and K is $\mathcal{F}_s$ -measurable. Fix any time $t < s$. Then, for $t \geq \tau$, we have $P(t, s) = 0$ and for $t \leq \tau$,*

$$P(t, s) = \mathbb{E}_t^{\mathbb{Q}} \{ e^{-\int_t^s (r_u + \lambda_u^{\mathbb{Q}}) du} K \} \quad (1.7)$$

The idea of representation (1.7) of the predefault price is that the discounting for default that occurs at an intensity is analogous to the discounting at the short rate r_t . Again a proof is given in section 2.2.

1.3.3 Defaultable Bond Pricing with Recovery

The next extension is to consider the recovery of some random payoff L at the default time τ , if default occurs before the maturity date s of the security. Adopting the assumptions of theorem 2 and adding the assumption that $L = l_{\tau}$ where l is a bounded predictable process that is also adopted to the underlying filtration $\mathcal{F}_t$. The market value at any time $t < \min(\tau, s)$ of any default recovery is, by definition of the equivalent martingale measure $\mathbb{Q}$, given by

$$R(t, s) = \mathbb{E}_t^{\mathbb{Q}} \{ e^{-\int_t^s r_u du} 1_{\{\tau \leq s\}} l_{\tau} \} \quad (1.8)$$

The assumption that τ is doubly-stochastic implies that it has a probability density under $\mathbb{Q}$, at any time u in $[t, s]$, conditional on $\mathcal{G}_t \vee \mathcal{F}_s$ and on the event that $\tau > t$

$$q(t, u) = \exp\left(-\int_t^u \lambda_z^Q dz\right) \lambda_u^Q$$

Thus again using the argument of iterated expectations as in the proof of the theorem 2, we have on the event of $\tau > t$

$$\begin{aligned} R(t, s) &= \mathbb{E}\{\mathbb{E}[e^{-\int_t^\tau r_z dz} 1_{\{\tau \leq s\}} l_\tau | \mathcal{G}_t \vee \mathcal{F}_s] | \mathcal{G}_t\} \\ &= \mathbb{E}\{\int_t^s e^{-\int_t^u r_z dz} q(t, u) l_u du | \mathcal{G}_t\} \\ &= \int_t^s \Phi(t, u) du \end{aligned}$$

using Fubini's theorem, where we call

$$\Phi(t, u) \equiv \mathbb{E}_t^Q\{\exp\left(-\int_t^u (r_z + \lambda_z^Q) dz\right) \lambda_u^Q l_u\}$$

We can summarize the main defaultable valuation result as follows.

Theorem 3 *Consider the security pays K at s if $\tau > s$, and otherwise pays l_τ at τ . Suppose l, K, r are all bounded. Suppose also that τ is doubly stochastic under $\mathbb{Q}$ driven by a filtration $\mathcal{F}_t$ with the property that r_t and l_t are $\mathcal{F}_t$ -adapted and K is $\mathcal{F}_s$ -measurable. Then, for $t \geq \tau$ we have $R(t, s) = 0$, and for $t < \tau$*

$$R(t, s) = \mathbb{E}_t^Q\{e^{-\int_t^s r_u + \lambda_u^Q du} K\} + \int_t^s \Phi(t, u) du \quad (1.9)$$

1.3.4 Credit Default Swaps

In this section we explain the basic definition of the most actively traded form of credit derivative, credit default swap(CDS). Since it is the most liquid derivative it has also been used as a benchmark for credit pricing.

A credit swap is a form of derivative security that can be considered as an insurance for defaults on bonds or loans. Credit default swaps pay the buyer of protection a given contingent amount at the time of the default of the bond or loan. This

contingent amount is usually the difference between the face value of a bond and its market value and it is paid at the time of the default. In exchange, the buyer of the protection pays a premium until the time of the credit event or until the maturity date of the credit default swap, whichever comes first.

Below we describe mathematically how one can compute the fair value of the CDS premium so that the contract is worth nothing at the beginning of the contract. The present value of the premium of CDS

$$E^Q \left\{ \int_t^T e^{\int_t^u -r_s ds} (1 - \chi(u)) p du \middle| \mathcal{G}_t \right\}$$

where p is the continuous premium paid by the CDS buyer for the default swap contract with maturity T . The present value of the payoff at default can be expressed as

$$E^Q \left\{ \int_t^T e^{\int_t^u -r_s ds} (1 - \chi(u)) l_u \lambda_u du \middle| \mathcal{G}_t \right\}$$

Therefore, the fair value of the CDS premium is

$$p = \frac{E^Q \left\{ \int_t^T e^{\int_t^u -r_s ds} (1 - \chi(u)) l_u \lambda_u du \middle| \mathcal{G}_t \right\}}{E^Q \left\{ \int_t^T e^{\int_t^u -r_s ds} (1 - \chi(u)) du \middle| \mathcal{G}_t \right\}}$$

which by doubly stochasticity assumption turns out to be

$$p = \frac{E^Q \left\{ \int_t^T e^{\int_t^u -(r_s + \lambda_s) ds} l_u \lambda_u du \middle| \mathcal{G}_t \right\}}{E^Q \left\{ \int_t^T e^{\int_t^u -(r_s + \lambda_s) ds} du \middle| \mathcal{G}_t \right\}} \quad (1.10)$$

Equation (1.10) states that, given the processes for interest rate r_t , the default intensity λ_t , the expected loss at default l_t , the ratio of these two expectations gives the fair market CDS premium at the beginning of the contract. In chapter CDS, we develop a multi-factor model for pricing the CDS's.

1.3.5 Financial Interpretation and Multi-name Product Pricing

The intensity-based single-name credit risk models provide a more flexible framework to model the dynamics and the term structure of credit risk than the structural models. Usually they are based upon market variables such as credit spreads and most intensity models can be fitted easily to the term structures of credit spreads and they have effectively become market standard in the pricing of standard credit derivatives such as credit default swaps. In fact, we develop a general hybrid framework in chapter 2, we prefer a purely reduced-form model when we try to calculate CDS premiums in chapter 3. But although they match the market variables well from a calibration point of view, they lack the interpretation of the basic financial variables as opposed to the structural models. You model the credit spread on a single name but you do not have any other source than the individual credit spreads to estimate the model parameters. Hybrid approach developed in section 2.2 takes advantage of having a structural component in default modelling so that part of the parameters can be estimated using market data other than credit spreads.

In recent years, several new derivative securities have been developed whose payoffs depend on the overall default behavior of a whole portfolio of underlying loans or bonds. One of the important examples is basket credit derivatives. A consistent model for default correlation is crucial to price and hedge these structures. Although we do not tackle the problem of pricing basket credit derivatives or hedging, it is an important issue needs to be dealt.

So far there were two approaches that have been taken to extend the reduced-form models to incorporate default correlation and multiple defaults. The most natural approach is to introduce correlation in the dynamics of the default intensities of the names, but to keep the models unchanged otherwise. This approach suffers from several disadvantages. Most importantly the default correlations that can be reached with this approach are typically too low compared to the observed default correlations, and in addition to that it is very hard to derive and analyze the resulting default

dependency structure. In Appendix B we develop an approach modelling default intensity as function of both an exogenously defined stochastic process and a common state factor and obtained more realistic levels of correlation of defaults using a particular type of function for the intensity. Another approach is the infectious defaults model by Davis and Lo (1999, 2000) [12], [13], further developed by Jarrow and Yu (2000) [47]. There default intensities jointly jump upwards by a discrete amount at the onset of a credit crisis. While intuitively very attractive, deriving the default probabilities for a single obligor is already very hard problem in this model which makes calibration very difficult. Let alone the estimation of the jump factor of the default intensities, since it is not clear how this model can be calibrated to historical data.

1.4 Hybrid Model

In credit derivatives market there are quite a few securities that depend on more than one source of risk. Just to mention a few, corporate bonds (which depend on interest rate risk and on credit risk of the issuing firm) and convertible bonds (which depend, in addition, on equity risk). Hence, most attractive credit risk models should involve all these three sources of risk together, i.e. equity risk, credit risk, and interest-rate risk.

Our framework brings together these three standard building blocks. Although we do not include all three when it comes down to actual pricing of an instrument of interest, we provide the most general framework but then focus on the most relevant sources of risk. Also we introduce the appropriate time scales in our stochastic models at every step and exploit different order of time scales to price the instruments approximately as there are no known closed-form solutions most of the time.

The general framework for pricing derivatives on a defaultable security, where the price of the security is modelled as a Geometric Brownian Motion with stochastic volatility, and the default event is modelled by the stopping time τ , the first jump of the counting process N_t . We assume that the short term interest rate process is

constant and equal to r .¹ The stochastic volatility process is defined as a positive, bounded function of an Ornstein-Uhlenbeck process. The default intensity process is also modelled as a positive bounded function of an Ornstein-Uhlenbeck process, where the parameters of the process might depend on the state process that governs the volatility of the security price. The Brownian Motion that drives the dynamics of the security price is correlated with the Brownian Motion that drives the volatility process, allowing the model to mimic empirical features of the returns distribution such as kurtosis and skewness. We also introduce correlation between the security price and the intensity Brownian motions allowing that changes in prices influence the likelihood of default.

$$\begin{cases} dS_t &= \mu S_t dt + \sigma(Y_t) S_t dW_t^S \\ dY_t &= \alpha(m - Y_t)dt + \sqrt{\alpha}\beta_\sigma dW_t^\sigma & dW_t^\sigma = \rho_\sigma dW_t^S + \sqrt{1 - \rho_\sigma^2} dZ_t^\sigma \\ dX_t &= \kappa(b - X_t)dt + \sqrt{\kappa}\beta_\lambda dW_t^{\lambda_t} & dW_t^{\lambda_t} = \rho_\lambda dW_t^S + \sqrt{1 - \rho_\lambda^2} dZ_t^{\lambda_t} \\ \lambda_t &= g(X_t) \end{cases} \quad (1.11)$$

where S is the stock price, σ is the volatility, λ is the instantaneous probability of default of the stock, and g is a positive bounded function which is bounded away from zero.

Clearly, the way the problem is set up gives rise to an incomplete market model in the sense that there exist derivatives that can not be hedged by a portfolio of the basic securities. We carefully derive all possible risk-neutral pricing laws and arrive at the general form of the Radon-Nykodim derivative

$$\begin{aligned} \frac{dP^{**}}{dP} \Big|_{F_t} &= e^{-\int_0^t \theta_1(u) dW_u^S - \frac{1}{2} \int_0^t \theta_1^2(u) du} e^{-\int_0^t \theta_2(u) dZ_u^\sigma - \frac{1}{2} \int_0^t \theta_2^2(u) du} \\ &\quad e^{-\int_0^t \theta_3(u) dZ_u^{\lambda_t} - \frac{1}{2} \int_0^t \theta_3^2(u) du} e^{\int_0^t (1 - \phi_u) \lambda_u du} (1_{\{\tau > t\}} + \phi(\tau) 1_{\{\tau \leq t\}}) \end{aligned}$$

where

$$\theta_1 = \frac{\mu - r - \lambda_t^*}{\sigma(Y_t)}$$

$$\theta_2 = \gamma_t$$

¹This implies that the money market account, the usual instrument adopted for deflation, will be $B_t = e^{rt}$.

$\theta_3 = \delta_t$ where the parameters γ, δ and ϕ are free and corresponds to market prices due to volatility risk, default intensity risk and default time risk, respectively.

1.4.1 Default Intensity Effects

We first study of the simplest product, a defaultable zero coupon bond, and the implied credit yield spreads as a result. We start with the Classical Merton's Model and immediately extend it to a mathematical model where we also have a default intensity process that would allow sudden defaults. And observe the immediate implications on the credit yield spreads and not surprisingly realizing the non-zero spreads in short maturities as opposed to Merton's case and otherwise pretty much the same. Then, we went onto extend the model with a stochastic default intensity, in particular a mean reverting one and through simulations find out that the relevant time scale, i.e. if we would like to obtain an interesting effect of the stochastic intensity as opposed to the constant case, it should have a long time scale so that we have some freedom for the spreads over the long run as well as the non-zero start. It is essential for our analysis that the default intensity process is mean reverting since the notion of time scales and mean reversion are intimately related. The improvement is exhibited in figure 1.1.

Figure 1.1: Solid curve is the spreads produced by the simple hybrid model with constant default intensity and the volatility. Dashed curve is the spreads produced by the model where the default intensity process is a mean reverting stochastic process using the approximate solution derived in section 2.5.2. Qualitatively, the correction to the spreads does not have much short run effect but has a persistent change in the long run as expected from the simulated results. Quantitatively, although approximate formula goes only up to the first order, it still fits the simulated data points with a small least squares error with the calibration of very few parameters.

1.4.2 Stochastic Volatility Effects

We then on top of this framework laid out a stochastic volatility process that drives the equity price in the spirit of Fouque et al (2004)[34]. The class of the process is a Geometric Brownian Motion as usual with stochastic volatility. We introduce another mean reverting process for the volatility and through simulations observe that the fast mean reversion have dramatic effects on the short run which the model with just the stochastic default intensity is missing. Moreover, having different time scales for the two stochastic processes allowed us to exploit the asymptotic analysis. We were able to come up with closed-form approximate solutions for the bond prices and therefore the yield spreads. The improvement is exhibited in figure 1.2.

Figure 1.2: Solid curve is the spreads produced by the simple hybrid model with constant default intensity and the volatility. Dotted curve is the spreads produced by the model where the default intensity process is a mean reverting stochastic process using the approximate solution derived in section 2.7.3. Qualitatively, the correction to the spreads does not have much long run effect as we observed numerical experiments the fast mean reverting stochastic volatility converges to the constant volatility case as the maturity gets large but has a dramatic effect to the short end of the curve. Quantitatively, although approximate formula goes only up to the first order, it still fits the simulated data points with a small least squares error with the calibration of very few parameters.

1.4.3 Connection to Interest Rates

Finally, we explore a model with a two-factor interest rate process that captures the relation of the default risk to a global market variable, interest rates. We introduce the two-factor model in the spirit of Duffie, Pedersen and Singleton[18], that would capture the short-run and long-run term structures well. In order to accommodate for the interaction of each of these two factors we also introduce a two-factor model for the default intensity.

The resulting framework combines, in a single parsimonious model and accounting for interaction between the two. In the appendix we also outline a framework that would include all the three major sources of risk at once. Hence, default information in the model is extracted from equity, fixed income and credit market information rather than just from credit-market information (as in reduced-form credit-risk models) or from just equity-market information (as in structural credit-risk models) and in fact you can observe the effects of each of these risks separately as correction terms in our closed form formulas. In particular, default probabilities may be jointly calibrated to market prices of equity and risky debt. This allows valuation, in a single consistent framework, of hybrid debt-equity securities such as convertible bonds that are vulnerable to default, as well as of derivatives on interest rates, equity and credit. Finally, we estimate the model parameters using a simple least squares estimate for real-world CDS premium data for a fixed time and show that we could calibrate the model so that we could capture the term-structure successfully, as shown in figure 1.3.

Figure 1.3: Above is the default-swap rates for a selected convertible bond issued by an entity which is rated Aaa by Moody's in 1999. Data is available at the website www.neatideas.com. Solid curve is the CDS premium curves produced by the model (3.3) and (3.4) using the asymptotic approximation obtained in (3.19). Used adjusted average interest rate is 6% and expected recovery is 40% for the above curve. Our multi-factor CIR equipped with time scales captures the premium structure that the one factor CIR models would not be able to generate.

1.4.4 Multi-name Products and Correlated Defaults

In credit derivatives market there are lots of multi-name products whose pricing is critically dependent on the correlation of defaults of these different names. Basket default swaps are first-to-default swaps are such examples. The hybrid framework can also serve as a basis for valuing credit portfolios where correlated default is an important source of risk. Although it is not our main focus, in the appendix we try to develop a model to price this kind of products paying particular attention to the default correlations. We try to combine the capital structure models and reduced form models by modelling the default intensities of different names as both a function of the overall market and a function of its individual structure. Modelling the effect of overall market is done through a proxy like a big common index, e.g. S&P 500 and the effect of individual structure is like a surprise default.

The straight forward intuition behind the setting is when the overall market is not doing well, the default probability of each name tend to go up together, not necessarily with the same rate. Or there could be something happening not in the whole market but in a specific sector which would bump up all the default probabilities of names in that sector. In addition to that there could be also something happening within a firm which would only effect that particular firm but not the others. So it is natural to assume the default probabilities(intensities) have two different components, one for the overall market effect and one for the individual firm effect.

In the typical setting of the model, the proxy used to capture the overall market impact is modelled as a Geometric Brownian Motion with stochastic volatility. The stochastic volatility process is defined as a positive, bounded function of an OU process. All the default intensities are modelled as product of a state process, which is an OU process with appropriate parameters and a positive function of the index level above. The Brownian Motions that drive the dynamics of all the state processes that effect the intensities are correlated with each other. One can, in general, introduce the correlation between the Brownian Motions of the index level and state processes but we rather capture that effect in the specific form that we choose for the intensity processes. The generated default time correlations as almost a linear function of one single parameter is depicted in figure 1.4. The level of correlation obtained via this

parameter are much higher compared to a purely reduced-form model.

Figure 1.4: Correlation effect of the parameter in the exponent of the function g , without the effect of the volatility of the index process

1.5 Future Research

In the framework suggested by the general model described in section 2.2, the default information is revealed both by the equity and the debt market rather than just one of them. Hence, the default probabilities can be adjusted to market prices of both equity and risky debt. This allows us to price in a single parsimonious model the kind of hybrid debt-equity securities such as convertible bonds that are exposed to default risk besides other derivatives of interest rates, equity and credit. Immediate step after this work would be the estimation and calibration of the model parameters using actual market data and testing how accurate and stable the model works.

Another research area to pursue would be extending the asymptotic analysis to the general framework of the model where there are more than one obligors and using the method price the kind of derivatives where the payoffs depend on the overall default behavior of a whole portfolio of underlying loans or bonds such basket default swaps and collateralized debt obligations(CDO's).

Our model can also serve as a basis for valuing credit portfolios where correlated default is an important source of risk. Our framework has several close references in the literature. It is intimately linked to the two standard approaches to credit-risk modeling: the class of structural models (Merton (1974)[59], Black-Cox (1976)[6] and the class of reduced-form models (e.g., Duffie and Singleton (1999)[19], Madan and Unal (2000)[56]) and others.

After all the modelling aspects of the problem as a result of the asymptotic analysis we come up with approximate closed form formulas for the prices of instruments

or quantity of interest such as CDS spreads. Not only that the asymptotic results reveal that there are a few group market parameters that needs to be calibrated for pricing purposes. Though the overall models are pretty complicated and have a lot of parameters and even more importantly most of them are unobservable such as risk premia of three different kinds, the asymptotic results suggest a way of calibration of the models through the price of rather liquid derivatives such as CDS's.

Moreover, counter-party risk is also recognized as a major source of risk. So models that capture the correlation structure between default times of different entities are much needed. Incorporating the notion of copulas to the framework to obtain default dependencies is another research direction to pursue.

Chapter 2

Default and Equity Risk in Credit Derivatives

2.1 Stochastic Volatility Models

2.1.1 Stochastic Volatility Models in Finance

The class of stochastic volatility (SV) models has its roots both in mathematical finance and financial econometrics. In fact, several variations of SV models originated from research looking at very different issues. Clark (1973)[10], for instance, suggested to model asset returns as a function of a random process of information arrival. This so-called time deformation approach yielded a time-varying volatility model of asset returns. Later Tauchen and Pitts (1983)[66] refined this work proposing a mixture of distributions model of asset returns with temporal dependence in information arrivals. Hull and White (1987)[43] were not directly concerned with linking asset returns to information arrival but rather were interesting in pricing European options assuming continuous time SV models for the underlying asset. They suggested a diffusion for asset prices with volatility following a positive diffusion process. The diffusion process is a mean reverting one and they compare the price given by their model with the price given by the Black-Scholes when the variance rate in Black-Scholes is put equal to the expected average variance rate during the life of the option. They find that

Black-Scholes overprices that are at the money or close to money, and underprices options that are deep in or deep out of the money. This is also consistent with the pattern of implied volatilities observed for currency options, e.g. see Section 17.2 of [41]. ARCH models, invented by Engle (1982)[23] and a lot of extensions after that besides SV models are mainly built to capture the volatility clustering feature. ARCH models in general can be considered as filters to extract the continuous time conditional variance process from discrete time data. And the seminal contribution of Nelson (1990) [62] is to show that ARCH models converge weakly to a diffusion process and therefore bringing the ARCH and SV models together.

And yet another approach emerged from the work of Taylor (1986) [67] who formulated a discrete time SV model as an alternative to Autoregressive Conditional Heteroskedasticity (ARCH) models. Until recently estimating Taylor's model, or any other SV model, remained almost infeasible. Recent advances in econometric theory have made estimation of SV models much easier. As a result, they have become an attractive class of models and an alternative to other classes such as ARCH. Also, Duan (1995)[14] shows that it is possible to use GARCH(1,1) as the basis for an internally consistent option pricing model.

For derivatives that last less than a year, the pricing impact of stochastic volatility is fairly small in absolute terms (although in percentage terms it could be quite large especially for deep out of the money options). It becomes progressively larger as the life of the derivative increases. In that case one feature that the models seem to like is mean reversion. The term mean reverting refers to the average time it takes a process to pull back to its mean level of its invariant long-run distribution. Empirical studies and common experience show that actually there are volatility clusters, i.e. there is tendency of high volatility coming in bursts. And the concept of burstiness is closely related to mean reversion. A bursty process is returning to its mean and the shorter the periods of the bursts, the more often it returns. These are the main features in a series of papers by Fouque et al. [11],[29],[30],[31].

2.1.2 Stochastic Volatility Models in Credit Risk

Originally, Sharpe (1963) [65] stated the dependence of stocks returns systematic (i.e., market or undiversifiable) risk and idiosyncratic (i.e., specific or diversifiable) risk. Indeed, systematic risk is known to be common to any risky asset in the financial market whereas idiosyncratic risk is peculiar to the asset under consideration. Therefore, credit risky assets (e.g., corporate bonds or debt) should satisfy such a dependence, which is our very first motivation in chapter 3 for a two factor default intensity.

The documented research shed light on the typology and components of credit risk. Given the state of the art, credit risk has to be envisioned along with systematic risk and idiosyncratic risk. Such a typology is used by Gatfaoui (2003) [35] to price risky debt in a Merton (1974)[59] framework where diffusion parameters are constant. However, under its constant parameter assumptions, Mertons model leads to implied spreads which are too low in comparison with observed credit spreads. Indeed, Eom et al. (2003)[25] show that adding stochastic interest rates correlated with the firm value in Mertons model fails to offset this prediction problem about implied credit spreads. To solve this problem, Hull et al. (2003)[42] study the implications of Mertons model about implied at-the-money volatility and volatility skews. Their results lead to several findings which are supported by empirical data. First, implied volatility is sufficient to predict credit spreads. Second, there is a positive relationship between credit spreads and implied volatility, and between volatility skews and both implied credit spreads and implied volatility. Third, implied volatility plays a major role in explaining credit spreads. Finally, as historical volatility leads to implied credit spreads which underestimate their observed counterparts, the implied volatility approach exhibits a superior performance in predicting credit spreads over time. Such findings are consistent with Black and Scholes (1973)[7] option pricing type models. Specifically, such models exhibit a volatility smile effect (i.e., the implied volatility is a U-shaped function of the options moneyness) which is determined by stochastic volatility, maturity and systematic risk among others (see Duque & Lopes (2000)[22] for details, and Psychoyios et al. (2003) [63] for a survey about stylized facts of volatility as well as stochastic volatility models). In the light of such results, Gatfaoui (2004)[36] extends the work of Gatfaoui (2003)[35] to stochastic parameters

in order to price risky debt in a Merton framework with stochastic volatility. Also, Fouque et al. (2004) [34] handle these challenges by introducing stochastic volatility in the dynamics of a defaultable asset and they use the framework of multi-scale stochastic volatility that they have developed in [31] both in equity markets and interest-rate derivatives.

2.2 General Framework under Physical and Risk-Neutral Measures

Let us fix a Probability space $(\Omega, \mathcal{F}, P)$ and the σ - algebra $\mathcal{F}_t = \sigma(W_t^S, Z_t^\sigma, Z^{\lambda_t})$ where $W_t^S, Z_t^\sigma, Z^{\lambda_t}$ are independent standard Brownian Motions. Let's also introduce the σ - algebra $\mathcal{G}_t = \sigma(\mathcal{F}_t \vee N_t)$ where N_t is a nonexplosive doubly stochastic (with respect to $\mathcal{F}_t$) counting process with intensity λ_t , i.e.

- i. λ_t is $\mathcal{F}_t$ predictable and $\int_0^t \lambda_s ds < \infty$ a.s.
- ii. $N_t - \int_0^t \lambda_s ds$ is a $\mathcal{G}_t$ local martingale
- iii. $P\{N_s - N_t = k | \mathcal{G}_t \vee \mathcal{F}_s\} = \frac{e^{-\int_t^s \lambda_u du} (\int_t^s \lambda_u du)^k}{k!}$

In this section we lay out the general framework for pricing derivatives on a defaultable security, where the price of the security is modelled as a Geometric Brownian Motion with stochastic volatility, and the default event is modelled by the stopping time τ , the first jump of the counting process N_t . We assume that the short term interest rate process is constant and equal to r .¹ The stochastic volatility process is defined as a positive, bounded function of an Ornstein-Uhlenbeck process. The default intensity process is also modelled as a positive bounded function of an Ornstein-Uhlenbeck process, where the parameters of the process might depend on the state process that governs the volatility of the security price. The Brownian Motion that drives the dynamics of the security price is correlated with the Brownian Motion that drives the volatility process, allowing the model to mimic empirical features of the returns

¹This implies that the money market account, the usual instrument adopted for deflation, will be $B_t = e^{rt}$.

distribution such as kurtosis and skewness. We also introduce correlation between the security price and the intensity Brownian motions allowing that changes in prices influence the likelihood of default.

$$\begin{cases} dS_t &= \mu S_t dt + \sigma(Y_t) S_t dW_t^S \\ dY_t &= \alpha(m - Y_t) dt + \sqrt{\alpha} \beta_\sigma dW_t^\sigma & dW_t^\sigma = \rho_\sigma dW_t^S + \sqrt{1 - \rho_\sigma^2} dZ_t^\sigma \\ dX_t &= \kappa(b - X_t) dt + \sqrt{\kappa} \beta_\lambda dW_t^{\lambda_t} & dW_t^{\lambda_t} = \rho_\lambda dW_t^S + \sqrt{1 - \rho_\lambda^2} dZ_t^{\lambda_t} \\ \lambda_t &= g(X_t) \end{cases} \quad (2.1)$$

where S is the stock price, σ is the volatility, λ is the instantaneous probability of default of the stock, and g is a positive bounded function which is bounded away from zero.

Clearly, the way the problem is set up gives rise to an incomplete market model in the sense that there exist derivatives that can not be hedged by a portfolio of the basic securities. Assumption of no arbitrage guarantees the existence of a set of equivalent martingale measures.² In this setting, an EMM P^* is a probability measure equivalent to P , under which the discounted price of the defaultable security, $e^{-rt} S_t 1_{\{\tau > t\}}$ is a $\mathcal{G}_t$ -martingale. At this point, we look for all possible EMM's P^* that allow us to write the price of a defaultable object as an expectation in terms of the intensity of the counting process N_t under P^* .³ Let us call the set of all such measures to be $\mathcal{S}$.

Hence, there is an equivalence between the discounted defaultable price being a P^* martingale and the process $e^{\int_0^t (r + \lambda_u) du} S_t$ being a P^* -martingale. And this last process can be interpreted as a non-defaultable security price discounted by interest rates adjusted by the intensity process. The intensity process has the rule of a spread process which increases interest rates due to the default risk.

In order to characterize all EMM's in the set $\mathcal{S}$, we make use of the two versions of the Girsanov's theorem, where one is for changes in the Brownian Filtration and one for the changes in the intensity process λ_t . In order to construct our argument we state the standard version of Girsanov's theorem for a d-dimensional Brownian filtration⁴

²non-empty, non-unitary.

³More general versions, useful for instance in markets with multiple defaults, would allow the intensity to depend not only on a Brownian filtration but also on default events.

⁴For the proof see Karatzas and Shreve [1991].

and also the Girsanov's theorem version for counting processes.⁵

2.2.1 Girsanov's Theorem for Diffusion Processes:

Given $\theta \in (\mathcal{L}^2)^d$, assume that $\xi_t^\theta = e^{-\int_0^t \theta_s dW_s - \frac{1}{2} \int_0^t \theta_s \cdot \theta_s ds}$ is a martingale (Novikov's condition is sufficient.) Then the process W^θ defined as

$$W_t^\theta = W_t + \int_0^t \theta_s ds, \quad 0 \leq t \leq T$$

is a Standard Brownian Motion under the new measure P^* .

Moreover, W^θ has the martingale representation theorem under the new measure P^* where $\frac{dP^*}{dP} = \xi_T^\theta$. Hence, any P^* martingale can be represented as

$$M_t = M_0 + \int_0^t \phi_s dW_s^\theta, \quad t \leq T$$

for some $\phi \in (\mathcal{L}^2)^d$

2.2.2 Girsanov's Theorem for Counting Processes:

Suppose N_t is a nonexplosive counting process with intensity λ_t , and ϕ is a strictly positive predictable process such that, for some fixed T , $\int_0^T \phi_s \lambda_s ds < \infty$ almost surely. Then,

$$\xi_t^\phi = e^{\int_0^t (1-\phi_s) \lambda_s ds} \prod_{\{i: \tau(i) \leq t\}} \phi_{\tau(i)}$$

is a well defined local martingale where $\tau(i)$ is the i^{th} jump time of N_t . In addition, if ξ_t^ϕ is a martingale (bounded ϕ suffices), then an equivalent martingale measure P^* is defined by $\frac{dP^*}{dP} = \xi_T^\phi$. Under this new martingale measure, N_t is still a nonexplosive counting process with intensity $\lambda_t \phi_t$.

Choosing any probability measure P^* under which N_t is still doubly stochastic with respect to $\mathcal{F}_t$, it is true for $s > t$ that: Suppose the counting process N_t is doubly

⁵For the proof see the Appendix A.

stochastic with respect to $\mathcal{F}_t$ under measure P^* , say with an intensity λ^* . Then one can show that:

$$\begin{aligned} E^*\{1_{\{\tau>s\}}|\mathcal{G}_t \vee \mathcal{F}_s\} &= E^*\{1_{\{\tau>t\}}1_{\{N_s-N_t=0\}}|\mathcal{G}_t \vee \mathcal{F}_s\} \\ &= 1_{\{\tau>t\}}E^*\{1_{\{N_s-N_t=0\}}|\mathcal{G}_t \vee \mathcal{F}_s\} \\ &= 1_{\{\tau>t\}}e^{-\int_t^s \lambda_u^* du} \end{aligned} \quad (2.2)$$

If we go from P^* to a measure P^{**} making use of G1, then one can prove that there is an equivalence between the discounted defaultable price being a $\mathcal{G}_t$ P^{**} -martingale and the process $e^{\int_0^t (r+\lambda_u^*)du} S_t$ being a $\mathcal{G}_t$ P^{**} -martingale as follows

$$\begin{aligned} E^{**}\{e^{-rs}S_s1_{\{\tau>s\}}|\mathcal{G}_t\} &= E^*\{\frac{\xi_s}{\xi_t}e^{-rs}S_s1_{\{\tau>s\}}|\mathcal{G}_t\} \\ &= E^*\{E^*\{\frac{\xi_s}{\xi_t}e^{-rs}S_s1_{\{\tau>s\}}|\mathcal{G}_t \vee \mathcal{F}_s\}|\mathcal{G}_t\} \\ &= E^*\{\frac{\xi_s}{\xi_t}e^{-rs}S_sE^*\{1_{\{\tau>s\}}|\mathcal{G}_t \vee \mathcal{F}_s\}|\mathcal{G}_t\} \\ &= 1_{\{\tau>t\}}e^{\int_0^t \lambda_u du}E^*\{\frac{\xi_s}{\xi_t}e^{-\int_0^s (r+\lambda_u^*)du}S_s|\mathcal{G}_t\} \\ &= 1_{\{\tau>t\}}e^{\int_0^t \lambda_u du}E^{**}\{e^{-\int_0^s (r+\lambda_u^*)du}S_s|\mathcal{G}_t\} \end{aligned} \quad (2.3)$$

where we used Bayes rule and equation (2.2).

Let's first take a look at the problem of analyzing what happens to the counting process N_t when we make a change of measure just changing the Brownian Motion Z^{λ_t} . Let P^* be a probability measure equivalent to P defined by the following change in the Brownian filtration: $W_t^{*S} = W_t^S, Z_t^{*\sigma} = Z_t^\sigma, Z_t^{*\lambda} = Z^{\lambda_t} + \int_0^t \theta_u du$, where θ_t is $\mathcal{F}_t$ -measurable and in $\mathcal{L}^2$.

Using Bayes rule, the $\mathcal{F}_t$ -measurability of the process θ_t and the law of iterated expectations we obtain:

$$\begin{aligned} P^*\{N_s - N_t = k|\mathcal{G}_t\} &= \frac{E\{\eta_s 1_{(N_s-N_t=k)}|\mathcal{G}_t\}}{E\{\eta_s E\{1_{(N_s-N_t=k)}|\mathcal{G}_t \vee \mathcal{F}_s\}|\mathcal{G}_t\}} \\ &= \frac{E\{\eta_s \frac{e^{-\int_t^s \lambda_u du} \int_t^s \lambda_u du^k}{k!}|\mathcal{G}_t\}}{E\{\eta_t \frac{e^{-\int_t^s \lambda_u du} \int_t^s \lambda_u du^k}{k!}|\mathcal{G}_t\}} \\ &= E^*\{\frac{e^{-\int_t^s \lambda_u du} \int_t^s \lambda_u du^k}{k!}|\mathcal{G}_t\} \end{aligned} \quad (2.4)$$

Conditioning on $\mathcal{G}_t \vee \mathcal{F}_s$ we see that the process $N_s - N_t$ is Poisson with parameter $\int_t^s \lambda_u du$.

Hence, provided that θ_t is $\mathcal{F}_t$ -measurable and in L^2 , the process N_t preserves its doubly stochasticity property with respect to $\mathcal{F}_t$, for this simple change of measure.

2.2.3 Market Prices of Risk

How do we characterize the elements in $\mathcal{S}$? The idea is to realize a two step change of measure, where we first change the intensity of the counting process going from P to P^* , using a particular case of G2 where N_t counts up to 1. Then, we apply G1 changing the measure from P^* to P^{**} . The Radon-Nikodym derivative from P to P^{**} is, by construction, the product of the two Radon-Nikodym derivatives from P to P^* and from P^* to P^{**} . By theorems G1 and G2

$$\frac{dP^*}{dP} = e^{\int_0^t (1-\phi_s)\lambda_s ds} (1_{\{\tau > t\}} + \phi(\tau)1_{\{\tau \leq t\}}) \quad \text{and} \quad \frac{dP^{**}}{dP^*} = e^{-\int_0^t \theta_s dW_s - \frac{1}{2} \int_0^t \theta_s \cdot \theta_s ds}$$

In general, ϕ and θ are free parameters, but for simplicity we assume that ϕ is deterministic and to guarantee that the process $e^{-rt}1_{\{\tau > t\}}S_t$ is a $\mathcal{G}_t$ martingale, we choose the θ as follows:

$$\theta_t = \begin{bmatrix} \frac{\mu - r - \lambda_t^*}{\sigma(Y_t)} \\ \gamma_t \\ \delta_t \end{bmatrix}$$

where the parameters γ and δ are free. Then the Radon-Nikodym Derivative of the new measure becomes

$$\begin{aligned} \frac{dP^{**}}{dP} \Big|_{F_t} &= e^{-\int_0^t \theta_1(u) dW_u^S - \frac{1}{2} \int_0^t \theta_1^2(u) du} e^{-\int_0^t \theta_2(u) dZ_u^\sigma - \frac{1}{2} \int_0^t \theta_2^2(u) du} \\ &\quad e^{-\int_0^t \theta_3(u) dZ_u^\lambda - \frac{1}{2} \int_0^t \theta_3^2(u) du} e^{\int_0^t (1-\phi_u)\lambda_u du} (1_{\{\tau > t\}} + \phi(\tau)1_{\{\tau \leq t\}}) \end{aligned}$$

where

$$\theta_1 = \frac{\mu - r - \lambda_t^*}{\sigma(Y_t)}$$

$$\theta_2 = \gamma_t$$

$$\theta_3 = \delta_t$$

Then the system (2.1) becomes:

$$\begin{cases} dS_t &= (r + \lambda_t^*)S_t dt + \sigma(Y_t)S_t dW_t^{*S} \\ dY_t &= [\alpha(m - Y_t) - \sqrt{\alpha}\beta_\sigma(\rho_\sigma \frac{\mu-r-\lambda_t^*}{\sigma(Y_t)} + \gamma_t \sqrt{1-\rho_\sigma^2})]dt + \sqrt{\alpha}\beta_\sigma dW_t^{*\sigma} \\ dX_t &= [\kappa(b - X_t) - \sqrt{\kappa}\beta_\lambda(\rho_\lambda \frac{\mu-r-\lambda_t^*}{\sigma(Y_t)} + \theta_t \sqrt{1-\rho_\lambda^2})]dt + \sqrt{\kappa}\beta_\lambda dW_t^{*\lambda} \\ \lambda_t^* &= \phi_t g(X_t) \end{cases}$$

Let's introduce the 3 dimensional vector process U_t as follows.

$$U_t = \begin{bmatrix} S_t \\ Y_t \\ X_t \end{bmatrix}, \text{ then the process followed by } U_t \text{ can be written as:}$$

$$\begin{aligned} dU_t &= \begin{bmatrix} r + \lambda_t^* \\ \alpha(m - Y_t) - \sqrt{\alpha}\beta_\sigma(\rho_\sigma \frac{\mu-r-\lambda_t^*}{\sigma(Y_t)} + \gamma_t \sqrt{1-\rho_\sigma^2}) \\ \kappa(b - X_t) - \sqrt{\kappa}\beta_\lambda(\rho_\lambda \frac{\mu-r-\lambda_t^*}{\sigma(Y_t)} + \theta_t \sqrt{1-\rho_\lambda^2}) \end{bmatrix} dt \\ &+ \begin{bmatrix} \sigma(Y_t)S_t & 0 & 0 \\ \sqrt{\alpha}\rho_\sigma\beta_\sigma & \sqrt{\alpha}\sqrt{1-\rho_\sigma^2}\beta_\sigma & 0 \\ \sqrt{\kappa}\beta_\lambda\rho_\lambda & 0 & \sqrt{\kappa}\sqrt{1-\rho_\lambda^2}\beta_\lambda \end{bmatrix} \begin{bmatrix} dW_t^{*S} \\ dZ_t^{*\sigma} \\ dZ_t^{*\lambda} \end{bmatrix} \end{aligned}$$

Then the Feynman-Kac PDE for the function $P(u, t) = E\{e^{-\int_0^t (r+\lambda_u^*)du} l(U_t)\}$, where $l(U_t) = h(S_t)$, is

$$\begin{aligned} P_t &+ (r + \lambda_t^*)S_t P_S + [\alpha(m - Y_t) - \sqrt{\alpha}\beta_\sigma(\rho_\sigma \frac{\mu-r-\lambda_t^*}{\sigma(Y_t)} + \gamma_t \sqrt{1-\rho_\sigma^2})]P_Y \\ &+ [(\kappa(b - X_t) - \sqrt{\kappa}\beta_\lambda(\rho_\lambda \frac{\mu-r-\lambda_t^*}{\sigma(Y_t)} + \theta_t \sqrt{1-\rho_\lambda^2}))]P_X \\ &+ \frac{1}{2}\sigma^2(Y_t)S_t^2 P_{SS} + \frac{1}{2}\alpha\beta_\sigma^2 P_{YY} + \frac{1}{2}\kappa\beta_\lambda^2 P_{XX} \\ &+ \sqrt{\alpha}\sigma(Y_t)S_t\rho_\sigma\beta_\sigma P_{SY} + \sqrt{\kappa}\sigma(Y_t)S_t\rho_\lambda\beta_\lambda P_{SX} + \sqrt{\alpha\kappa}\rho_\sigma\beta_\sigma\rho_\lambda\beta_\lambda P_{YX} = 0 \end{aligned} \tag{2.5}$$

with the boundary condition $P(U_T, T) = h(S_T)$. Let's introduce the parameters $\Lambda_1 = \rho_\sigma \frac{\mu-r-\lambda_t^*}{\sigma(Y_t)} + \gamma_t \sqrt{1-\rho_\sigma^2}$ and $\Lambda_2 = \rho_\lambda \frac{\mu-r-\lambda_t^*}{\sigma(Y_t)} + \theta_t \sqrt{1-\rho_\lambda^2}$ for notational convenience.

2.3 Immediate Improvements: Simplistic Case

We first recall how the price of a defaultable bond is computed in BS setting.

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

with exogenous default intensity λ , where both σ and λ are constants and the bond pays no dividend. Using the results from section 1, under the risk neutral measure, the asset price becomes

$$S_t = S_0 \exp\left((r + \lambda - \frac{1}{2}\sigma^2)t + \sigma W_t^*\right)$$

where W_t^* is a standard BM under the risk neutral measure, and $\lambda^* = \phi\lambda$, here we assume ϕ is also constant.

In the Merton approach, default occurs if $S_T < K$ for some threshold value K . In this case the price at time t of a defaultable bond is simply the price of a European digital option which pays 1 if S_T exceeds the threshold or 0 otherwise. It is explicitly given by $P^d(t, S_t)$ where

$$\begin{aligned} P^d(t, S_t) &= E^*\{e^{-(r+\lambda^*)(T-t)} 1_{\{S_T > K\}} | S_t = s\} \\ &= e^{-(r+\lambda^*)(T-t)} P^*\{S_T > K | S_t = s\} \\ &= e^{-(r+\lambda^*)(T-t)} P^*\{(r + \lambda - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T^* - W_t^*) > \log \frac{K}{s}\} \\ &= e^{-(r+\lambda^*)(T-t)} P^*\left\{\frac{W_T^* - W_t^*}{\sqrt{T-t}} > -\frac{\log \frac{s}{K} + (r + \lambda - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right\} \\ &= e^{-(r+\lambda^*)\tau} N(d_2(\tau)) \end{aligned} \tag{2.6}$$

with the common notation $\tau = T - t$ and the distance to default

$$d_2(\tau) = \frac{\log \frac{s}{K} + (r + \lambda^* - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \tag{2.7}$$

In the Black & Cox generalization, the default occurs the first time the underlying hits the threshold K as described in [6]. From a probabilistic point of view we have

$$E^*\{1_{\inf_{t \leq u \leq T} S_u > K} | \mathcal{G}_t\} = P^*\left\{\inf_{t \leq u \leq T} \left((r + \lambda^* - \frac{1}{2}\sigma^2)(T - t) + \sigma(W_T^* - W_t^*)\right) > \log \frac{K}{s}\right\}$$

which can be calculated by using the distribution of the minimum of a (non-standard) BM. From a PDE point of view, we have

$$E^*\{e^{-(r+\lambda^*)(T-t)} 1_{\{S_T > K\}} | S_t = s\} = P(t, S_t)$$

where $P(t, s)$ is the solution of the following problem.

$$\begin{cases} \mathcal{L}_{BS}(\sigma, r + \lambda^*)P &= 0 & \text{on } s > K, t < T \\ P(t, K) &= 0 & \text{for } t \leq T \\ P(T, s) &= 1 & \text{for } s > K \end{cases} \quad (2.8)$$

which is to be solved for $s > K$. This problem can be solved by introducing the solution $P^d(t, s)$ of the corresponding digital option problem

$$\begin{cases} \mathcal{L}_{BS}(\sigma, r + \lambda^*)P^d &= 0 & \text{on } s > 0, t < T \\ P(T, s) &= 1 & \text{for } s > K, \quad 0 \text{ otherwise} \end{cases}$$

The price of an European digital option which pays \$1 at maturity if $S_T > K$ and nothing otherwise, is given by the $P^d(t, s)$ at time $t < T$ where $P^d(t, s)$ is computed explicitly in (2.6). It can be checked that the solution $P(t, s)$ of the system (2.8) can be written

$$P(t, s) = P^d(t, s) - \left(\frac{s}{K}\right)^{1 - \frac{2(r+\lambda^*)}{\sigma^2}} P^d\left(t, \frac{K^2}{s}\right) \quad (2.9)$$

The formula was derived in Willmott et al. [69]. Combining (2.9) with (2.6) we get

$$P(t, s) = e^{-(r+\lambda^*)(T-t)} (N(d_+(\tau)) - \left(\frac{s}{K}\right)^{1 - \frac{2(r+\lambda^*)}{\sigma^2}} N(d_-(\tau)))$$

where we denote

$$d_{\pm} + (\tau) = \frac{\pm \frac{s}{K} + (r + \lambda - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

Recall that the yield spread $S(0, T)$ at time 0 is defined by

$$\exp(-S(0, T)T) = \frac{P(0, T)}{B(0, T)}$$

where $B(0, T)$ is the default free zero-coupon bond price given here, in the case of a constant interest rate r , by $B(0, T) = \exp(-rT)$ and $P(0, T) = P(0, T, s)$ which yields the formula

$$S(0, T) = -\frac{1}{T} \log[N(d_+(T)) - (\frac{s}{K})^{1-\frac{2(r+\lambda^*)}{\sigma^2}} N(d_-(T))] + \lambda^* \quad (2.10)$$

In figure 2.3 we show the yield spread curve $S(0, T)$ as a function of maturity T for some typical values of the constant volatility and default intensity, the other parameters are the constant interest rate r and the ratio of initial value to default level $\frac{s}{K}$. As is well documented in the literature, in this first passage model the likelihood of default is essentially zero for short maturities even for highly levered firms, corresponding to $\frac{s}{K}$ close to one, as illustrated in figure 2.1. Long term behavior of the spreads resulting from the hybrid model are clear from equation (2.10) converges back to the initial level λ^* as opposed to the classical first passage case where it goes to 0. As discussed in the first section, the challenge for theoretical pricing models is to raise the average predicted spread relative to crude models such as the constant volatility model presented in this section, without overstating the risks associated with volatility or leverage, but many of the extended models proposed in the literature still have difficulties in predicting realistic credit spreads.

We propose to handle this challenge by introducing a hybrid model with a stochastic exogenous default intensity and stochastic volatility in the dynamics of the defaultable asset at the same time. We explain in the following sections that a naive introduction of stochastic intensity or volatility may modify the credit spreads a little bit, however, a careful modeling of the time scale content of the default intensity and the volatility gives the desired modification in the yield spread at both short and long maturities.

Figure 2.1: Above are two spread curves versus time to maturity. Solid one is the credit yield spreads produced by the purely structural, first passage model and the dashed one is the spreads produced by the hybrid model with a first passage type of structural component and a reduced-form component with a constant default intensity. Additional exogenous default shifts the whole spread curve upwards starting from time 0, hence takes care of the non-zero short term spreads issue and the magnitude of the shift increases for longer maturities. Parameters used are: leverage value is $\frac{s}{K} = 1.3$, the interest rate is 6% and the used volatility level is 12%. Clear from the initial value of the dashed curve the level of default intensity is 3%.

Figure 2.2: As the level of the intensity increases, the hybrid model produces yield spreads that are higher both in the short run and the long run. The effect is persistent and grows with the length of the maturity. While the yield spreads for first passage model, solid curve in this case, converge to 0 for long maturities, they converge to the initial intensity level for the hybrid model. We use a leverage level of $\frac{s}{K} = 1.2$, interest rate is 6%, level of default intensity for the curves from bottom to top 0%,2%,4% and 6%.

Figure 2.3: As the level of the volatility increases, the hybrid model produces yield spreads that emerge from the same level but in the short run there is rapid increase due to the structural part of the defaults. For longer maturities the difference fades out and they converge to the initial level. We use a leverage level of 1.3, interest rate is 6%, level of default intensity is 2% and the level of volatility from bottom to top 10%,15%,20% and 25%.

2.4 Default Intensity with Slow Mean Reversion

In the context of credit markets and derivatives pricing, intensity based models are widely used and in particular stochastic intensity processes are believed to be the appropriate framework for the underlying dynamics. For an extended discussion we refer to Duffie and Singleton[19]. In order to illustrate our approach we consider first the case where the exogenous default intensity process is driven by one factor which we assume to be a mean-reverting Gaussian diffusion, i.e. an Ornstein-Uhlenbeck process. The dynamics under the physical measure is described by the following pair of SDEs

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t^S \\ dX_t &= \kappa(b - X_t)dt + \beta_\lambda \sqrt{2\kappa} dW_t^\lambda \end{aligned}$$

where the actual default intensity is $\lambda_t = g(X_t)$ for a general function g with some assumptions listed below.

Main assumptions of our framework are

- i. The intensity function g is positive, non-decreasing and bounded away and above from 0.
- ii. The invariant distribution of the intensity factor X is the Gaussian distribution with mean b and standard deviation β_λ and it is independent of the parameter κ .
- iii. The important parameter $\kappa > 0$ is the rate of mean reversion of the process X_t . In other words $1/\kappa$ is the time scale of this process, meaning that it reverts to its mean over times of order $1/\kappa$. Small values of κ correspond to slow mean reversion and large values of κ correspond to fast mean reversion.

Also, the standard Brownian motions W^S and W^λ are correlated as

$$E\{dW_t^S dW_t^\lambda\} = \rho_\lambda dt$$

where ρ_λ is a constant correlation coefficient satisfying $\rho_\lambda < 1$. We remark that for the purpose of illustration we choose the intensity factor to be an Ornstein- Uhlenbeck process, however, one could choose different processes as well. Moreover, in our simulations we choose particular volatility functions $g(x) = x^2$ so it is almost like a *CIR* process.

In order to price defaultable bonds under this model for the underlying we rewrite it under a risk neutral measure using previous section's results, chosen by the market through the market price of volatility risk Λ_2 and default risk of ϕ_t , as follows

$$\begin{cases} dS_t &= (r + \lambda_t^*)S_t dt + \sigma S_t dW_t^{*S} \\ dX_t &= [\kappa(b - X_t) - \sqrt{\kappa}\Lambda_2]dt + \beta_\lambda \sqrt{2\kappa} dW_t^{*\lambda} \\ \lambda_t^* &= \phi_t g(X_t) \end{cases} \quad (2.11)$$

Here, W_t^{*S} and $W_t^{*\lambda}$ are standard Brownian motions under the risk-neutral measure and correlated under the physical measure. We assume that the market price of intensity risk Λ_2 is constant and ϕ_t is deterministic.

In section 2.5, we compute the yield spreads that results when we use the stochastic intensity model in (2.11). Our focus is the combined role of the mean reversion time $\frac{1}{\kappa}$ and the correlation ρ_λ on the yield spread curve. We use various values for κ , corresponding to stochastic intensity that range from slowly mean reverting $\kappa = 0.05$ to fast mean reverting $\kappa = 10$. For each value of κ we present the uncorrelated case $\rho_\lambda = 0$ and a negatively correlated case which is $\rho_\lambda = -0.5$. In each figure we plot the yield spread curves as functions of time to maturity, and the solid curve corresponds to a constant intensity. The dashed curve is the yield curve under the stochastic intensity model (2.11), where the initial intensity level $g(X_0)$ and reverting mean level of the intensity are set to the constant intensity case which is 0.01. The constant volatility yields are computed using the explicit formula (2.10). The stochastic intensity yields are computed using Monte Carlo simulations of trajectories for the model (2.11). For these illustrations we choose the following parameter values: $\Lambda_2 = 0$, $\frac{s}{K} = 1.3$, $g(Y_0) = 0.01$, $r = 0.06$, $g(b) = 0.01$, $\beta_\lambda = 0.05$.

Figure 2.4 illustrates the effects of a fast mean reverting default intensity with no

correlation between the intensity process and the underlying process and 2.5 is the same setting with negative correlation which is almost the same. The yields for long maturities are not significantly affected. There is a mild spread increase for shorter maturities and this increase is slightly lower with zero correlation. But this feature of having nonzero short-term spreads are already captured through the nature of the reduced form models. Figure 2.6 illustrates the effects of stochastic default intensity that runs on a medium time scale. We observe that the effect is similar to an increase in constant intensity case as shown in figure 2.2. This effect is again enhanced by negative correlation as seen in figure 2.7.

Figure 2.8 illustrates the effects of a slow mean reverting intensity, without and figure 2.9 with negative correlation. In this case the yields for short maturities are not significantly affected and the effect is enhanced by the presence of a small negative correlation. But in the long run, there is a remarkable difference to the constant intensity case. And this effect is qualitatively and quantitatively very important for the corporate bond type of securities which have longer maturities. This feature of the curve will be captured in our analysis of the default intensity model with a slow mean reverting default intensity in the following section 2.5. We do not conclude from these numerical experiments that the time scale content of default intensity is crucial in the shaping of the yield spread curve but at least have some evidence that if anybody would like to introduce the notion of time scales to the reduced-form models framework, more interesting case would be the long time scale effects. In particular, a long time scale with a negative correlation gives rise to a lot of flexibility for the maturities, as compared to the constant intensity case.

Figure 2.4: Solid curve is the benchmark credit yield curve with constant default intensity and volatility. Dashed curve is the spread curve produced by the hybrid model when the default intensity is a fast mean reverting OU process. Introducing a fast mean reverting stochastic default intensity, i.e. $\kappa = 10$ in system (2.11) increases the credit yield curve slightly for shorter maturities and converges to the constant case for longer maturities. In this experiment we have no correlation between the intensity and the asset process, i.e. $\rho_\lambda = 0$

Figure 2.5: Solid curve is the benchmark credit yield curve with constant default intensity and volatility. Dashed curve is the spread curve produced by the hybrid model when the default intensity is a fast mean reverting OU process. Dotted curve is the spread curve produced by the hybrid model when the default intensity is a fast mean reverting OU process that is also negatively correlated to the asset price process. Introducing a fast mean reverting stochastic default intensity, i.e. $\kappa = 10$ in system (2.11) also having a negative correlation between the asset level process and the default intensity gives a little more flexibility for shorter maturities and again converges to the constant intensity spreads for longer maturities just like the no correlation case. In this experiment we have negative correlation between the intensity and the asset process, i.e. $\rho_\lambda = -0.5$ for the dotted curve

Figure 2.6: Solid curve is the benchmark credit yield curve with constant default intensity and volatility. Dashed curve is the spread curve produced by the hybrid model when the default intensity is a mean reverting OU process. Introducing a mean reverting stochastic default intensity where the speed of mean reversion is medium, i.e. $\kappa = 0.5$ in system (2.11) increases the credit yield curve slightly for mid-range maturities and is not all that different from the constant intensity case for short or long maturities. In this experiment we have no correlation between the intensity and the asset process, i.e. $\rho_\lambda = 0$

Figure 2.7: Solid curve is the benchmark credit yield curve with constant default intensity and volatility. Dashed curve is the spread curve produced by the hybrid model when the default intensity is a mean reverting OU process with a medium speed of mean reversion. Dotted curve is the spread curve produced by the hybrid model when the default intensity is still the same OU process but it is negatively correlated to the asset price process. Introducing a mean reverting stochastic default intensity, i.e. $\kappa = 0.5$ in system (2.11) and having a negative correlation between the asset level process and the default intensity again changes the structure of the credit yield curve only in the mid-range maturities. In this experiment we have negative correlation between the intensity and the asset process, i.e. $\rho_\lambda = -0.5$ for the dotted curve

Figure 2.8: Solid curve is the benchmark credit yield curve with constant default intensity and volatility. Dashed curve is the spread curve produced by the hybrid model when the default intensity is a slow mean reverting OU process. Having a slow mean reverting stochastic default intensity, i.e. $\kappa = 0.05$ in system (2.11) increases the credit yield curve slightly for shorter maturities and keeps its influence for the longer maturities also as opposed to the fast mean reverting case, see figure 2.4. It suggests that the long time scale for the default intensity is more relevant than the short time scale in order to generate a broader class of spread curves. In this experiment we have no correlation between the intensity and the asset process, i.e. $\rho_\lambda = 0$

Figure 2.9: Solid curve is the benchmark credit yield curve with constant default intensity and volatility. Dashed curve is the spread curve produced by the hybrid model when the default intensity is a slow mean reverting OU process. Dotted curve is the spread curve produced by the hybrid model when the default intensity is a slow mean reverting OU process that is also negatively correlated to the asset price process. Having a slow mean reverting stochastic default intensity, i.e. $\kappa = 0.05$ in system (2.11) gives some room for flexibility to the long end of the curve and the effect is mildly boosted with the addition of the negative correlation between the asset level process and the default intensity. The parameter value for correlation between the intensity and the asset process, i.e. $\rho_\lambda = -0.5$ for the dotted curve

2.5 Modelling Default Risk

2.5.1 Merton's Model with Surprise Defaults

Assuming the setting of system (2.11) for the asset price process and the intensity process. In this section we define the structural component of the default as in Merton's case i.e. default happens if S_t is below the default boundary K at time T and also we have a stochastic, mean reverting exogenous default intensity. Motivated by the results of numerical experiments of the previous section 2.4, we choose a slowly mean reverting one i.e. the system (2.11) looks like

$$\begin{cases} dS_t &= (r + \lambda_t^*)S_t dt + \sigma S_t dW_t^{*S} \\ dX_t &= [\delta(b - X_t) - \sqrt{\delta}\Lambda_2]dt + \sqrt{\delta}\beta_\lambda\sqrt{2\kappa}dW_t^{*\lambda} \\ \lambda_t^* &= \phi_t g(X_t) \end{cases} \quad (2.12)$$

where δ is a small parameter. So the price of a defaultable zero-coupon bond in this case would be (2.11).

$$P^\delta(t, s, x) = E^*\{e^{-(r+\lambda(x))(T-t)}h(S_T)|S_t = s, X_t = x\} \quad (2.13)$$

where we denote the δ dependence as a super script and also we drop the $*$ notation for λ for the rest of the section keeping in mind that we are already under the risk neutral measure. As a particular case of the formula (2.5) we obtain

$$\begin{cases} \frac{\partial P^\delta}{\partial t} + \mathcal{L}_{(S,X)}P^\delta - (r + \lambda(x))P^\delta &= 0 \\ P^\delta(T, s, x) &= h(s) \end{cases} \quad (2.14)$$

where $\mathcal{L}_{(S,X)}$ is the infinitesimal generator of the two dimensional Markovian Process (S_t, X_t) . Then we define the operator $\mathcal{L}^\delta$ as follows

$$\mathcal{L}^\delta = \frac{\partial}{\partial t} + \mathcal{L}_{(S,X)} - (r + \lambda(x))(\cdot)$$

so that the system (2.14) becomes:

$$\begin{cases} \mathcal{L}^\delta P^\delta(t, s, x) &= 0 \\ P^\delta(T, s, x) &= h(s) \end{cases} \quad (2.15)$$

If we choose in particular $h(s) = 1$ then we are in the case of a defaultable bond.

In order to calculate the approximate solutions for P^δ we decompose the operator $\mathcal{L}^\delta$ according to the powers of $\sqrt{\delta}$ as follows:

$$\begin{cases} \mathcal{L}^\delta &= \mathcal{L}_2 + \sqrt{\delta}\mathcal{M}_1 + \delta\mathcal{M}_2 \\ \mathcal{L}_2 &= \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2}{\partial s^2} + (r + \lambda(x))(s \frac{\partial}{\partial s} - \cdot) \\ \mathcal{M}_1 &= \beta_\lambda(\rho_\lambda \sigma s \frac{\partial^2}{\partial s \partial x} - \Lambda_2 \frac{\partial}{\partial x}) \\ \mathcal{M}_2 &= \delta(b - x) \frac{\partial}{\partial x} + \frac{1}{2}\beta_\lambda^2 \frac{\partial^2}{\partial x^2} \end{cases} \quad (2.16)$$

We expand the P^δ in powers of $\sqrt{\delta}$

$$P^\delta = P_0 + \sqrt{\delta}P_1 + \delta P_2 + \dots$$

plugging this in equation (2.15) and using the operator notation introduced in (2.16) we get

$$\mathcal{L}_2 P_0 + \sqrt{\delta}(\mathcal{L}_2 P_1 + \mathcal{M}_1 P_0) + \delta(\mathcal{L}_2 P_2 + \mathcal{M}_1 P_1 + \mathcal{M}_2 P_0) + \dots = 0$$

Hence, matching the first two terms and the terminal conditions we can define the P_0 & P_1 as a solution of the following two systems of equations:

$$\begin{cases} \mathcal{L}_2 P_0 &= 0 \\ P_0(T, s, x) &= h(s) \end{cases} \quad (2.17)$$

$$\begin{cases} \mathcal{L}_2 P_1 + \mathcal{M}_1 P_0 &= 0 \\ P_1(T, s, x) &= 0 \end{cases}$$

Both P_0 & P_1 depend on x as a parameter since the operators $\mathcal{L}_2 = \mathcal{L}_{BS}(\sigma, r + \lambda(x))$ do not take derivatives with respect to x variable. Hence, the leading term is the

solution of the system

$$\begin{cases} \mathcal{L}_{BS}(\sigma, r + \lambda(x))P_0 &= 0 \\ P_0(T, s, x) &= h(s) \end{cases}$$

Note that P_0 depends on x only through the adjusted default rate, so x is only a parameter that defines P_0 . So the leading term $P_0 = P_{BS}(\sigma, r + \lambda(x))$ which is given by equation (2.6). Next, we derive an expression for P_1 .

Proposition 1:

Let

$$\mathcal{M}_1 = V_1^\delta s \frac{\partial}{\partial s} \left(\frac{\partial}{\partial x} \right) + V_0^\delta \frac{\partial}{\partial x} \quad (2.18)$$

The function $P_1(t, s, x)$ is explicitly given by

$$P_1(t, s, x) = (T - t) \left(V_1^\delta s \frac{\partial}{\partial s} \left(\frac{\partial P_0}{\partial x} \right) + V_0^\delta \frac{\partial P_0}{\partial x} \right) - \frac{1}{2} (T - t)^2 \left(\frac{\partial \lambda}{\partial x} \left(s \frac{\partial}{\partial s} - \cdot \right)^2 P_0 \right) \quad (2.19)$$

For a proof and the explicit form of the correction term see section *Appendix D*.

2.5.2 First Passage Model with Surprise Defaults

Now we change the problem slightly. We adopt the setting of the section 2.5.1 and system (2.12). But we extend the definition of the structural component of the default to any time the asset process S_t hits the default boundary level K until maturity, T , as opposed to just at time T . Therefore, the price of a defaultable zero-coupon bond in this case would be

$$P^\delta(t, s, x) = E \{ e^{-(r+\lambda(x))(T-t)} h(S_T) 1_{\{\inf_{t \leq u \leq T} S_u > K\}} | S_t = s, X_t = x \}$$

The price function satisfies almost the same PDE system as in the Merton's case except for the additional boundary condition at the default barrier.

$$\begin{cases} \frac{\partial P^\delta}{\partial t} + \mathcal{L}_{(S,X)}P^\delta - (r + \lambda(x))P^\delta &= 0 \\ P^\delta(t, K, x) &= 0 \\ P^\delta(T, s, x) &= h(s) \end{cases}$$

Using barrier options approach, we introduce the function $u(t, s, x)$ as

$$P^\delta(t, s, x) = 1_{\{\inf_{0 \leq u \leq t} S_u > K\}} u(t, S_t = s, X_t = x)$$

where the function $u(t, s, x)$ satisfies, for $s \geq K$, the system

$$\begin{cases} \frac{\partial u}{\partial t} + \mathcal{L}_{(S,X)}u - (r + \lambda(x))u &= 0 & \text{on } s > K, t < T \\ u(t, K, x) &= 0 & \text{for } t \leq T \\ u(T, s, x) &= h(s) & \text{for } s > K \end{cases}$$

The equation solved by the first term of the asymptotic expansion

$$\begin{cases} \mathcal{L}_{BS}(\sigma, r + \lambda(x))u_0 &= 0 & \text{on } s > K, t < T \\ u_0(t, K, x) &= 0 & \text{for } t \leq T \\ u_0(T, s, x) &= h(s) & \text{for } s > K \end{cases}$$

for a frozen value of x . Equation solved by the first correction term

$$\begin{cases} \mathcal{L}_{BS}(\sigma, r + \lambda(x))u_1 + \mathcal{M}_1 u_0 &= 0 & \text{on } s > K, t < T \\ u_1(t, K) &= 0 & \text{for } t \leq T \\ u_1(T, s) &= 0 & \text{for } s > K \end{cases} \quad (2.20)$$

The closed-form solution of the system (2.20) is given in Appendix D. In consequence, we approximate the price of a defaultable zero-coupon bond

$$P^\delta(0, T) \sim u_0(0, s) + \sqrt{\delta} u_1(0, s)$$

Figure 2.10: Solid curve is the spreads produced by the simple hybrid model with constant default intensity and the volatility. Dashed curve is the spreads produced by the model where the default intensity process is a mean reverting stochastic process using the approximate solution derived in section 2.5.2. Qualitatively, the correction to the spreads does not have much short run effect but has a persistent change in the long run as expected from the simulated results. Quantitatively, although approximate formula goes only up to the first order, it still fits the simulated data points with a small least squares error with the calibration of very few parameters. Parameter values are $V_0^\delta = 0.002$, $V_1^\delta = -0.008$, $\lambda^* = 0.01$, $\sigma^* = 0.12$ and $\frac{s}{K} = 1.3$

Resulting spreads from the approximate formula with the fitted parameters are compared to the spreads obtained from the hybrid model with constant intensity in figure 2.10. The difference between the two curves is due to the stochasticity of the default intensity. As expected from the slowly mean reverting choice for the default intensity, the spreads are almost the same in the short term but rather different for longer maturities.

2.6 Stochastic Volatility with Fast Mean Reversion

In the context of equity markets and derivatives pricing and hedging, stochastic volatility is recognized as an essential feature in the modelling of the underlying dynamics. For an extended discussion we refer to Willmott et al.[69] and the references in there. In order to illustrate our approach we consider first the case where volatility is driven by one factor which we assume to be a mean-reverting Gaussian diffusion, i.e. an Ornstein-Uhlenbeck process. The dynamics under the physical measure is described by the following pair of SDEs

$$dS_t = \mu S_t dt + \sigma(Y_t) S_t dW_t^S$$

$$dY_t = \alpha(m - Y_t)dt + \nu\sqrt{2\alpha}dW_t^Y$$

with a constant exogenous default intensity λ . Main assumptions of our framework are

- i. The volatility function σ is positive, non-decreasing and bounded away and above from 0.
- ii. The invariant distribution of the volatility factor Y is the Gaussian distribution with mean m and standard deviation ν and it is independent of the parameter α .
- iii. The important parameter $\alpha > 0$ is the rate of mean reversion of the process Y . In other words $1/\alpha$ is the time scale of this process, meaning that it reverts to its mean over times of order $1/\alpha$. Small values of α correspond to slow mean reversion and large values of α correspond to fast mean reversion.

Also, the standard Brownian motions W^S and W^Y are correlated as

$$E\{dW_t^S dW_t^Y\} = \rho_\sigma dt$$

where ρ_σ is a constant correlation coefficient satisfying $\rho_\sigma < 1$. We remark that for the purpose of illustration we choose the volatility factor to be an Ornstein-Uhlenbeck process, however, in our approach, Y could be any ergodic diffusion with a unique invariant distribution, as explained in more detail in Fouque et al (2000)[27]. Moreover, in our simulations we choose particular volatility functions $\sigma(y)$ as being proportional to $\max(c1, \min(c2, \exp(y)))$, that is the exponential function with lower and upper cutoffs.

In order to price defaultable bonds under this model for the underlying we rewrite it under a risk neutral measure using previous section's results, chosen by the market through the market price of volatility risk Λ_1 and default risk of ϕ , as follows

$$\begin{cases} dS_t &= (r + \lambda^*)S_t dt + \sigma(Y_t)S_t dW_t^{*S} \\ dY_t &= (\alpha(m - Y_t) - \nu\sqrt{2\alpha}\Lambda_1(Y_t))dt + \nu\sqrt{2\alpha}dW_t^{*Y} \end{cases} \quad (2.21)$$

with an exogenous default intensity $\lambda^* = \phi\lambda$. Here, W_t^{*S} and W_t^{*Y} are standard Brownian motions under the risk-neutral measure and correlated under the physical measure. We assume that the market price of volatility risk Λ_1 is bounded and a function of y only whereas ϕ is just a constant.

In this section, we compute the yield spreads that results when we use the stochastic volatility model in (2.21). Our focus is the combined role of the mean reversion time $\frac{1}{\alpha}$ and the correlation ρ_σ on the yield spread curve. We use various values for α , corresponding to volatility factors that range from slowly mean reverting $\alpha = .05$ to fast mean reverting $\alpha = 10$. For each value of α we present the uncorrelated case $\rho_\sigma = 0$ and a negatively correlated case which is $\rho_\sigma = -0.5$. In each figure we plot the yield spread curves as functions of time to maturity, and the solid curve corresponds to a constant volatility. The dashed curve is the yield curve under the stochastic volatility model (2.21), where the initial volatility level $\sigma(Y_0)$ and the long-run average volatility (see (2.22) below) coincide with the volatility level for the constant volatility case. The constant volatility yields are computed using the explicit formula (2.10). The stochastic volatility yields are computed using Monte Carlo simulations of trajectories for the model (2.21). For these illustrations we choose the following parameter values: $\Lambda_1 = 0$, $\frac{s}{K} = 1.2$, $\sigma(Y_0) = 0.12$, $r = 0.06$, $m = 0.12$, $\nu = 0.6$.

Figure 2.11 illustrates the effects of a slowly mean reverting volatility with no correlation between the volatility and the underlying processes and 2.12 is the same setting with negative correlation which is almost the same. The yields for short maturities are not significantly affected. There is a mild spread increase for longer maturities and this increase is slightly lower with zero correlation. This feature of the curve was already captured with the setting of section 2.5.2.

Figure 2.13 illustrates the effects of stochastic volatility that runs on a medium time scale. We observe that the effect is similar to an increase in volatility as shown in Figure 2.3. This effect is again enhanced by negative correlation as seen in figure 2.14. This feature of the curve will be captured in the leading order term by choosing an appropriate effective volatility level σ^* as explained in section 2.7.3.

Figure 2.15 illustrates the effects of a fast mean reverting volatility, without and and figure 2.16 with negative correlation. In this case the yields for short maturities are significantly affected and the effect is enhanced by the presence of a small negative correlation. It is remarkable that this effect is qualitatively and quantitatively very different from the effect resulting from an increase in the volatility level as shown in figure 2.3. This feature of the curve will be captured in our analysis of the stochastic volatility model with a fast mean reverting volatility factor in the following section 2.7.3. We conclude from these numerical experiments that the time scale content of stochastic volatility is crucial in the shaping of the yield spread curve. In particular, a short time scale with a negative correlation gives enhanced spreads at short maturities, as compared with the constant volatility case.

A well separated fast volatility time scale has been observed in equity [30] and fixed income [11] markets. A main feature of this short time scale is that it can be treated by singular perturbation techniques as described in detail in [29]. This leads to a description where the effects of the stochastic volatility can be summarized in terms of three group market parameters, an effective constant volatility σ^* , leverage $\frac{s}{K}$ and a skew parameter V_3^ϵ . Below we will generalize these results to the case of defaultable bonds.

Figure 2.11: Solid curve is the benchmark credit yield curve with constant default intensity and volatility. Dashed curve is the spread curve produced by the hybrid model when the volatility of the underlying asset level process is a slow mean reverting OU process. Introducing a slow mean reverting stochastic volatility, i.e. $\alpha = 0.05$ in system (2.21) increases the credit yield curve slightly for shorter maturities but lifts it up more for longer maturities. In this experiment we have no correlation between the volatility and the asset level process, i.e. $\rho_\sigma = 0$. We also take $\Lambda_1 = 0$ so there is no risk premium due to volatility risk

Figure 2.12: Solid curve is the benchmark credit yield curve with constant default intensity and volatility. Dashed curve is the spread curve produced by the hybrid model when the volatility of the underlying asset level process is an OU process with slow mean reversion. Dotted curve is the spread curve produced by the hybrid model when the volatility is a slow mean reverting OU process that is also negatively correlated to the asset price process. Introducing a slow mean reverting stochastic volatility, i.e. $\alpha = 0.05$ in system (2.21) also having a negative correlation between the asset level process and the volatility pushed the spread curve further up for longer maturities as expected while keeping the short term structure unchanged from the no correlation case. In this experiment we have negative correlation between the volatility and the asset process, i.e. $\rho_\sigma = -0.5$ for the dotted curve.

2.7 Modelling Equity Risk

2.7.1 Merton's Model with Stochastic Volatility

We first review the singular perturbation results in the case of a European option. Let the payoff function at the maturity time T be $h(s)$. The price of the option is obtained as the expected value of the discounted payoff under the risk neutral measure in the stochastic volatility model (2.21):

$$P^\epsilon(t, s, y) = E\{e^{-(r+\lambda^*)(T-t)}h(S_T)|S_t = s, Y_t = y\}$$

In Fouque et al. (2000)[27] it is shown that in the limit of the volatility time scale going to zero, or equivalently *alpha* goes to infinity, the price P^ϵ converges to the

Figure 2.13: Solid curve is the benchmark credit yield curve with constant default intensity and volatility. Dashed curve is the spread curve produced by the hybrid model when the volatility of the underlying asset level process is a mean reverting OU process where the level mean reversion speed is medium, i.e. $\alpha = 0.5$ in system (2.21). The shift in the overall shape is spread out and the lift is larger than the slow mean reverting volatility case which gives more variety overall. In this experiment we have no correlation between the volatility and the asset level process, i.e. $\rho_\sigma = 0$.

Figure 2.14: Solid curve is the benchmark credit yield curve with constant default intensity and volatility. Dashed curve is the spread curve produced by the hybrid model when the the volatility of the underlying asset level process is a mean reverting OU process where the level of mean reversion is medium, i.e. $\alpha = 0.5$ in system (2.21). Having mean reversion in the volatility process combined with the effect of the correlation with the asset level process yields more flexibility to the spread curve for mid-range maturities while not changing too many things about the short or long term structure from the no correlation case. In this experiment we have negative correlation between the volatility and the asset process, i.e. $\rho_\sigma = -0.5$ which resulted the dotted curve.

Figure 2.15: Solid curve is the benchmark credit yield curve with constant default intensity and volatility. Dashed curve is the spread curve produced by the hybrid model when the volatility of the underlying asset level process is a a fast mean reverting OU process, i.e. $\alpha = 10$ in system (2.21). The short term spreads increase much more rapidly allowing for sudden changes in short period of times. Although the idea of having the hybrid model approach takes care of non-zero initial yield spreads, introduction of fast mean reverting stochastic volatility help generating quickly changing term structures. The setting is not any different than the constant case for longer maturities. In this experiment we have no correlation between the volatility and the asset level process, i.e. $\rho_\sigma = 0$.

Figure 2.16: Solid curve is the benchmark credit yield curve with constant default intensity and volatility. Dashed curve is the spread curve produced by the hybrid model when the volatility of the underlying asset level process is a mean reverting OU process where the level of mean reversion is fast, i.e. $\alpha = 10$ in system (2.21). Having fast mean reversion in the volatility process combined with the effect of the correlation with the asset level process yields more dramatic changes to the short end of the spread curve but still converges to the constant volatility case as the maturities get larger. In this experiment we have negative correlation between the volatility and the asset process, i.e. $\rho_\sigma = -0.5$ which pushed up the dashed and resulted the dotted curve.

Black-Scholes price computed with an effective constant volatility $\bar{\sigma}$ given by

$$\bar{\sigma}^2 = \langle \sigma^2(\cdot) \rangle = \frac{1}{\sqrt{2\pi\nu}} \int \sigma^2(y) e^{-\frac{(y-m)^2}{2\nu^2}} dy \quad (2.22)$$

where σ^2 is averaged with respect to the invariant distribution of the OU process. This limiting price, denoted by $\bar{P}_{BS}(t, s)$ satisfies the following system

$$\mathcal{L}_{BS}(\bar{\sigma}, r + \lambda^*) \bar{P}_{BS} = 0$$

$$\bar{P}_{BS}(T, s) = h(s)$$

where $\mathcal{L}_{BS}(\bar{\sigma}, r + \lambda^*)$ is given by

$$\mathcal{L}_{BS}(\bar{\sigma}, r + \lambda^*) = \frac{\partial}{\partial t} + \frac{1}{2} \bar{\sigma}^2 s^2 \frac{\partial^2}{\partial s^2} + (r + \lambda^*) \left(s \frac{\partial}{\partial s} - \cdot \right)$$

The main effects of stochastic volatility are captured by the first order correction proportional to $1/\alpha = \epsilon$ and denoted by $\bar{P}_1(t, s)$. It is given as the solution of the problem

$$\begin{cases} \mathcal{L}_{BS}(\bar{\sigma}, r + \lambda^*) \bar{P}_1 &= -V_2^\epsilon s^2 \frac{\partial^2 \bar{P}_{BS}}{\partial s^2} - V_3^\epsilon s \frac{\partial}{\partial s} \left(s^2 \frac{\partial^2 \bar{P}_{BS}}{\partial s^2} \right) \\ \bar{P}_1(T, s) &= 0 \end{cases}$$

where the parameters V_2^ϵ and V_3^ϵ are small of order ϵ , and are complicated functions of the original model parameters. Note that the first order price approximation

$$P^\epsilon(t, s, y) \sim \bar{P}_{BS}(t, s) + \bar{P}_1(t, s)$$

does not depend on the current level y of the volatility factor which is not directly observed. The calibration is simplified by introducing the corrected effective volatility:

$$\sigma^{*2} = \bar{\sigma}^2 + 2V_2^\epsilon \quad (2.23)$$

and the BS price $\bar{P}_{BS}^*$ computed at the volatility level σ^* . We define the correction term $\bar{P}_1^*$ by

$$\begin{cases} \mathcal{L}_{BS}(\sigma^*, r + \lambda^*) \bar{P}_1^* &= -V_3 s \frac{\partial}{\partial s} (s^2 \frac{\partial^2 \bar{P}_{BS}^*}{\partial s^2}) \\ \bar{P}_1^*(T, s) &= 0 \end{cases}$$

so that

$$\bar{P}(t, s, y) \sim \bar{P}_{BS}^*(t, s) + \bar{P}_1^*(t, s)$$

The accuracy of this approximation is of order $\frac{1}{\alpha}$ in the case of a smooth payoff h , and of order $\frac{\log \alpha}{\alpha}$ in the case of call options as proved in Fouque et al (2003)[31].

Observe that σ^*, λ^* and V_3^ϵ are the only parameters needed to compute this approximation, in fact, they can be calibrated from implied volatilities as explained in Fouque (2004)[34].

2.7.2 First Passage Model with Stochastic Volatility

Here we consider an option that pays $h(S_T)$ at maturity time T if the underlying stays above a level K before time T and zero otherwise. Under the model (2.21) for the underlying, the price at time zero of this down and out barrier option is given by

$$P^\epsilon(0, s, y) = e^{-(r+\lambda^*)T} E^* \{ h(S_T) 1_{\{\inf_{0 \leq u \leq T} S_u > K\}} | S_0 = s, Y_0 = y \}$$

We define $u(t, s, y)$ by

$$u(t, s, y) = e^{-(r+\lambda^*)(T-t)} E^* \{h(S_T) 1_{\{\inf_{t \leq u \leq T} S_u > K\}} | S_t = s, Y_t = y\}$$

so that the price of the barrier option at time t is given by

$$P^\epsilon(t) = 1_{\{\inf_{0 \leq u \leq t} S_u > K\}} u(t, S_t, Y_t)$$

The function $u(t, s, y)$ satisfies for $s \geq K$ the problem

$$\begin{cases} (\frac{\partial}{\partial t} + \mathcal{L}_{S,Y} - (r + \lambda^*))u &= 0 & \text{on } s > K, t < T \\ u(t, K) &= 0 & \text{for } t \leq T \\ u(T, s) &= h(s) & \text{for } s > K \end{cases}$$

where $\mathcal{L}_{S,Y}$ is the infinitesimal generator of the two dimensional process (S, Y) given by (2.21).

Following Fouque et al (2000)[27], in the limit as ϵ approaches zero we find that $u(t, s, y)$ converges to $u_0(t, s)$ which solves the constant volatility problem

$$\begin{cases} \mathcal{L}_{BS}(\bar{\sigma}, r + \lambda^*)u_0 &= 0 & \text{on } s > K, t < T \\ u_0(t, K) &= 0 & \text{for } t \leq T \\ u_0(T, s) &= h(s) & \text{for } s > K \end{cases}$$

As in the European case, for calibration purposes, it is convenient to use the adjusted effective volatility σ^* defined in (2.23). Hence, we define $u_0^*(t, s)$ as the solution of the problem

$$\begin{cases} \mathcal{L}_{BS}(\sigma^*, r + \lambda^*)u_0^* &= 0 & \text{on } s > K, t < T \\ u_0^*(t, K) &= 0 & \text{for } t \leq T \\ u_0^*(T, s) &= h(s) & \text{for } s > K \end{cases} \quad (2.24)$$

and we define the correction $u_1^*(t, s)$ by

$$\left\{ \begin{array}{ll} \mathcal{L}_{BS}(\sigma^*, r + \lambda^*)u_1^* &= -V_3^\epsilon s \frac{\partial}{\partial s} (s^2 \frac{\partial^2 u_0^*}{\partial s^2}) \quad \text{on } s > K, t < T \\ u_1^*(t, K) &= 0 \quad \text{for } t \leq T \\ u_1^*(T, s) &= 0 \quad \text{for } s > K \end{array} \right. \quad (2.25)$$

Note that, the small parameter V_3^ϵ is the same as in the European case.

2.7.3 Pricing Formulas and Implied Spreads

In this section we consider the case $h(s) = 1$ corresponding to a defaultable zero-coupon bond. From formula (2.6) (2.25) in section 2.7.2 we find that u_0^* defined in (2.24) is explicitly given by

$$u_0^*(t, s) = e^{-(r+\lambda^*)(T-t)} (N(d_+) - (\frac{s}{K})^p N(d_-)) \quad (2.26)$$

with the notation

$$p = 1 - \frac{2(r + \lambda^*)}{\sigma^{*2}}, \quad d_{\pm} = \frac{\pm \log(\frac{s}{K}) - \frac{p\sigma^{*2}}{2}\tau}{\sigma^* \sqrt{\tau}}$$

where we dropped the $*$ and τ inside $d_{\pm}$ for notational simplicity. As a result of all, the price of the defaultable bond at time 0 is approximated by

$$\bar{P}(0, T) \sim u_0^*(0, s) + \sqrt{\epsilon} u_1^*(0, s)$$

where $u_0^*(t, s)$ is given in equation (2.26) and $u_1^*(t, s) = y_1(t, s) + y_2(t, s)$ which are derived in Appendix E. Here we state the final answer.

$$\begin{aligned} y_1(t, s) &= (T - t) V_3^\epsilon e^{-(r+\lambda(x))(T-t)} [N'(d_+) [\frac{d_+^2 - 1}{(\sigma^* \sqrt{\tau})^3} + \frac{d_+}{(\sigma^* \sqrt{\tau})^2}] \\ &+ N'(d_-) [\frac{d_-^2 - 1}{(\sigma^* \sqrt{\tau})^3} + \frac{(3p-1)d_-}{(\sigma^* \sqrt{\tau})^2} + \frac{p(3p-2)}{(\sigma^* \sqrt{\tau})}] (\frac{s}{K})^p \\ &+ N(d_-) [(1-p)p^2] (\frac{s}{K})^p] \end{aligned}$$

Figure 2.17: Solid curve is the spreads produced by the simple hybrid model with constant default intensity and the volatility. Dotted curve is the spreads produced by the model where the default intensity process is a mean reverting stochastic process using the approximate solution derived in section 2.7.3. Qualitatively, the correction to the spreads does not have much long run effect as we observed numerical experiments the fast mean reverting stochastic volatility converges to the constant volatility case as the maturity gets large but has a dramatic effect to the short end of the curve. Quantitatively, although approximate formula goes only up to the first order, it still fits the simulated data points with a small least squares error with the calibration of very few parameters. Used parameter values are $V_3^\epsilon = -0.003$ and $\sigma^* = 0.12$ and $\frac{s}{K} = 1.2$

and

$$y_2(t, s) = \frac{(\frac{s}{K})^{p/2}}{\sigma^* \sqrt{2\pi}} \int_t^T \frac{\log \frac{s}{K}}{(z-t)^{3/2}} e^{-\frac{\log(\frac{s}{K})^2}{2\sigma^{*2}(z-t)}} e^{(\frac{(\sigma^* p)^2}{8} + r + \lambda^*)(z-t)} g(z) dz$$

Figure 2.18: Solid curve is the spreads produced by the simple hybrid model with constant default intensity and the volatility. Dotted curve is the spreads produced by the model where the stochastic volatility process is a mean reverting one using with a speed of mean reversion parameter $\alpha = 10$. Qualitatively, the correction to the spreads does not have much long run effect and converges to the constant volatility case as the maturity gets large but has a dramatic effect to the short end of the curve. Dotted curve is the exact same setting with an even faster mean reverting process where $\alpha = 20$. although the short term yields increase a little more, the long run behavior is still the same. Existence of a long time scale is important in having long term flexibility for credit yield spreads.

2.8 Modelling Default and Equity Risk:

2.8.1 Combined Framework and Mathematical Derivations

Just like the previous section's approach, we start from the case of pricing of a European derivative this time under the setting of (2.5) where we have both stochastic volatility and the intensity at the same time. Therefore we have the slow and fast time scales together, we will carry out a two dimensional version of the asymptotic analysis done in sections 2.5 and 2.7. As a result of the analysis we will have two correction terms, one due to short scale and another one due to long. And they would separately represent the additional default risk for the stochastic intensity and the addition equity risk for the stochastic volatility.

Before we move on to the mathematical analysis part, we run simulations to test the flexibility of each factor and the interaction between the two factors. As seen in figure 2.18, no matter how big the speed of mean reversion for the volatility factor it always converges to a constant volatility model in the long run. Although it pushed the short term spreads upwards, has almost no long term effect.

On the other hand as figure 2.19 suggests, the slow mean reversion factor takes over in the long run and provides some freedom to the long end of the spread curve without changing it much in the short end. And the effect is boosted by the negative correlation parameter as in figure 2.20 We start with computing

Figure 2.19: Solid curve is the spreads produced by the simple hybrid model with constant default intensity and the volatility. Dotted curve is the spreads produced by the model where the stochastic volatility process is a mean reverting one using with a speed of mean reversion parameter $\alpha = 10$. Dotted curve is the hybrid model where we also a stochastic default intensity. A slow mean reverting process is used, $\kappa = 0.1$ and we observe the effect of the addition to the long end of the curve without changing too much the short end of the curve.

Figure 2.20: Solid curve is the spreads produced by the simple hybrid model with constant default intensity and the volatility. Dashed curve is the hybrid model where we have a fast mean reverting stochastic volatility and slow mean reverting default intensity, that is $\alpha = 10$ and $\kappa = 0.1$ in (2.5). We further boost the effect of the stochastic default intensity with the introduction of a negative correlation of 50% with the underlying state process. It effects the curve overall both in the long run and short run. The combination of the time scales and the correlation parameters yields a broad set of credit yield spreads with realistic features. Parameter value for correlation is $\rho_\lambda = -0.5$

$$P^{\epsilon, \delta}(t, S_t, Y_t, X_t) = E^* \{ e^{-(r+\lambda^*)(T-t)} h(S_T) | S_t, Y_t, X_t \} \quad (2.27)$$

$$\begin{cases} \frac{\partial P^{\epsilon, \delta}}{\partial t} + \mathcal{L}_{(S, Y, X)} P^{\epsilon, \delta} - (r + \lambda^*) P^{\epsilon, \delta} &= 0 \\ P^{\epsilon, \delta}(T, s, y, x) &= h(s) \end{cases} \quad (2.28)$$

where $\mathcal{L}_{(S, Y, X)}$ is the infinitesimal generator of the Markovian Process (S_t, Y_t, X_t) . Then we define the operator $\mathcal{L}^{\epsilon, \delta}$ as follows

$$\mathcal{L}^{\epsilon, \delta} = \frac{\partial}{\partial t} + \mathcal{L}_{(S, Y, X)} - (r + \lambda(x))(\cdot)$$

so that the system (2.28) becomes:

$$\begin{cases} \mathcal{L}^{\epsilon, \delta} P^{\epsilon, \delta} &= 0 \\ P^{\epsilon, \delta}(T, s, y, x) &= h(s) \end{cases} \quad (2.29)$$

If we choose in particular $h(s) = 1$ then we are in the case of defaultable bond.

In order to calculate the approximate solutions for $P^{\epsilon, \delta}$ we decompose the operator $\mathcal{L}^{\epsilon, \delta}$ according to the powers of $\sqrt{\epsilon}$ and $\sqrt{\delta}$ as follows:

$$\left\{ \begin{array}{lcl} \mathcal{L}^{\epsilon, \delta} & = & \frac{1}{\epsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}} \mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\delta} \mathcal{M}_1 + \delta \mathcal{M}_2 + \sqrt{\frac{\delta}{\epsilon}} \mathcal{M}_3 \\ \mathcal{L}_0 & = & (m - y) \frac{\partial}{\partial y} + \frac{1}{2} \beta_\sigma^2 \frac{\partial^2}{\partial y^2} \\ \mathcal{L}_1 & = & \beta_\sigma (\rho_\sigma \sigma(y) s \frac{\partial^2}{\partial s \partial y} - \Lambda_1(y, x) \frac{\partial}{\partial y}) \\ \mathcal{L}_2 & = & \frac{\partial}{\partial t} + \frac{1}{2} \sigma^2(y) s^2 \frac{\partial^2}{\partial s^2} + (r + \lambda(x)) (s \frac{\partial}{\partial s} - \cdot) \\ \mathcal{M}_1 & = & \beta_\lambda (\rho_\lambda \sigma(y) s \frac{\partial^2}{\partial s \partial x} - \Lambda_2(y, x) \frac{\partial}{\partial x}) \\ \mathcal{M}_2 & = & \delta (b - x) \frac{\partial}{\partial x} + \frac{1}{2} \beta_\lambda^2 \frac{\partial^2}{\partial x^2} \\ \mathcal{M}_3 & = & \beta_\sigma \rho_\sigma \rho_\lambda \beta_\lambda \frac{\partial^2}{\partial y \partial x} \end{array} \right. \quad (2.30)$$

We expand the $P^{\epsilon, \delta}$ in powers of $\sqrt{\delta}$ first

$$P^{\epsilon, \delta} = P_0^\epsilon + \sqrt{\delta} P_1^\epsilon + \delta P_2^\epsilon + \dots$$

plugging this in equation (2.29) and using the operator notation introduced in (2.30) we get

$$\left(\frac{1}{\epsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}} \mathcal{L}_1 + \mathcal{L}_2 \right) P_0^\epsilon + \sqrt{\delta} \left\{ \left(\frac{1}{\epsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}} \mathcal{L}_1 + \mathcal{L}_2 \right) P_1^\epsilon + \left(\mathcal{M}_1 + \frac{1}{\sqrt{\epsilon}} \mathcal{M}_3 \right) P_0^\epsilon \right\} + \dots = 0 \quad (2.31)$$

Hence, matching the first two terms and the terminal conditions we can define the P_0^ϵ & P_1^ϵ as a solution of the following two systems of equations:

$$\left\{ \begin{array}{lcl} \left(\frac{1}{\epsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}} \mathcal{L}_1 + \mathcal{L}_2 \right) P_0^\epsilon & = & 0 \\ P_0^\epsilon(T, s, y, x) & = & h(s) \end{array} \right.$$

$$\left\{ \begin{array}{lcl} \left(\frac{1}{\epsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}} \mathcal{L}_1 + \mathcal{L}_2 \right) P_1^\epsilon + \left(\mathcal{M}_1 + \frac{1}{\sqrt{\epsilon}} \mathcal{M}_3 \right) P_0^\epsilon & = & 0 \\ P_1^\epsilon(T, s, y, x) & = & 0 \end{array} \right.$$

Both P_0^ϵ & P_1^ϵ depend on x as a parameter since the operators $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2$ do not take derivatives with respect to x . Consider the first term P_0^ϵ and expand it as

$$P_0^\epsilon = P_0 + \sqrt{\epsilon}P_{1,0} + \epsilon P_{2,0} + \epsilon^{3/2}P_{3,0} + \dots \quad (2.32)$$

In the notation $P_{i,j}$ i corresponds to power of $\sqrt{\epsilon}$ and j corresponds to the power of $\sqrt{\delta}$. Our ultimate goal in this section would be calculating the approximate formula

$$P^{\epsilon,\delta} \approx P_0 + \sqrt{\epsilon}P_{1,0} + \sqrt{\delta}P_{0,1}$$

Inserting (2.32) in (2.31) we get

$$\begin{aligned} & \frac{1}{\epsilon}\mathcal{L}_0P_0 + \frac{1}{\sqrt{\epsilon}}(\mathcal{L}_0P_{1,0} + \mathcal{L}_1P_0) \\ & + (\mathcal{L}_0P_{2,0} + \mathcal{L}_1P_{1,0} + \mathcal{L}_2P_0) \\ & + \sqrt{\epsilon}(\mathcal{L}_0P_{3,0} + \mathcal{L}_1P_{2,0} + \mathcal{L}_2P_{1,0}) \\ & + \dots = 0 \end{aligned} \quad (2.33)$$

Matching the first term and using (2.30) we get

$$\mathcal{L}_0P_0 = (\alpha(m-y)\frac{\partial}{\partial y} + \frac{1}{2}\beta_\sigma^2\frac{\partial^2}{\partial y^2})P_0 = 0$$

Here we choose P_0 to be independent of y so that we do not get any unreasonable growth, i.e. $P_0 = P_0(t, s, x)$. Also matching the second term gives us

$$\mathcal{L}_0P_{1,0} + \mathcal{L}_1P_0 = 0 \quad (2.34)$$

But since the operator $\mathcal{L}_1$ takes derivatives with respect to y and P_0 is constant in y $\mathcal{L}_1P_0 = 0$ and the equation (2.34) reduces to

$$\mathcal{L}_0P_{1,0} = 0$$

Now, we also choose $P_{1,0}$ not to depend on y , i.e. $P_{1,0} = P_{1,0}(t, s, x)$. Then, order 1 terms in (2.33) gives

$$\mathcal{L}_0 P_{2,0} + \mathcal{L}_2 P_0 = 0 \quad (2.35)$$

Notice that (2.35) is a Poisson equation in $P_{2,0}$ with respect to y variable. So only reasonable solution occurs in case of $\mathcal{L}_2 P_0$ being in the orthogonal complement of $\mathcal{L}_0^*$. Namely,

$$\langle \mathcal{L}_2 P_0 \rangle = 0 \quad (2.36)$$

where the $\langle \rangle$ denotes the integration with respect to the invariant distribution Φ of the process Y_t . Since, P_0 does not depend on y $\langle \mathcal{L}_2 P_0 \rangle = \langle \mathcal{L}_2 \rangle P_0$ and

$$\langle \mathcal{L}_2 \rangle = \frac{\partial}{\partial t} + \frac{1}{2} \langle \sigma^2(\cdot) \rangle s^2 \frac{\partial^2}{\partial s^2} + (r + \lambda(x)) \left(s \frac{\partial}{\partial s} - \cdot \right)$$

which is the $\mathcal{L}_{BS}(\bar{\sigma}, r + \lambda(x))$ where $\bar{\sigma}$ is the effective volatility and $r + \lambda(x)$ is the default adjusted short rate.

$$\bar{\sigma}^2 = \langle \sigma^2(\cdot) \rangle = \int \sigma^2(y) \Phi(dy)$$

Hence, the leading term is the solution of the system

$$\begin{cases} \mathcal{L}_{BS}(\bar{\sigma}, r + \lambda(x)) P_0 &= 0 \\ P_0(T, s, x) &= h(s) \end{cases} \quad (2.37)$$

Note that P_0 depends on x only through the adjusted default rate, so x is only a parameter that defines P_0 . Next, we derive an expression for $P_{1,0}$. $\sqrt{\epsilon}$ term in (2.33) gives

$$\mathcal{L}_0 P_{3,0} + \mathcal{L}_1 P_{2,0} + \mathcal{L}_2 P_{1,0} = 0$$

Then, the same averaging condition gives

$$\langle \mathcal{L}_1 P_{2,0} \rangle + \langle \mathcal{L}_2 \rangle P_{1,0} = 0$$

Then, (2.35) and (2.36) gives

$$P_{2,0} = -\mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle)P_0$$

Introduce the notation

$$\mathcal{A} = -\langle \mathcal{L}_1 \mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) \rangle \quad (2.38)$$

Then, we can define the $P_{1,0}$ as the solution of the system of system

$$\begin{cases} \mathcal{L}_{BS}(\bar{\sigma}, r + \lambda^*(x))P_{1,0} + \mathcal{A}P_0 &= 0 \\ P_{1,0}(T, s, x) &= 0 \end{cases}$$

Observe that, $P_{1,0}$ is the solution of a Black-Scholes equation with a source term and zero terminal condition.

Computation of the operator $\mathcal{A}$:

Let $\psi(y)$ be the solution of the Poisson Equation with respect to y

$$\mathcal{L}_0\psi(y) = \sigma^2(y) - \bar{\sigma}^2$$

then

$$\mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) = \frac{1}{2}\psi(y)s^2\frac{\partial^2}{\partial s^2}$$

and therefore

$$\begin{aligned} \mathcal{L}_1 \mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) &= \beta_\sigma(\rho_\sigma \sigma(y)s\frac{\partial^2}{\partial s \partial y} - \Lambda_1(y, x)\frac{\partial}{\partial y})\frac{1}{2}\psi(y)s^2\frac{\partial^2}{\partial s^2} \\ &= \frac{\rho_\sigma \beta_\sigma \sigma(y)}{2}\frac{\partial \psi}{\partial y}s\frac{\partial}{\partial s}(s^2\frac{\partial^2}{\partial s^2}) - \frac{\beta_\sigma \Lambda_1(y, x)}{2}\frac{\partial \psi}{\partial y}s^2\frac{\partial^2}{\partial s^2} \end{aligned} \quad (2.39)$$

By averaging in y with respect to the invariant distribution and using (2.38) of the operator $\mathcal{A}$ we get

$$\mathcal{A} = \frac{1}{2} \langle \beta_\sigma \Lambda_1(y, x) \frac{\partial \psi}{\partial y} \rangle s^2 \frac{\partial^2}{\partial s^2} - \frac{\rho_\sigma}{2} \langle \beta_\sigma \sigma(y) \frac{\partial \psi}{\partial y} \rangle s \frac{\partial}{\partial s} s^2 \frac{\partial^2}{\partial s^2} \quad (2.40)$$

$$\equiv V_3^\epsilon s^2 \frac{\partial^2}{\partial s^2} - V_2^\epsilon s \frac{\partial}{\partial s} s^2 \frac{\partial^2}{\partial s^2} \quad (2.41)$$

where the bracketed terms are functions of x

Proposition 1:

The function $P_{1,0}(t, s, x)$ is explicitly given by

$$P_{1,0} = (T - t) \mathcal{A} P_0$$

where the operator $\mathcal{A}$ is as in equation (2.38) and P_0 as in equation (2.37).

Consider the term P_1^ϵ and expand it as

$$P_1^\epsilon = P_{0,1} + \sqrt{\epsilon} P_{1,1} + \epsilon P_{2,1} + \epsilon^{3/2} P_{3,1} + \dots \quad (2.42)$$

Inserting (2.42) in (2.33) we get

$$\begin{aligned} & \frac{1}{\epsilon} \mathcal{L}_0 P_{0,1} + \frac{1}{\sqrt{\epsilon}} (\mathcal{L}_0 P_{1,1} + \mathcal{L}_1 P_{0,1} + \mathcal{M}_3 P_0) \\ & + (\mathcal{L}_0 P_{2,1} + \mathcal{L}_1 P_{1,1} + \mathcal{L}_2 P_{0,1}) + \mathcal{M}_1 P_0 + \mathcal{M}_3 P_{1,0} \\ & + \sqrt{\epsilon} (\mathcal{L}_0 P_{3,1} + \mathcal{L}_1 P_{2,1} + \mathcal{L}_2 P_{1,1} + \mathcal{M}_1 P_{1,0} + \mathcal{M}_3 P_{2,0}) \\ & + \dots = 0 \end{aligned} \quad (2.43)$$

Matching the first term and using (2.30) we get

$$\mathcal{L}_0 P_{0,1} = (\alpha(m - y) \frac{\partial}{\partial y} + \frac{1}{2} \beta_\sigma^2 \frac{\partial^2}{\partial y^2}) P_{0,1} = 0$$

Here we choose $P_{0,1}$ to be independent of y just like for P_0 . Also matching the second term gives us

$$\mathcal{L}_0 P_{1,1} + \mathcal{L}_1 P_{0,1} + \mathcal{M}_3 P_0 = 0 \quad (2.44)$$

But since the operator $\mathcal{L}_1$ takes derivatives with respect to y and $P_{0,1}$ is constant in y $\mathcal{L}_1 P_{0,1} = 0$. Similarly, $\mathcal{M}_3$ takes derivative with respect to y and P_0 does not depend on y . Hence, the equation (2.44) reduces to

$$\mathcal{L}_0 P_{1,1} = 0$$

Now, we also choose $P_{1,1}$ not to depend on y , i.e. $P_{1,1} = P_{1,1}(t, s, x)$. Then, order 1 terms in (2.43), using the facts $\mathcal{L}_1 P_{1,1} = \mathcal{M}_3 P_{1,0}$ gives us

$$\mathcal{L}_0 P_{2,1} + \mathcal{L}_2 P_{0,1} + \mathcal{M}_1 P_0 = 0 \quad (2.45)$$

Notice that (2.45) is a Poisson equation in $P_{2,1}$ with respect to y variable. Then, the averaging condition in this case turns out to be

$$\langle \mathcal{L}_2 P_{0,1} + \mathcal{M}_1 P_0 \rangle = \langle \mathcal{L}_2 \rangle P_{0,1} + \langle \mathcal{M}_1 \rangle P_0 = 0$$

where $\langle \mathcal{M}_1 \rangle$ is

$$\langle \mathcal{M}_1 \rangle = \rho_\lambda \langle \beta_\lambda(\cdot) \sigma(\cdot) \rangle s \frac{\partial^2}{\partial s \partial x} - \langle \beta_\lambda(\cdot) \Lambda_2(\cdot, x) \rangle \frac{\partial}{\partial x} \quad (2.46)$$

$$\equiv V_0^\delta s \frac{\partial^2}{\partial s \partial x} - V_1^\delta \frac{\partial}{\partial x} \quad (2.47)$$

since the terminal condition $h(x)$ is taken care of by the P_0 term, we can write the $P_{0,1}$ as a solution to the system

$$\begin{cases} \mathcal{L}_{BS}(\bar{\sigma}, r + \lambda(x)) P_{0,1} + \langle \mathcal{M}_1 \rangle P_0 &= 0 \\ P_{0,1}(T, s, x) &= 0 \end{cases}$$

In fact, we can explicitly calculate the term $P_{0,1}$ as in the following proposition. See Appendix D for the actual derivation.

Proposition 2:

The function $P_{0,1}(t, s, x)$ is explicitly given by

$$P_{0,1} = (T - t) \langle \mathcal{M}_1 \rangle P_0 - \frac{1}{2}(T - t)^2 \left(\frac{\partial \lambda}{\partial x} \left(s \frac{\partial}{\partial s} - \cdot \right)^2 P_0 \right) \quad (2.48)$$

We start writing the first order approximation again by using the propositions of the previous section

$$\begin{aligned} P^{\epsilon, \delta} &\approx P_0 + \sqrt{\epsilon} P_{1,0} + \sqrt{\delta} P_{0,1} \\ &= P_0 + (T - t) (\sqrt{\epsilon} \mathcal{A} + \sqrt{\delta} \langle \mathcal{M}_1 \rangle) P_0 - \frac{1}{2}(T - t)^2 \left(\frac{\partial \lambda}{\partial x} \left(s \frac{\partial}{\partial s} - \cdot \right)^2 P_0 \right) \end{aligned}$$

2.8.2 Risk due to Equity Risk

The short time scale contribution to $P^{\epsilon, \delta}$ is $(T - t) \sqrt{\epsilon} \mathcal{A} P_0$ so we consider the operator

$$\sqrt{\epsilon} \mathcal{A} = \frac{\sqrt{\epsilon}}{2} \langle \beta_\sigma \Lambda_1(\cdot, x) \frac{\partial \psi}{\partial y} \rangle s^2 \frac{\partial^2}{\partial s^2} - \frac{\rho_\sigma \sqrt{\epsilon}}{2} \langle \beta_\sigma \sigma(\cdot) \frac{\partial \psi}{\partial y} \rangle s \frac{\partial}{\partial s} s^2 \frac{\partial^2}{\partial s^2}$$

Then, we introduce the following notation for two parameters

$$\begin{cases} V_2^\epsilon &= \frac{\sqrt{\epsilon}}{2} \langle \beta_\sigma \Lambda_1(\cdot, x) \frac{\partial \psi}{\partial y} \rangle \\ V_3^\epsilon &= -\frac{\rho_\sigma \sqrt{\epsilon}}{2} \langle \beta_\sigma \sigma(\cdot) \frac{\partial \psi}{\partial y} \rangle \end{cases} \quad (2.49)$$

Now, we can rewrite the short time scale contribution to $P^{\epsilon, \delta}$ as

$$(T - t) \sqrt{\epsilon} \mathcal{A} P_0 = (T - t) (V_2^\epsilon s^2 \frac{\partial^2}{\partial s^2} + V_3^\epsilon s \frac{\partial}{\partial s} s^2 \frac{\partial^2}{\partial s^2}) P_0 \quad (2.50)$$

2.8.3 Risk due to Default Risk

The long time scale contribution to $P^{\epsilon, \delta}$ is $(T - t) \sqrt{\delta} \langle \mathcal{M}_1 \rangle P_0 - \frac{1}{2}(T - t)^2 \left(\frac{\partial \lambda}{\partial x} \left(s \frac{\partial}{\partial s} - \cdot \right)^2 P_0 \right)$ so we consider the operator

$$\sqrt{\delta} \langle \mathcal{M}_1 \rangle = \left[\frac{\rho_\lambda \sqrt{\delta}}{2} \langle \beta_\lambda \sigma(\cdot) \rangle \right] s \frac{\partial}{\partial s} \left(\frac{\partial}{\partial x} \right) + \left[-\frac{\sqrt{\delta}}{2} \langle \beta_\lambda \Lambda_2(\cdot, x) \rangle \right] \frac{\partial}{\partial x} \quad (2.51)$$

Then, we introduce the following notation for two parameters

$$\begin{cases} V_0^\delta &= -\frac{\sqrt{\delta}}{2} < \beta_\lambda \Lambda_2(\cdot, x) > \\ V_1^\delta &= \frac{\rho_\lambda \sqrt{\delta}}{2} < \beta_\lambda \sigma(\cdot) > \end{cases}$$

Now, we can rewrite the long time scale contribution to $P^{\epsilon, \delta}$ as

$$(T-t)\sqrt{\epsilon} < \mathcal{M}_1 > P_0 = (T-t)(V_1^\delta s \frac{\partial}{\partial s} (\frac{\partial}{\partial x}) + V_0^\delta \frac{\partial}{\partial x}) P_0 \quad (2.52)$$

In summary, for the European case we calculate the default risk contribution as follows

$$(T-t)\sqrt{\epsilon} < \mathcal{M}_1 > P_0 - \frac{1}{2}(T-t)^2 \left(\frac{\partial \lambda}{\partial x} \left(s \frac{\partial}{\partial s} - \cdot \right)^2 P_0 \right)$$

where the three terms on RHS are respectively given by the equations (2.51) and (D.10).

2.8.4 Bond Price and Yield Approximations

Again, we go to the case of pricing of the defaultable zero coupon bond where the default could occur any time until the maturity. This time we simply write out the final equations as we derived them separately in sections 2.5 and 2.7. Objective is to compute the expectation

$$P^{\epsilon, \delta}(t, S_t, Y_t, X_t) = E^* \{ e^{-(r+\lambda^*)(T-t)} h(S_T) 1_{\inf_{t \leq u \leq T} S_u > K} | S_t, Y_t, X_t \}$$

We introduce the new function $u(t, s, y, x)$

$$P(t) = 1_{\inf_{0 \leq u \leq t} S_u > K} u(t, S_t, Y_t, X_t)$$

where the function it satisfies, for $s \geq K$, the system

$$\left\{ \begin{array}{ll} \frac{\partial u}{\partial t} + \mathcal{L}_{(S,Y,X)}u - (r + \lambda(x))u &= 0 \quad \text{on } s > K, t < T \\ u(t, K) &= 0 \quad \text{for } t \leq T \\ u(T, s) &= h(s) \quad \text{for } s > K \end{array} \right.$$

The equation solved by the first term of the asymptotic expansion

$$\left\{ \begin{array}{ll} \mathcal{L}_{BS}(\bar{\sigma}, r + \lambda(x))u_0 &= 0 \quad \text{on } s > K, t < T \\ u_0(t, K) &= 0 \quad \text{for } t \leq T \\ u_0(T, s) &= h(s) \quad \text{for } s > K \end{array} \right.$$

for a frozen value of x . Equation solved by the first correction term from the short time scale contribution

$$\left\{ \begin{array}{ll} \mathcal{L}_{BS}(\bar{\sigma}, r + \lambda(x))u_{1,0} + \mathcal{A}u_0 &= 0 \quad \text{on } s > K, t < T \\ u_{1,0}(t, K) &= 0 \quad \text{for } t \leq T \\ u_{1,0}(T, s) &= 0 \quad \text{for } s > K \end{array} \right. \quad (2.53)$$

Introducing the $u_{1,0}^* = u_{1,0} + (T - t)\mathcal{A}u_0$ the system becomes

$$\left\{ \begin{array}{ll} \mathcal{L}_{BS}(\bar{\sigma}, r + \lambda(x))u_{1,0}^* &= 0 \quad \text{on } s > K, t < T \\ u_{1,0}^*(t, K) &= m(t) \quad \text{for } t \leq T \\ u_{1,0}^*(T, s) &= 0 \quad \text{for } s > K \end{array} \right. \quad (2.54)$$

where $m(t) = \lim_{s \downarrow K} (T - t)\mathcal{A}u_0$ Equation solved by the first correction term from the long time scale contribution

$$\left\{ \begin{array}{ll} \mathcal{L}_{BS}(\bar{\sigma}, r + \lambda^*(x))u_{0,1} + \langle \mathcal{M}_1 \rangle u_0 &= 0 \quad \text{on } s > K, t < T \\ u_{0,1}(t, K) &= 0 \quad \text{for } t \leq T \\ u_{0,1}(T, s) &= 0 \quad \text{for } s > K \end{array} \right. \quad (2.55)$$

Solution of the system (2.55) is a bit more involved than the (2.53) which we do in Appendix D, but at the end it boils down to solving a similar homogeneous system

$$\begin{cases} \mathcal{L}_{BS}(\bar{\sigma}, r + \lambda(x))u_{0,1}^* &= 0 & \text{on } s > K, t < T \\ u_{0,1}^*(t, K) &= n(t) & \text{for } t \leq T \\ u_{0,1}^*(T, s) &= 0 & \text{for } s > K \end{cases} \quad (2.56)$$

for some function $n(t)$ defined in (D.18). In consequence, we approximate the price of a defaultable zero-coupon bond

$$\begin{aligned} P(0, T) &\sim u_0(0, s) + u_{1,0}^*(0, s) - (T - t)\mathcal{A}u_0(0, s) \\ &+ u_{0,1}^*(0, s) - (T - t) < \mathcal{M}_1 > u_0(0, s) \\ &+ \frac{1}{2}(T - t)^2 \left(\frac{\partial \lambda}{\partial x} \left(s \frac{\partial}{\partial s} - \cdot \right)^2 P_0 \right) \end{aligned}$$

The integral formulas

$$u_{1,0}^*(t, s) = \frac{\left(\frac{s}{K}\right)^{p/2}}{\sigma^* \sqrt{2\pi}} \int_t^T \frac{\log \frac{s}{K}}{(z - t)^{3/2}} e^{-\frac{\log(\frac{s}{K})^2}{2\sigma^{*2}(z-t)}} e^{(\frac{(\sigma^* p)^2}{8} + r + \lambda^*)(z-t)} m(z) dz \quad (2.57)$$

and

$$u_{0,1}^*(t, s) = \frac{\left(\frac{s}{K}\right)^{p/2}}{\sigma^* \sqrt{2\pi}} \int_t^T \frac{\log \frac{s}{K}}{(z - t)^{3/2}} e^{-\frac{\log(\frac{s}{K})^2}{2\sigma^{*2}(z-t)}} e^{(\frac{(\sigma^* p)^2}{8} + r + \lambda^*)(z-t)} n(z) dz \quad (2.58)$$

along with the functions $m(t)$ and $n(t)$ are calculated explicitly in Appendix D.

Figure 2.21: Solid curve is the spreads produced by the simple hybrid model with constant default intensity and the volatility. Dotted curve is the spreads produced by the model where both the default intensity and the stochastic volatility are mean reverting stochastic processes with slow and fast mean reversions respectively using the approximate solution derived in section 2.8.4. Qualitatively, the correction to the spreads both have short term and long term effects. We have the non-zero short term spreads due to default intensity, quick changes in the short run due to fast mean reverting stochastic volatility and the necessary flexibility over the long run due to slow fast mean reverting stochastic intensity. And quantitatively, all the pricing and parameter fitting are done through approximate formula which goes only up to the first order and it still fits the simulated data points with a small least squares error with the calibration of the few parameters. The parameters obtained for the dotted curve $V_3^\epsilon = -0.003, V_0^\delta = 0.0003, V_1^\delta = -0.0005$ and $\sigma^* = 0.12$ and $\frac{s}{K} = 1.2$

Chapter 3

Credit Default Swap Pricing

This problem explores the expectation of the credit market by developing a parsimonious four-factor credit default swap model. In particular, it is tempting to ask following questions:

- i. What were the default probabilities, both risk-neutral and physical, expected by the credit market during different periods?
- ii. What was the expected rate of recovery in the underlying reference given default?
- iii. How did a default swap model perform over different phases of period?
- iv. What economic and financial factors are potentially important in pricing default swaps?

To be able to answer these type of problems, we propose a valuation framework for credit default swaps that admits flexible correlation between underlying state variable processes. The framework is versatile enough to separate the the default probabilities and the expected recovery rates. We develop a four-factor default swap model, where the first two factors are economy-wide ones intended to capture the dynamics of the U.S. term structure of interest rates and the third factor is name-specific to the credit risk of a particular entity. The fourth factor represents an exogenous default source which is global or market-wide. In the model, we relate the hazard rate to the four

state variables, and explicitly specify market prices of risk so that the parameters of the underlying factor dynamics and that of the market risk premiums can be identified individually.

In this chapter we propose a relatively simple closed form approximations to the credit spreads and bond prices with realistic short maturity spreads that are related to the structural characteristics of the firm's economic environment and accommodates stochastic interest rates. The distinguishing feature of the model is that it incorporates the attractive features of the Longstaff-Schwartz(1995)[52] with hazard rate approach Jarrow-Turnbull(1995)[45], Madan-Unal(1998)[55] and Duffie-Singleton(1999)[19].

Diffusion-based models of pricing risky debt define default as occurring either at maturity (Merton(1974)[59]) or when the firm's asset value diffuses to a prespecified default boundary for the first time(Longstaff-Schwartz(1995)[52]). An attractive feature of these models is that they express the default time in terms of firm specific structural variables. These models can then answer questions about the implications of debt pricing of changes in firm specific variables such as capital structure reorganizations. However, this important feature is compromised by their inability to generate realistic credit spreads in the shorter term although Longstaff-Schwartz(1995)[52] did succeed in obtaining such spreads in the medium term. In these models, time needs to pass to allow assets to diffuse for the default probability to materialize. Equivalently, the probability of a positive-equity firm defaulting in the near term is negligible leading to near zero spreads for short maturities.

The recent hazard rate approach to pricing risky debt of Jarrow and Turnbull(1995)[45], Madan and Unal(1998)[55] and Duffie and Singleton(1999)[19], develops a class of models that allow for the possibility of default in the immediate future. This literature proposes an exogenous model for the hazard rate, which we also adopt in this chapter, which is the likelihood of the firm defaulting over the next period.

A major advantage of this approach is that they generate realistic short maturity credit spreads. However, these models lack a structural definition of the default event. As a consequence, the resulting hazard rate model is a reduced form with parameters that lack a structural interpretation and hence offers no guidance in the presence of a structural change in firm specific variables.

This chapter seeks to propose a two factor hazard rate process in approximate closed form. The key difference of the model is the interpretation of these two factors. The default occurs due to one of these factors, One of them is the firm-specific default probability. For instance, the default is a consequence of a single jump loss event that drives the equity value to zero and requires cash outlays that cannot be externally financed. A case in point is the near default of Long Term Capital Management, resulting from an adverse movement in interest rates. Or another case could be of Barings where a large trading loss forced bankruptcy. Both these examples illustrate the phenomena of default arising from the arrival of an unforeseen loss. Such a sudden fatal loss can be caused by numerous surprise events including the outcome of lawsuits, unexpected devaluations, sudden default of a creditor, supplier or a customer, and catastrophes in production lines.

The model has a number of attractive features. First, consistent with the hazard-rate literature, the probability of such sudden loss arriving unexpectedly is captured in the pricing equations by discounting the promised payments by the hazard rate. Also the default intensities are in correlation with the stochastic interest rates. To achieve this we simply correlate the stochastic differential equations that drive the factors through their Brownian Motions.

Second, our treatment of the interest rate risk differ from the literature that explicitly allows for the relationship between credit spreads and default-free interest rates(Longstaff-Schwartz(1995)[52] and Kim, Ramaswamy and Sundaresan(1993)[49]). Because in this literature current asset values are assumed not to be interest sensitive. And these models would predict that an increase in interest rates benefits the firm's equity and reduces credit spreads. This is an overly simplified assumption and for some specific firms it simply would not be correct. We allow in our model for both negative and positive correlation between the interest rates and default intensities. Finally, the proposed model, by virtue of its closed form, enables the researcher to undertake comparative statistics analysis and enhances the empirical applicability of the model.

3.1 Credit Default Swap Valuation

Let us fix a Probability space $(\Omega, \mathcal{F}, P)$ and the σ - algebra $\mathcal{F}_t = \sigma(W_t^S, Z_t^\sigma, Z_t^{\lambda_t})$ where $W_t^S, Z_t^\sigma, Z_t^{\lambda_t}$ are independent standard Brownian Motions. Let's also introduce the σ - algebra $\mathcal{G}_t = \sigma(\mathcal{F}_t \vee N_t)$ where N_t is a nonexplosive doubly stochastic (with respect to $\mathcal{F}_t$) counting process with intensity λ_t , i.e.

- i. λ_t is $\mathcal{F}_t$ predictable and $\int_0^t \lambda_s ds < \infty$ a.s.
- ii. $N_t - \int_0^t \lambda_s ds$ is a $\mathcal{G}_t$ local martingale
- iii. $P\{N_s - N_t = k | \mathcal{G}_t \vee \mathcal{F}_s\} = \frac{e^{-\int_t^s \lambda_u du} (\int_t^s \lambda_u du)^k}{k!}$

We first derive the formula for CDS premium, where the short rate process is modelled as a two factor process and the default event is modelled by the stopping time τ , the first jump of the counting process N_t , $\chi(t) = 1_{\tau \leq t}$. The default intensity process is modelled as a positive bounded function of an Ornstein-Uhlenbeck process. The Brownian Motions that drive the dynamics of the short rate process are correlated among themselves and with the Brownian Motion that drives the default intensity. The present value of the premium of CDS

$$E^Q \left\{ \int_t^T e^{\int_t^u -r_s ds} (1 - \chi(u)) p du | \mathcal{G}_t \right\}$$

where p is the continuous premium paid by the CDS buyer for the default swap contract with maturity T . The present value of the payoff at default can be expressed as

$$E^Q \left\{ \int_t^T e^{\int_t^u -r_s ds} (1 - \chi(u)) l_u \lambda_u du | \mathcal{G}_t \right\}$$

Therefore, the fair value of the CDS premium is

$$p = \frac{E^Q \left\{ \int_t^T e^{\int_t^u -r_s ds} (1 - \chi(u)) l_u \lambda_u du | \mathcal{G}_t \right\}}{E^Q \left\{ \int_t^T e^{\int_t^u -r_s ds} (1 - \chi(u)) du | \mathcal{G}_t \right\}}$$

which by doubly stochasticity assumption turns out to be

$$p = \frac{E^Q\{\int_t^T e^{\int_t^u -(r_s+\lambda_s)ds} l_u \lambda_u du | \mathcal{G}_t\}}{E^Q\{\int_t^T e^{\int_t^u -(r_s+\lambda_s)ds} du | \mathcal{G}_t\}} \quad (3.1)$$

Equation (3.1) states that, given the processes for interest rate r_t , the default intensity λ_t , the expected loss at default l_t , the ratio of these two expectations gives the fair market CDS premium at the beginning of the contract.

3.2 Parametric Credit Default Swap Model

Following Duffee (1999) and Zhang(2003), we model the default free interest rate process as sum of a constant and two economic stochastic variables, X_t and Y_t , that each follow a CIR process. X_t represents the short term component of the interest rates and Y_t represents the long term component. And they are correlated through their Brownian Motions by a constant parameter ρ_1

$$\begin{cases} r_t &= c_0 + c_1 X_t + c_2 Y_t \\ dX_t &= \frac{1}{\epsilon}(\bar{x} - X_t)dt + \frac{1}{\sqrt{\epsilon}}\sigma_X \sqrt{X_t} dW_t^X \\ dY_t &= \delta(\bar{y} - Y_t)dt + \sqrt{\delta}\sigma_Y \sqrt{Y_t} dW_t^Y \end{cases}$$

The default intensity is also a function of two state processes. The instantaneous likelihood of default depend both on firm-specific distress variables, represented by Z_t and systematic risk U_t . The firm-specific component is correlated with X_t by a constant parameter ρ_2 and U_t is uncorrelated with the rest since it is related to actuarial defaults, not necessarily related to firm specific or market specific issues. Hence the overall setting is

$$\begin{cases} \lambda_t &= g(Z_t) + h(U_t) \\ dZ_t &= \frac{1}{\epsilon}(\bar{z} - Z_t)dt + \frac{1}{\sqrt{\epsilon}}\sigma_Z \sqrt{Z_t} dW_t^Z \\ dU_t &= \delta(\bar{u} - U_t)dt + \sqrt{\delta}\sigma_U \sqrt{U_t} dW_t^U \end{cases}$$

where r_t is the short rate process and λ_t is the instantaneous probability of default of the bond, g and h are positive bounded functions which are bounded away from zero.

Following Madan and Zhang(2001) we model the recovery rate related to the underlying processes of the hazard rate.

$$l_t = l_0 + l_1 e^{-g(Z_t)} \quad (3.2)$$

We may note that as $g(Z) \rightarrow 0, l \rightarrow l_0 + l_1$ and $g(Z) \rightarrow \inf, l \rightarrow l_0$. Hence, we require the restrictions that $l_0 \geq 0, l_1 \geq 0$ and $0 \leq l_0 + l_1 \leq 1$. Equation (3.2) is attractive from both theoretical and empirical viewpoints. First, consistent with extant empirical evidence, recovery is negatively related to default probability. That is, $\frac{\partial l}{\partial z} = -l_1 e^{-g(Z)} \leq 0$. The set-up is guided by the belief that financial distress can diminish the ability of the borrower to pay its creditors in the event of default. In the paper they model the interest rate by a one factor CIR process and the default intensity as a linear function of the interest rate.

Under risk-neutral measure the above systems looks like

$$\begin{cases} r_t &= c_0 + c_1 X_t + c_2 Y_t \\ dX_t &= \left(\frac{1}{\epsilon}(\bar{x} - X_t) - \frac{1}{\sqrt{\epsilon}} \eta_x \sigma_X \sqrt{X_t} \right) dt + \frac{1}{\sqrt{\epsilon}} \sigma_X \sqrt{X_t} dW_t^X \\ dY_t &= \left(\delta(\bar{y} - Y_t) - \sqrt{\delta} \Lambda_y \sigma_Y \sqrt{Y_t} \right) dt + \sqrt{\delta} \sigma_Y \sqrt{Y_t} dW_t^Y \end{cases} \quad (3.3)$$

and

$$\begin{cases} \lambda_t &= \phi(g(Z_t) + h(U_t)) \\ dZ_t &= \left(\frac{1}{\epsilon}(\bar{z} - Z_t) - \frac{1}{\sqrt{\epsilon}} \Lambda_z \sigma_Z \sqrt{Z_t} \right) dt + \frac{1}{\sqrt{\epsilon}} \sigma_Z \sqrt{Z_t} dW_t^Z \\ dU_t &= \left(\delta(\bar{u} - U_t) - \sqrt{\delta} \eta_u \sigma_U \sqrt{U_t} \right) dt + \sqrt{\delta} \sigma_U \sqrt{U_t} dW_t^U \end{cases} \quad (3.4)$$

We will try to calculate the numerator and denominator of expression (3.1) separately, which are very similar. First consider the denominator and call $f(t, x, y, z, u; T) = E\{\int_t^T e^{-\int_t^u (r_s + \lambda_s) ds} du\}$. Then, the Feynman-Kac PDE satisfied by f is

Figure 3.1: CDS Premiums produced by the one factor reduced-form model where the default intensity processe is a one factor CIR process with level of mean reversion 0.01. The structure of the curve for the premiums is either decreasing to its mean level as in the solid case or decreasing to its mean level depending on the initial value of the intensity. Interest rate is assumed to be constant to derive the results used to generate the above curves.

Figure 3.2: CDS Premiums produced by the two factor reduced-form model where the total mean level of the default intensity processe is low, i.e. $\bar{z} + \bar{u} = .006$ in system (3.4). Hence, the structure of the curve for the premiums is likely to be one of the highly rated bonds. With the two factor model we can generate structures of premiums that could change direction along the way as opposed to the monotonic premiums induced by the one-factor CIR models.

$$\begin{aligned}
& f_t + \left(\frac{1}{\epsilon}(\bar{x} - x) - \frac{1}{\sqrt{\epsilon}}\eta_x\sigma_X\sqrt{x}\right)f_x + \left(\delta(\bar{y} - y) - \sqrt{\delta}\Lambda_y\sigma_Y\sqrt{y}\right)f_y \\
& + \left(\frac{1}{\epsilon}(\bar{z} - z) - \frac{1}{\sqrt{\epsilon}}\Lambda_z\sigma_Z\sqrt{z}\right)f_z + \left(\delta(\bar{u} - u) - \sqrt{\delta}\eta_u\sigma_U\sqrt{u}\right)f_u \\
& + \frac{1}{\epsilon}\frac{1}{2}\sigma_X^2xf_{xx} + \sqrt{\frac{\delta}{\epsilon}}\rho_1\sigma_X\sigma_Y\sqrt{xy}f_{xy} \\
& + \frac{1}{\epsilon}\rho_2\sigma_X\sigma_Z\sqrt{xz}f_{xz} + \delta\frac{1}{2}\sigma_Y^2yf_{yy} + \sqrt{\frac{\delta}{\epsilon}}\rho_1\rho_2\sigma_Y\sigma_Z\sqrt{yz}f_{yz} \\
& + \frac{1}{\epsilon}\frac{1}{2}\sigma_Z^2zf_{zz} + \delta\frac{1}{2}\sigma_U^2uf_{uu} - (c_0 + c_1x + c_2y + \phi g(z) + \phi h(u))f + 1 = 0
\end{aligned} \tag{3.5}$$

with the boundary condition $f(T, x, y, z, u; T) = 0$.

Figure 3.3: CDS Premiums produced by the two factor reduced-form model where the total mean level of the default intensity processe is high, i.e. $\bar{z} + \bar{u} = 0.03$ in system (3.4). Hence, the structure of the curve for the premiums is likely to be one of the bonds that is considered rather risky. With the two factor model we can generate structures of premiums that are monotonic like the one-factor CIR models. In this case a financial interpretation of the curve would be that the default risk that the bond carries is high and so accumulating over the time.

Figure 3.4: Blue curve is our benchmark curve where the used mean interest rate level is 4%. Above dashed curve is the CDS premiums generated when the mean interest rate level is relatively low, 2% and the below dashed curve when the mean interest rate level is 6%. As the interest rates go down the effect of the default intensity takes over. Used parameters for the simulations are $\bar{x} + \bar{y} = 0.02, 0.04, 0.06$ from top to bottom in system (3.3)

Figure 3.5: Blue curve is our benchmark curve where the used mean interest rate level is 4%. The dashed one is the structure of the CDS Premiums when the mean interest rate level is 6% and there is no correlation between the short rate and default intensity processes. We already know as the interest rates go up the CDS premiums tend to decrease and the effect is amplified with the introduction of 50% positive correlation with the fast mean reverting factor Z_t in (3.4). Used parameters for the simulations are $\bar{x} + \bar{y} = 0.04$ for the solid curve, $\bar{x} + \bar{y} = 0.06$ for the dashed and dotted curves, and $\rho_Z = 0$ for the dashed and $\rho_Z = 0.5$ for the dotted curve.

Figure 3.6: Blue curve is our benchmark curve where the used mean interest rate level is 4%. The dashed one is the structure of the CDS Premiums when the mean interest rate level is 6% and there is no correlation between the short rate and default intensity processes. We already know as the interest rates go up the CDS premiums tend to decrease and the effect is amplified with the introduction of 50% positive correlation with the fast mean reverting factor Z_t in (3.4) and the dashed curve is the resulting premium curve. We further introduce 50% correlation also in the long scale. Dotted curve is a little different than the dashed one due to that correlation. Used parameters for the simulations are $\bar{x} + \bar{y} = 0.04$ for the solid curve, $\bar{x} + \bar{y} = 0.06$ for the dashed and dotted curves, and $\rho_Z = 0.5$ for the dashed and $\rho_Z = \rho_U = 0.5$ for the dotted curve.

3.3 Closed-form Approximate Solutions and Implied Spreads

We rewrite the system (3.5) as

$$\begin{cases} \mathcal{L}^{\epsilon, \delta} f^{\epsilon, \delta} = 0 \\ f^{\epsilon, \delta}(T, x, y, z, u; T) = 0 \end{cases} \quad (3.6)$$

In order to calculate the approximate solutions for $f^{\epsilon, \delta}$ we decompose the operator $\mathcal{L}^{\epsilon, \delta}$ according to the powers of $\sqrt{\epsilon}$ and $\sqrt{\delta}$ as follows:

$$\begin{cases} \mathcal{L}^{\epsilon, \delta} = \frac{1}{\epsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}} \mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\delta} \mathcal{M}_1 + \delta \mathcal{M}_2 + \sqrt{\frac{\delta}{\epsilon}} \mathcal{M}_3 \\ \mathcal{L}_0 = [(\bar{x} - x) \frac{\partial}{\partial x} + \frac{1}{2} \sigma_X^2 x \frac{\partial^2}{\partial x^2}] + [(\bar{z} - z) \frac{\partial}{\partial z} + \frac{1}{2} \sigma_Z^2 z \frac{\partial^2}{\partial z^2}] + \rho_2 \sigma_X \sigma_Z \sqrt{xz} \frac{\partial^2}{\partial x \partial z} \\ \mathcal{L}_1 = -(\eta_x \sigma_X \sqrt{x} \frac{\partial}{\partial x} + \Lambda_z \sigma_Z \sqrt{z} \frac{\partial}{\partial z}) \\ \mathcal{L}_2 = \frac{\partial}{\partial t} - (c_0 + c_1 x + c_2 y + \phi g(z) + \phi h(u))(\cdot) + 1 \\ \mathcal{M}_1 = -(\Lambda_y \sigma_Y \sqrt{y} \frac{\partial}{\partial y} + \eta_u \sigma_U \sqrt{u} \frac{\partial}{\partial u}) \\ \mathcal{M}_2 = [(\bar{y} - y) \frac{\partial}{\partial y} + \frac{1}{2} \sigma_Y^2 y \frac{\partial^2}{\partial y^2}] + [(\bar{u} - u) \frac{\partial}{\partial u} + \frac{1}{2} \sigma_U^2 u \frac{\partial^2}{\partial u^2}] \\ \mathcal{M}_3 = \rho_1 \sigma_X \sigma_Y \sqrt{xy} \frac{\partial^2}{\partial x \partial y} + \rho_1 \rho_2 \sigma_Y \sigma_Z \sqrt{yz} \frac{\partial^2}{\partial y \partial z} \end{cases} \quad (3.7)$$

3.3.1 The Long Term Interaction

We expand the $f^{\epsilon, \delta}$ in powers of $\sqrt{\delta}$

$$f^{\epsilon, \delta} = f_0^\epsilon + \sqrt{\delta} f_1^\epsilon + \delta f_2^\epsilon + \dots \quad (3.8)$$

plugging (3.8) in equation (3.6) and using the operator notation introduced in (3.7) we get

$$(\frac{1}{\epsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}} \mathcal{L}_1 + \mathcal{L}_2) f_0^\epsilon + \sqrt{\delta} \{ (\frac{1}{\epsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}} \mathcal{L}_1 + \mathcal{L}_2) f_1^\epsilon + (\mathcal{M}_1 + \frac{1}{\sqrt{\epsilon}} \mathcal{M}_3) f_0^\epsilon \} + \dots = 0$$

Hence, matching the first two terms and the terminal conditions we can define the f_0^ϵ & f_1^ϵ as a solution of the following two systems of equations:

$$\begin{cases} (\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2)f_0^\epsilon = 0 \\ f_0^\epsilon(T, x, y, z, u; T) = 0 \end{cases} \quad (3.9)$$

$$\begin{cases} (\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2)f_1^\epsilon + (\mathcal{M}_1 + \frac{1}{\sqrt{\epsilon}}\mathcal{M}_3)f_0^\epsilon = 0 \\ f_1^\epsilon(T, x, y, z, u; T) = 0 \end{cases}$$

3.3.2 The Short Term Interaction

Consider the first term f_0^ϵ and expand it as

$$f_0^\epsilon = f_0 + \sqrt{\epsilon}f_{1,0} + \epsilon f_{2,0} + \epsilon^{3/2}f_{3,0} + \dots \quad (3.10)$$

In the notation $f_{i,j}$ i corresponds to power of $\sqrt{\epsilon}$ and j corresponds to the power of $\sqrt{\delta}$. We will first derive the first order approximation

$$f^{\epsilon,\delta} \approx f_0 + \sqrt{\epsilon}f_{1,0} + \sqrt{\delta}f_{0,1}$$

Inserting (3.10) into (3.9) we get

$$\begin{cases} \frac{1}{\epsilon}\mathcal{L}_0 f_0 + \frac{1}{\sqrt{\epsilon}}(\mathcal{L}_0 f_{1,0} + \mathcal{L}_1 f_0) \\ + (\mathcal{L}_0 f_{2,0} + \mathcal{L}_1 f_{1,0} + \mathcal{L}_2 f_0) \\ + \sqrt{\epsilon}(\mathcal{L}_0 f_{3,0} + \mathcal{L}_1 f_{2,0} + \mathcal{L}_2 f_{1,0}) \\ + \dots = 0 \end{cases} \quad (3.11)$$

Matching the first term and using (3.7) we get

$$\mathcal{L}_0 f_0 = [(\bar{x} - x)\frac{\partial}{\partial x} + \frac{1}{2}\sigma_X^2 x \frac{\partial^2}{\partial x^2}] + [(\bar{z} - z)\frac{\partial}{\partial z} + \frac{1}{2}\sigma_Z^2 z \frac{\partial^2}{\partial z^2}] + \rho_2 \sigma_X \sigma_Z \sqrt{xz} \frac{\partial^2}{\partial x \partial z} f_0 = 0$$

Here we choose f_0 to be independent of both x and z so that we do not get any unreasonable growth, i.e. $f_0 = f_0(t, y, u; T)$. Also matching the second term gives us

$$\mathcal{L}_0 f_{1,0} + \mathcal{L}_1 f_0 = 0 \quad (3.12)$$

But since the operator $\mathcal{L}_1$ takes derivatives with respect to x and z , hence $\mathcal{L}_1 f_0 = 0$ and the equation (3.12) reduces to

$$\mathcal{L}_0 f_{1,0} = 0$$

So we also choose $f_{1,0}$ not to depend on x and z , i.e. $f_{1,0} = f_{1,0}(t, y, u; T)$. Then, order 1 terms in (3.11) gives

$$\mathcal{L}_0 f_{2,0} + \mathcal{L}_2 f_0 = 0 \quad (3.13)$$

Notice that (3.13) is a Poisson equation in $f_{2,0}$ with respect to x and z variables. So only reasonable solution occurs in case of $\mathcal{L}_2 f_0$ being in the orthogonal complement of $\mathcal{L}_0^*$. Namely,

$$\langle \mathcal{L}_2 f_0 \rangle = \langle \mathcal{L}_2 \rangle f_0 = 0 \quad (3.14)$$

where the $\langle \rangle$ denotes the integration with respect to the invariant distribution Φ of the two dimensional process (X_t, Z_t) . Since, f_0 depends neither on x nor on z $\langle \mathcal{L}_2 f_0 \rangle = \langle \mathcal{L}_2 \rangle f_0$ and

$$\langle \mathcal{L}_2 \rangle = \frac{\partial}{\partial t} + -(c_0 + c_1 \langle x \rangle + c_2 y + \phi \langle g(z) \rangle + \phi h(u))(\cdot) + 1$$

Hence to get the f_0 we need to solve the ODE

$$\begin{cases} f_t - (c_0 + c_1 \langle x \rangle + c_2 y + \phi \langle g(z) \rangle + \phi h(u))f + 1 = 0 \\ f_0(t, y, u; T) = 0 \end{cases}$$

Therefore,

$$\begin{cases} \alpha = \alpha(y, u) \equiv -(c_0 + c_1 \langle x \rangle + c_2 y + \phi \langle g(z) \rangle + \phi h(u)) \\ f_0(t, y, u; T) = \frac{e^{\alpha(T-t)} - 1}{\alpha} \end{cases}$$

Next, we derive an expression for $f_{1,0}$. $\sqrt{\epsilon}$ term in (3.11) gives

$$\mathcal{L}_0 f_{3,0} + \mathcal{L}_1 f_{2,0} + \mathcal{L}_2 f_{1,0} = 0$$

Then, the same averaging condition gives

$$\langle \mathcal{L}_1 f_{2,0} \rangle + \langle \mathcal{L}_2 \rangle f_{1,0} = 0$$

Then, by (3.13) and (3.14) we get

$$f_{2,0} = -\mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) f_0$$

Introducing the notation

$$\mathcal{A} = -\langle \mathcal{L}_1 \mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) \rangle$$

Then, we can define the $f_{1,0}$ as the solution of the system of system

$$\begin{cases} \mathcal{L}_2 f_{1,0} + \mathcal{A} f_0 = 0 \\ f_{1,0}(T, y, u; T) = 0 \end{cases}$$

And the solution to that is

$$f_{1,0} = (T - t) \mathcal{A} f_0$$

3.3.3 Calculated CDS Premiums

Consider the term f_1^ϵ and expand it as

$$f_1^\epsilon = f_{0,1} + \sqrt{\epsilon} f_{1,1} + \epsilon f_{2,1} + \epsilon^{3/2} f_{3,1} + \dots \quad (3.15)$$

Inserting (3.15) in (3.6) we get

$$\begin{cases} \frac{1}{\epsilon}\mathcal{L}_0 f_{0,1} + \frac{1}{\sqrt{\epsilon}}(\mathcal{L}_0 f_{1,1} + \mathcal{L}_1 f_{0,1} + \mathcal{M}_3 f_0) \\ + (\mathcal{L}_0 f_{2,1} + \mathcal{L}_1 f_{1,1} + \mathcal{L}_2 f_{0,1}) + \mathcal{M}_1 f_0 + \mathcal{M}_3 f_{1,0} \\ + \sqrt{\epsilon}(\mathcal{L}_0 f_{3,1} + \mathcal{L}_1 f_{2,1} + \mathcal{L}_2 f_{1,1} + \mathcal{M}_1 f_{1,0} + \mathcal{M}_3 f_{2,0}) \\ + \dots = 0 \end{cases} \quad (3.16)$$

Matching the first term, we get

$$\mathcal{L}_0 f_{0,1} = 0$$

Hence, we choose $f_{0,1}$ to be independent of x and z just like for f_0 . Also matching the second term gives us

$$\mathcal{L}_0 f_{1,1} + \mathcal{L}_1 f_{0,1} + \mathcal{M}_3 f_0 = 0 \quad (3.17)$$

But since the operator $\mathcal{L}_1$ takes derivatives with respect to x and z , and $f_{0,1}$ is constant in those variables $\mathcal{L}_1 f_{0,1} = 0$. Similarly, $\mathcal{M}_3$ takes derivative with respect to x and z , and f_0 does not depend on those. Hence, the equation (3.17) reduces to

$$\mathcal{L}_0 f_{1,1} = 0$$

Now, we also choose $f_{1,1}$ not to depend on x or z , i.e. $f_{1,1} = f_{1,1}(t, y, u; T)$. Then, order 1 terms give, using the facts $\mathcal{L}_1 f_{1,1} = \mathcal{M}_3 f_{1,0} = 0$

$$\mathcal{L}_0 f_{2,1} + \mathcal{L}_2 f_{0,1} + \mathcal{M}_1 f_0 = 0 \quad (3.18)$$

Notice that (3.18) is a Poisson equation in $f_{2,1}$ with respect to x and z variables. Then, the solubility condition in this case turns out to be

$$\langle \mathcal{L}_2 f_{0,1} + \mathcal{M}_1 f_0 \rangle = \langle \mathcal{L}_2 \rangle f_{0,1} + \mathcal{M}_1 f_0 = 0$$

Therefore we can write $f_{0,1}$ as a solution of the system

$$\begin{cases} <\mathcal{L}_2> f_{0,1} + \mathcal{M}_1 f_0 = 0 \\ f_{0,1}(T, y, u; T) = 0 \end{cases}$$

And solution of that system is given by the formula

$$\begin{cases} f_{0,1} = -f_0 + K(y, u) \left[-\frac{e^{\alpha(T-t)}T-t}{\alpha} + \frac{f_0}{\alpha} + \frac{f_0}{\alpha^2} + \frac{e^{\alpha(T-t)}(T-t)}{\alpha^2} \right] \\ \alpha = -(c_0 + c_1 <x> + c_2 y + \phi <g(z)> + \phi h(u)) \\ f_0 = \frac{e^{\alpha(T-t)}-1}{\alpha} \\ K^\delta(y, u) = -\sqrt{\delta}(c_2 \Lambda_y \sigma_Y \sqrt{y} + \phi h'(u) \Lambda_u \sigma_U \sqrt{u}) \end{cases}$$

Therefore we got the approximate result for the denominator. A similar calculation can be done for the numerator of the term in (3.1). And we get the approximate formula for the CDS premium p as follows:

$$p = \frac{EQ\{\int_t^T e^{\int_t^u -(r_s + \lambda_s)ds} l_u \lambda_u du | \mathcal{G}_t\}}{EQ\{\int_t^T e^{\int_t^u -(r_s + \lambda_s)ds} du | \mathcal{G}_t\}} \equiv \frac{\text{top}}{\text{bottom}} \quad (3.19)$$

$$\begin{aligned} \text{top} &\sim (V_1^\epsilon(T-t) + V_2^\epsilon) \left(\beta \frac{e^{\alpha(T-t)}-1}{\alpha} \right. \\ &\quad \left. + K^\delta \left[-\frac{e^{\alpha(T-t)}T-t}{\alpha} + \frac{e^{\alpha(T-t)}-1}{\alpha^2} + \frac{e^{\alpha(T-t)}(T-t)}{\alpha^2} + \frac{e^{\alpha(T-t)}-1}{\alpha^3} \right] \right) \\ \text{bottom} &\sim (V_1^\epsilon(T-t) + V_2^\epsilon) \left(\frac{e^{\alpha(T-t)}-1}{\alpha} \right. \\ &\quad \left. + K^\delta \left[-\frac{e^{\alpha(T-t)}T-t}{\alpha} + \frac{e^{\alpha(T-t)}-1}{\alpha^2} + \frac{e^{\alpha(T-t)}(T-t)}{\alpha^2} + \frac{e^{\alpha(T-t)}-1}{\alpha^3} \right] \right) \end{aligned}$$

where

$$\begin{aligned} \alpha(y, u) &= -(c_0 + c_1 <x> + c_2 y + \phi <g(z)> + \phi h(u)) \\ \beta(u) &= <l(z)\phi g(z)> + <l(z)\phi> h(u) \\ K^\delta(y, u) &= -\sqrt{\delta}(c_2 \Lambda_y \sigma_Y \sqrt{y} + \phi h'(u) \Lambda_u \sigma_U \sqrt{u}) \end{aligned}$$

Figure 3.7: Above is the default-swap rates for a selected convertible bond issued by an entity which is rated Aaa by Moody's in 1999. Data is available at the website www.neatideas.com. Solid curve is the CDS premium curves produced by the model (3.3) and (3.4) using the asymptotic approximation obtained in (3.19). Fitted parameter values are $\alpha = -0.06$, $\beta = 0.6$, $V_1^\epsilon = 0.004$, $V_2^\epsilon = 0.002$ and $K^\delta = -0.0091$. The parameters α corresponds to the adjusted average interest rate of 6% and β corresponds to 40% expected recovery. The superscript ϵ refers to short time scale correction and δ refers to long time scale correction. Our multi-factor CIR equipped with time scales captures the premium structure that the one factor CIR models would not be able to generate.

Appendix A

Girsanov's Theorem for Counting Processes

Claim : Suppose N_t is a nonexplosive counting process with intensity λ_t , and ϕ_t is a strictly positive predictable process such that, for some fixed T , $\int_0^T \phi_s \lambda_s ds < \infty$ almost surely. Then,

$$\xi_t^\phi = e^{\int_0^t (1-\phi_s) \lambda_s ds} \prod_{\{i: \tau(i) \leq t\}} \phi_{\tau(i)}$$

is a well defined local martingale where $\tau(i)$ is the i^{th} jump time of N_t .

Proof : Define

$$X_t = e^{\int_0^t (1-\phi_s) \lambda_s ds} \quad \text{and} \quad Y_t = \prod_{\{i: \tau(i) \leq t\}} \phi_{\tau(i)} \quad \text{and} \quad M_t = N_t - \lambda_t dt$$

Then

- i. $\xi_t^\phi = X_t Y_t$
- ii. M_t is a local martingale
- iii. $dX_t = (1 - \phi_t) \leq \int_0^t (1-\phi_s) \lambda_s ds \, dt = (1 - \phi_t) \lambda_t X_t dt$
- iv. $dY_t = (\prod_{\{i: \tau(i) < t\}} \phi_{\tau(i)}) (\phi_t - 1) dN_t = Y_t (\phi_t - 1) dN_t$
- v. $dM_t = dN_t - \lambda_t dt$

By the above five facts and general Ito formula with jumps, ξ_t^ϕ is calculated as:

$$\begin{aligned}
d\xi_t^\phi &= d(X_t Y_t) \\
&= dX_t Y_{t-} + X_{t-} dY_t + \Delta X_t \Delta Y_t \\
&= (1 - \phi_t) \lambda_t X_t Y_{t-} dt + X_{t-} Y_{t-} (\phi_t - 1) dN_t \\
&= (1 - \phi_t) \lambda_t \xi_{t-}^\phi + (\phi_t - 1) \xi_{t-}^\phi dN_t \\
&= (1 - \phi_t) \lambda_t \xi_{t-}^\phi + (\phi_t - 1) \xi_{t-}^\phi (dM_t + \lambda_t dt) \\
&= (\phi_t - 1) \xi_{t-}^\phi dM_t
\end{aligned}$$

In the third equation we made use of the fact that X_t is a continuous process which implies $\Delta X_t = 0$. Since M_t is a local martingale, we know that an integral against a local martingale is also a local martingale under certain conditions(?) for the integrand $\diamond$

Claim : If ξ_t^ϕ is a martingale, then an equivalent martingale measure P^* is defined by $\frac{dP^*}{dP} = \xi_T^\phi$. Under this new martingale measure, N_t is still a nonexplosive counting process with intensity $\lambda_t \phi_t$.

Proof : To say that N_t is counting process with intensity $\lambda_t \phi_t$ what we need to show is $A_t = N_t - \int_0^t \lambda_s \phi_s$ is P^* local martingale where $\frac{dP^*}{dP} = \xi_T^\phi$. Or equivalently we can show that the process $Z_t = \xi_t^\phi A_t$ is a P local martingale.

By the first claim

$$dA_t = dN_t - \lambda_t \phi_t dt \quad \text{and} \quad d\xi_t^\phi = (\phi_t - 1) \xi_{t-}^\phi dM_t$$

Then by the Ito's formula with jumps we get

$$\begin{aligned}
dZ_t &= d\xi_t^\phi A_{t-} + \xi_{t-}^\phi dA_t + \Delta \xi_t^\phi \Delta A_t \\
&= (\phi_t - 1) \xi_{t-}^\phi A_{t-} dM_t + \xi_{t-}^\phi (dN_t - \lambda_t \phi_t dt) + (\phi_t - 1) \xi_{t-}^\phi dN_t \\
&= (\phi_t - 1) \xi_{t-}^\phi A_{t-} dM_t - \xi_{t-}^\phi \lambda_t \phi_t dt + \phi_t \xi_{t-}^\phi dN_t \\
&= (\phi_t - 1) \xi_{t-}^\phi A_{t-} dM_t - \xi_{t-}^\phi \lambda_t \phi_t dt + \phi_t \xi_{t-}^\phi (dM_t + \lambda_t dt) \\
&= [(\phi_t - 1) \xi_{t-}^\phi A_{t-} + \phi_t \xi_{t-}^\phi] dM_t
\end{aligned}$$

Hence Z_t can be written as an integral against a local martingale, which would imply Z_t itself is a P local martingale. Therefore, A_t is a P^* local martingale and therefore N_t is a counting process with intensity $\lambda_t \phi_t$ under the new measure P^* $\diamond$

Appendix B

Defaultable Bond Pricing via HJM

An alternative approach to the one given in section 1.1.2 for pricing of defaultable bonds is based on the term structure forward spread rates as in HJM [39] type models. Let us consider a defaultable zero-coupon bond maturing at s has a price at time t , as in section 1.1.2, of the form

$$P(t, s) = \exp\left(-\int_t^s (F(t, u) + S(t, u))du\right) \quad (\text{B.1})$$

where $F(t, s)$ is the default free forward rate in HJM sense, and likewise $S(t, u)$ is the credit yield spread forward rate in HJM sense. For a fixed maturity date s , we assume that the process $\{F(t, s) : 0 \leq t \leq s\}$ follows

$$dF(t, s) = \mu_F(t, s)ds + \sigma_F(t, s)dW_t$$

Assuming μ_F and σ_F satisfying the HJM technical conditions, we have the drift restriction

$$\mu_F(t, s) = \sigma_F(t, s) \int_t^s \sigma_F(t, u)du$$

We also assume the spread forward rates process follow a similar model

$$dS(t, s) = \mu_S(t, s)ds + \sigma_S(t, s)dW_t$$

By using Stochastic Calculus, we can compute an implied default intensity $\lambda_t = S(t, t)/l_t$, l_t being the lost ratio of market value of the bond in case of a default. And the corresponding drift restriction turns out

$$\mu_S(t, s) = \sigma_S(t, s) \int_t^s \sigma_F(t, u) du + \sigma_F(t, s) \int_t^s \sigma_S(t, u) du$$

One might ask about the asymmetry of the drift conditions for the default free forward rate and the spread forward rate despite the symmetric appearance in the expression (B.1). But they are different because they do not appear symmetrically in the gain process which has to be a martingale under the risk neutral measure. We remind you that the gain process, G_t looks like

$$G_t = (1 - \Lambda_t)B(0, t)P(t, T) + \int_0^t (1 - l_s)B(0, s)P(s, T)d\Lambda_s$$

Having calculated the implied default intensity, one can simulate the forward rates and the actual default times at the same time, which could be important for example pricing of spread options. Obviously, the simulation of default times is not necessary to price defaultable bonds. Hence, given the forward rate F , the volatility of spread process σ_s and the initial yield spread curve $\{S(0, t) : t \geq 0\}$, $S(t, u)$ is known. There are some restrictions on the initial yield curve to get non-negative spread all the time. Bielecki and Rutkowski (2000) [4] takes this approach and extends to model spreads by rating.

Extended Framework with Stochastic Interest Rates

Let us consider a more generalized version of the same problem with stochastic interest rate. Let us assume instead of a constant r , we have an $\mathcal{F}_t$ measurable process r_t .

Then, our SDE system for the prices will be

$$\begin{cases} dS_t &= \mu S_t dt + \sigma_t S_t dW_t^S \\ \sigma_t &= f(Y_t) \\ dY_t &= \alpha(m - Y_t)dt + \beta_\sigma dW_t^\sigma & dW_t^\sigma = \rho_\sigma dW_t^S + \sqrt{1 - \rho_\sigma^2} dZ_t^\sigma \\ d\lambda_t &= a(b - \lambda_t)dt + \beta_\lambda dW_t^{\lambda_t} & dW_t^{\lambda_t} = \rho_\lambda dW_t^S + \sqrt{1 - \rho_\lambda^2} dZ_t^{\lambda_t} \\ dr_t &= \alpha_r(m_r - r_t)dt + \beta_r dW_t^r & dW_t^r = \rho_r dW_t^S + \sqrt{1 - \rho_r^2} dZ_t^r \end{cases}$$

We would like to find a measure P^* under which the process $e^{-rt} S_t 1_{\{r > t\}}$ is a $\mathcal{G}_t$ -martingale. As we showed earlier this is equivalent to have the process $e^{-\int_0^t (r_u + \lambda_u) du} S_t$ a $\mathcal{G}_t$ -martingale.

Using this and the two step change of measure described in the notes we obtain the system under consideration under P^* using the following change in the Brownian filtration

$$W_t^* = W_t + \int_0^t \theta_u du$$

where

$$W_t = \begin{bmatrix} W_t^S \\ Z_t^\sigma \\ Z_t^{\lambda_t} \\ Z_t^r \end{bmatrix} \quad \text{and} \quad \theta_t = \begin{bmatrix} \frac{\mu - r - \lambda_t^*}{f(Y_t)} \\ \gamma_t \\ \delta_t \\ \kappa_t \end{bmatrix}$$

where the parameters γ , δ and κ are free. Finally the system becomes,

$$\begin{cases} dS(t) &= (r + \lambda_t^*) S_t dt + \sigma_t S_t dW_t^{*S}(t) \\ \sigma_t &= f(Y_t) \\ dY(t) &= [\alpha(m - Y_t) - \beta_\sigma(\rho_\sigma \frac{\mu - r - \lambda_t^*}{f(Y_t)} + \gamma_t \sqrt{1 - \rho_\sigma^2})] dt + \beta_\sigma dW_t^{*\sigma} \\ d\lambda_t^* &= [(a - \frac{\phi_t'}{\phi_t})(\frac{ab\phi_t}{a - \frac{\phi_t'}{\phi_t}} - \lambda_t^*) - \beta_\lambda \phi_t (\rho_\lambda \frac{\mu - r - \lambda_t^*}{f(Y_t)} + \delta_t \sqrt{1 - \rho_\lambda^2})] dt + \beta_\lambda \phi_t dW_t^{*\lambda} \\ dr_t &= [\alpha_r(m_r - r_t) - \beta_r(\rho_r \frac{\mu - r_t - \lambda_t^*}{f(Y_t)} + \kappa_t \sqrt{1 - \rho_r^2})] dt + \beta_r dW_t^{*r} \end{cases}$$

Appendix C

Extended Framework, Intensity as a Function of Underlying

Let us consider a more generalized version of the same problem where intensity rate process depends also upon the process of the underlying. Let us assume that $\lambda_t = g(X_t, S_t)$ where the function g has a certain form, but kept general for now, and X_t is some state process. Then, our SDE system for the prices will be

$$\left\{ \begin{array}{ll} dS_t = \mu S_t dt + \sigma_t S_t dW_t^S \\ \sigma_t = f(Y_t) \\ dY_t = \alpha(m - Y_t)dt + \beta_\sigma dW_t^\sigma & dW_t^\sigma = \rho_\sigma dW_t^S + \sqrt{1 - \rho_\sigma^2} dZ_t^\sigma \\ dX_t = a(b - X_t)dt + \beta_\lambda dW^{\lambda_t} & dW^{\lambda_t} = \rho_\lambda dW_t^S + \sqrt{1 - \rho_\lambda^2} dZ^{\lambda_t} \\ \lambda_t = g(X_t, S_t) \end{array} \right.$$

We would like to find a measure P^* under which the process $e^{-rt} S_t 1_{\{\tau > t\}}$ is a $\mathcal{G}_t$ -martingale. As we showed earlier this is equivalent to have the process $e^{-\int_0^t (r + \lambda_u^*) du} S_t$ a $\mathcal{G}_t$ -martingale, where λ_t^* is the intensity process under the measure P^* .

Using this and the two step change of measure described in the notes we obtain the system under consideration under P^* using the following change in the Brownian filtration

$$W_t^* = W_t + \int_0^t \theta_u du$$

where

$$W_t = \begin{bmatrix} W_t^S \\ Z_t^\sigma \\ Z_t^{\lambda_t} \end{bmatrix} \quad \text{and} \quad \theta_t = \begin{bmatrix} \frac{\mu - r - \phi_t g(X_t, S_t)}{f(Y_t)} \\ \gamma_t \\ \delta_t \end{bmatrix}$$

where the parameters γ , δ and ϕ are free. Finally the system becomes,

$$\begin{cases} dS_t &= (r + \phi_t g(X_t, S_t))S_t dt + \sigma_t S_t dW_t^{*S} \\ \sigma_t &= f(Y_t) \\ dY_t &= [\alpha(m - Y_t) - \beta_\sigma(\rho_\sigma \frac{\mu - r - \lambda_t^*}{f(Y_t)} + \gamma_t \sqrt{1 - \rho_\sigma^2})]dt + \beta_\sigma dW_t^{*\sigma} \\ dX_t &= a(b - X_t) - \beta_\lambda(\rho_\lambda \frac{\mu_t - r - \phi_t g(X_t, S_t)}{f(Y_t)} + \delta_t \sqrt{1 - \rho_\lambda^2})dt + \beta_\lambda dW_t^{*\lambda} \end{cases}$$

Modelling the Default Correlation in Multi-Name Products

In today's financial markets there are lots of multi-name products whose pricing is critically dependent on the correlation of defaults of these different names. Basket default swaps, CDO's, CBO's are such examples. In this section we try to develop a model to price this kind of products paying particular attention to the default correlations. We try to combine the capital structure models and reduced form models by modelling the default intensities of different names as both a function of the overall market and a function of its individual structure. Modelling the effect of overall market is done through a proxy like a big common index, e.g. S&P 500 and the effect of individual structure is like a surprise default.

The straight forward intuition behind the setting is when the overall market is not doing well, the default probability of each name tend to go up together, not necessarily with the same rate. Or there could be something happening not in the whole market but in a specific sector which would bump up all the default probabilities of names in that sector. In addition to that there could be also something happening within a firm which would only effect that particular firm but not the others. So it is natural to assume the default probabilities(intensities) have two different components, one for the overall market effect and one for the individual firm effect.

In the typical setting of the model, the proxy used to capture the overall market impact is modelled as a Geometric Brownian Motion with stochastic volatility. The

stochastic volatility process is defined as a positive, bounded function of an OU process. All the default intensities are modelled as product of a state process, which is an OU process with appropriate parameters and a positive function of the index level above. The Brownian Motions that drive the dynamics of all the state processes that effect the intensities are correlated with each other. One can, in general, introduce the correlation between the Brownian Motions of the index level and state processes but we rather capture that effect in the specific form that we choose for the intensity processes.

Now suppose we try to price a product that depends on N different names. Then the SDE's that describe the event look as follows:

$$\begin{cases} dS_t &= \mu S_t dt + \sigma_t S_t dW_t^S \\ \lambda_t^i &= X_t^i g_i(S_t, K^i) \quad \text{for } i = 1, 2, \dots, N \\ dX_t^i &= a^i(b^i - X_t^i)dt + \beta_X^i dW_t^i \quad E\{dW_t^i dW_t^j\} = \rho_{ij}^X dt \end{cases}$$

where S is the common index level, σ is the volatility of the index level, λ^i is the instantaneous probability of default of name i . X is the state process that effects all the intensity processes and the function g is some power function which blows up at a certain fixed boundary level K^i . Although one can keep the function g general for the rest of the section we will assume that $g(S_t, K) = (\frac{S_t - K}{S_0 - K})^n$ where $S_0 - K$ is a normalization factor and S_0 is the initial value of the index in the period of interest. Although there seems to be a lot of parameters in the general setting, as far as the correlation of default times are concerned there are just a few key parameters. The most important one is the actual exponent in the function g . Clearly a positive power corresponds to a positive and negative power corresponds to a negative correlation between the market and the default intensity of the individual name. If we assume same type of g , i.e. the same power and the same boundary level for two different names and keeping the other variables fixed we observe that the correlation is almost a linear function of the square root of this power parameter. (See figure C.1)¹. And the difference between the two pictures is the level of the volatility of the common

¹At this point all our observations are based on simulated data with parameters chosen from the published literature (see Duffie and Singleton (2003)).

index process. So the first one corresponds to a market with high volatility and the second one with a low volatility. As we can see from the figures above through this

Figure C.1: Correlation effect of the parameter in the exponent of the function g , under low volatile and high volatile environments

model we get correlation between default times up to 90%. Here the correlation is defined in the classical sense. In the market, it is also of interest the correlation between the consecutive defaults, i.e. the defaults happened within the same year. But one get just similar results for that definition of correlation too.

At this point we see that in the above pair of figures although each of them are almost straight lines, the slopes of those lines are different. This means that the level of correlation introduced by the specific form of the function g creates different effects in different regimes in the market. Note that, besides the exponent and the volatility parameters, another very effective parameter is the explosion boundary K in the function g . We call it explosion boundary because once the value of the common index gets close to this level, it increases all the intensities by incredible amount and we get simultaneous defaults. Also the more we are further away from this level, the smaller the intensities are, i.e. when the market is doing well all the default intensities tend to go lower. And the closeness of this level to the index level is basically the sensitivity of the individual to the overall market. But in order to create a uniform effect of this exponent under different regimes of the process S_t , we also define this explosion boundary as a function of S_0 and σ and we let $K = S_0 - L * \sigma$. Then if we generate the first two pictures with this new definition of the boundary we observe the same level of correlation effect under both regimes, as shown in figure C.2. Regarding the other set of parameters that could possibly effect the default time

Figure C.2: Correlation effect of the parameter in the exponent of the function g , without the effect of the volatility of the index process

correlation², we experimented the same phenomena under different sets of values of all those parameters. We observe that in all possible values of these parameters we get the same effect on the correlation of the default times. For instance, below in figure C.3, we show the effect of the correlation parameter between the state processes to the premium of a First-to-Default insurance contract which is explained in more details in the following example. As it is clear from the picture, there is almost no effect of the parameter to the price.

Figure C.3: Correlation between Brownian Motions vs. the premium of a First-to-Default contract

Example: First to default valuation

Under our general setting, we now price a simple product that is a contingent claim that pays off at the time of the first of the n names defaults. And the payoff of the product is exactly \$1 at the time of the default. Although the example is pretty simple we still do not have a closed form answer or some kind of theoretical result because of the way we described the defaults. But intuitively we have an idea of the correlation to the price of this simple derivative. Namely, if we have n perfectly uncorrelated default intensities then the intensity of the first-to-default event is the sum of all intensities. On the other hand if they are all perfectly correlated then having an insurance against the first-to-default or the any one of them would be the same if we assume that each name has the same default intensity. Therefore the price of the contract should be much less in the case of the perfectly correlated case then the uncorrelated case. By a similar argument, one can convince himself that actually the price of this contract is a decreasing function of the correlation of the default times. Hence in our model a decreasing function of the square root of the exponent parameter. Figure C.4 shows the impact of the exponent on the price of

²for example, the correlation parameter of the Brownian Motions in the state processes or the reverting mean level of the state processes

the first-to-default contract.

Figure C.4: Correlation of Default Times vs. the premium of a First-to-Default contract

Appendix D

Slow Scale Correction Formulas without Default Boundary

In this appendix we derive the explicit solution for the slowly mean reverting default intensity, when there is no default boundary. Using straight forward calculus we compute the following five quantities which are needed. Using the shorthand notation $\tau = T - t$

$$\left\{ \begin{array}{ll} e^{(r+\lambda(x))\tau} P_0 &= N(d_+) \\ e^{(r+\lambda(x))\tau} \frac{\partial P_0}{\partial s} &= \frac{1}{s} \frac{N'(d_+)}{\sigma\sqrt{\tau}} \\ e^{(r+\lambda(x))\tau} \frac{\partial^2 P_0}{\partial s^2} &= \frac{1}{s^2} \left[\frac{-d_+ N'(d_+)}{(\sigma\sqrt{\tau})^2} - \frac{N'(d_+)}{\sigma\sqrt{\tau}} \right] \\ e^{(r+\lambda(x))\tau} \frac{\partial P_0}{\partial x} &= -\frac{\partial \lambda}{\partial x} \tau \left[N(d_+) - \frac{N'(d_+)}{\sigma\sqrt{\tau}} \right] \\ \frac{\partial}{\partial s} (e^{(r+\lambda(x))\tau} \frac{\partial P_0}{\partial x}) &= -\frac{\partial \lambda}{\partial x} \tau \frac{1}{s} \left[\frac{N'(d_+)}{\sigma\sqrt{\tau}} + \frac{d_+ N'(d_+)}{(\sigma\sqrt{\tau})^2} \right] \end{array} \right. \quad (\text{D.1})$$

We first show how we convert the system (2.17) to a homogeneous system and how to solve that. First through a change of variables we convert (2.17) to another non-homogeneous system (D.4) and then through another change of variable to a homogeneous system, (D.8), where the solution is known. First define

$$w_1(t, s) = (T - t) \mathcal{M}_1 P_0 = (T - t) (V_1^\delta s \frac{\partial}{\partial s} (\frac{\partial P_0}{\partial x}) + V_0^\delta \frac{\partial P_0}{\partial x}) \quad (\text{D.2})$$

and then consider

$$w_2(t, s) = P_1(t, s) - w_1(t, s) \quad (\text{D.3})$$

Note that,

$$\begin{cases} \mathcal{L}_{BS}(\sigma, r + \lambda(x))(\frac{\partial P_0}{\partial x}) &= \frac{\partial \lambda}{\partial x}(s \frac{\partial P_0}{\partial s} - P_0) \\ \mathcal{L}_{BS}(\sigma, r + \lambda(x))(s \frac{\partial}{\partial s}(\frac{\partial P_0}{\partial x})) &= \frac{\partial \lambda}{\partial x} s \frac{\partial}{\partial s}(s \frac{\partial P_0}{\partial s} - u_0) \end{cases} \quad (\text{D.4})$$

Therefore w_2 solves the system

$$\begin{cases} \mathcal{L}_{BS}(\bar{\sigma}, r + \lambda(x))w_2 &= (T - t) \frac{\partial \lambda}{\partial x}(s \frac{\partial}{\partial s} - \cdot)^2 P_0 \quad \text{on } t < T \\ P_1(T, s) &= 0 \end{cases} \quad (\text{D.5})$$

Now we are in position to eliminate the inhomogeneous term as in the fast scale case because we do not have derivatives with respect to x variable anymore in the source term. All the partial derivative operators commute with the BS operator. So following the same idea we introduce

$$w_3(t, s) = -\frac{1}{2}(T - t)^2 \left(\frac{\partial \lambda}{\partial x} (s \frac{\partial}{\partial s} - \cdot)^2 P_0 \right) \quad (\text{D.6})$$

and then consider

$$w_4(t, s) = w_2(t, s) - w_3(t, s) \quad (\text{D.7})$$

Finally, w_4 solves the system

$$\begin{cases} \mathcal{L}_{BS}(\sigma, r + \lambda(x))w_4 &= 0 \\ P_1(T, s) &= 0 \end{cases} \quad (\text{D.8})$$

But clearly because of the terminal condition the solution is the trivial one which is 0. Hence

$$P_1 = w_1 + w_3$$

Let us calculate w_1 and w_3 as they are explicitly given by the equations (D.2) and (D.6). Also using (D.1)

$$\begin{aligned} w_1(t, s) &= e^{-(r+\lambda(x))(T-t)} (T - t)^2 \left(-\frac{\partial \lambda}{\partial x} \right) \left[V_1^\delta \left(\frac{N'(d_+)}{\sigma \sqrt{T-t}} \right) \right. \\ &\quad \left. + \frac{d_+ N'(d_+)}{(\sigma^* \sqrt{T-t})^2} \right] + V_0^\delta \left(N(d_+) - \frac{N'(d_+)}{\sigma^* \sqrt{T-t}} \right) \end{aligned} \quad (\text{D.9})$$

Similarly, we obtain

$$\begin{aligned} w_3(t, s) &= -\frac{1}{2}e^{-(r+\lambda(x))(T-t)}(T-t)^2\frac{\partial\lambda}{\partial x}\left[N(d_+) - \frac{N'(d_+)}{\sigma^*\sqrt{T-t}}\right. \\ &\quad \left.+ \frac{-d_+N'(d_+)}{(\sigma^*\sqrt{T-t})^2} - \frac{N'(d_+)}{\sigma^*\sqrt{T-t}}\right] \end{aligned} \quad (\text{D.10})$$

Slow Scale Correction Formulas with Default Boundary:

In this appendix we derive the integral formulas for the correction terms rising both in section 2.5.2 and 2.8.4. Using straight forward calculus we compute the following five quantities which are needed.

$$\begin{aligned} e^{(r+\lambda(x))(T-t)}u_0 &= N(d_+) - \left(\frac{s}{K}\right)^p N(d_-) \\ e^{(r+\lambda(x))(T-t)}\frac{\partial u_0}{\partial s} &= \frac{1}{s}\left[\frac{N'(d_+)}{\sigma^*\sqrt{T-t}} - p\left(\frac{s}{K}\right)^p N(d_-) + \left(\frac{s}{K}\right)^p \frac{N'(d_-)}{\sigma^*\sqrt{T-t}}\right] \\ e^{(r+\lambda(x))(T-t)}\frac{\partial^2 u_0}{\partial s^2} &= \frac{1}{s^2}\left[\frac{-d_+N'(d_+)}{(\sigma^*\sqrt{T-t})^2} - \frac{N'(d_+)}{\sigma^*\sqrt{T-t}}\right] + \frac{1}{s^2}\left(\frac{s}{K}\right)^p[(p-p^2)N(d_-) \\ &\quad + \frac{(2p-1)N'(d_-)}{\sigma^*\sqrt{T-t}} + \frac{d_-N'(d_-)}{(\sigma^*\sqrt{T-t})^2}] \\ e^{(r+\lambda(x))(T-t)}\frac{\partial u_0}{\partial x} &= -\frac{\partial\lambda}{\partial x}(T-t)\left[N(d_+) - \left(\frac{s}{K}\right)^p N(d_-) - \frac{N'(d_+)}{\sigma^*\sqrt{T-t}}\right. \\ &\quad \left.- \frac{2N(d_-)}{(\sigma^*\sqrt{T-t})^2}\log\left(\frac{s}{K}\right)\left(\frac{s}{K}\right)^p + \frac{N'(d_-)}{\sigma^*\sqrt{T-t}}\left(\frac{s}{K}\right)^p\right] \\ \frac{\partial}{\partial s}(e^{(r+\lambda(x))(T-t)}\frac{\partial u_0}{\partial x}) &= -\frac{\partial\lambda}{\partial x}(T-t)\frac{1}{s}\left[\frac{N'(d_+)}{\sigma^*\sqrt{T-t}} - p\left(\frac{s}{K}\right)^p N(d_-) + \left(\frac{s}{K}\right)^p \frac{N'(d_-)}{\sigma^*\sqrt{T-t}}\right. \\ &\quad + \frac{d_+N'(d_+)}{(\sigma^*\sqrt{T-t})^2} + \frac{2N'(d_-)}{(\sigma^*\sqrt{T-t})^3}\left(\frac{s}{K}\right)^p\log\left(\frac{s}{K}\right) - \frac{2pN(d_-)}{(\sigma^*\sqrt{T-t})^2}\left(\frac{s}{K}\right)^p\log\left(\frac{s}{K}\right) \\ &\quad \left.- \frac{2N(d_-)}{(\sigma^*\sqrt{T-t})^2}\left(\frac{s}{K}\right)^p - \frac{d_-N'(d_-)}{(\sigma^*\sqrt{T-t})^2}\left(\frac{s}{K}\right)^p + \frac{pN'(d_-)}{\sigma^*\sqrt{T-t}}\left(\frac{s}{K}\right)^p\right] \end{aligned} \quad (\text{D.11})$$

We first show how the solution of the inhomogeneous PDE system (2.55) to the solution of the homogeneous system (2.56) and then by the help of the above set of formulas we calculate the resulting $n(t)$ appearing in the solution. And finally the integral formula for the solution of the final homogeneous system is calculated as in section E. First define

$$v_1(t, s) = (T-t)\mathcal{M}_1 u_0 = (T-t)(V_1^\delta s \frac{\partial}{\partial s}(\frac{\partial u_0}{\partial x}) + V_0^\delta \frac{\partial u_0}{\partial x}) \quad (\text{D.12})$$

and then consider

$$v_2(t, s) = u_{0,1}(t, s) - v_1(t, s) \quad (\text{D.13})$$

Note that,

$$\begin{cases} \mathcal{L}_{BS}(\bar{\sigma}, r + \lambda(x))(\frac{\partial u_0}{\partial x}) &= \frac{\partial \lambda}{\partial x}(s \frac{\partial u_0}{\partial s} - u_0) \\ \mathcal{L}_{BS}(\bar{\sigma}, r + \lambda(x))(s \frac{\partial u_0}{\partial s}(\frac{\partial u_0}{\partial x})) &= \frac{\partial \lambda}{\partial x} s \frac{\partial}{\partial s}(s \frac{\partial u_0}{\partial s} - u_0) \end{cases}$$

Therefore v_2 solves the system

$$\begin{cases} \mathcal{L}_{BS}(\bar{\sigma}, r + \lambda^*(x))v_2 &= (T - t) \frac{\partial \lambda}{\partial x}(s \frac{\partial}{\partial s} - \cdot)^2 u_0 & \text{on } s > K, t < T \\ u_{0,1}(t, K) &= n_2(t) & \text{for } t \leq T \\ u_{0,1}(T, s) &= 0 & \text{for } s > K \end{cases} \quad (\text{D.14})$$

where $n_2(t) = \lim_{s \downarrow K} v_2(t, s) = \lim_{s \downarrow K} u_{0,1}(t, s) - v_1(t, s) = -\lim_{s \downarrow K} v_1(t, s)$ since the first part of the limit is 0 by equation (2.55). Now we are in position to eliminate the inhomogeneous term as in the fast scale case because we do not have derivatives with respect to x variable anymore in the source term. All the partial derivatives commute with the BS operator. So following the same idea we introduce

$$v_3(t, s) = -\frac{1}{2}(T - t)^2 \left(\frac{\partial \lambda}{\partial x} (s \frac{\partial}{\partial s} - \cdot)^2 u_0 \right) \quad (\text{D.15})$$

and then consider

$$v_4(t, s) = v_2(t, s) - v_3(t, s) \quad (\text{D.16})$$

Finally, v_4 solves the system

$$\begin{cases} \mathcal{L}_{BS}(\bar{\sigma}, r + \lambda^*(x))v_4 &= 0 \\ u_{0,1}(t, K) &= n(t) & \text{for } t \leq T \\ u_{0,1}(T, s) &= 0 & \text{for } s > K \end{cases} \quad (\text{D.17})$$

where

$$n(t) = n_2(t) - \lim_{s \downarrow K} v_3(t, s) \quad (\text{D.18})$$

And once we get the result for v_4

$$u_{0,1} = v_1 + v_3 + v_4$$

Let us calculate v_1 and v_3 as they are explicitly given by the equations (D.12) and (D.15). Also using (D.11)

$$\begin{aligned}
v_1(t, s) &= e^{-(r+\lambda(x))(T-t)}(T-t)^2 \left(-\frac{\partial \lambda}{\partial x} \right) \left(V_1^\delta \left(\frac{N'(d_+)}{\sigma^* \sqrt{T-t}} - p \left(\frac{s}{K} \right)^p N(d_-) + \left(\frac{s}{K} \right)^p \frac{N'(d_-)}{\sigma^* \sqrt{T-t}} \right. \right. \\
&+ \frac{d_+ N'(d_+)}{(\sigma^* \sqrt{T-t})^2} + \frac{2N'(d_-)}{(\sigma^* \sqrt{T-t})^3} \left(\frac{s}{K} \right)^p \log \left(\frac{s}{K} \right) - \frac{2pN(d_-)}{(\sigma^* \sqrt{T-t})^2} \left(\frac{s}{K} \right)^p \log \left(\frac{s}{K} \right) \\
&- \frac{2N(d_-)}{(\sigma^* \sqrt{T-t})^2} \left(\frac{s}{K} \right)^p - \frac{d_- N'(d_-)}{(\sigma^* \sqrt{T-t})^2} \left(\frac{s}{K} \right)^p + \frac{pN'(d_-)}{\sigma^* \sqrt{T-t}} \left(\frac{s}{K} \right)^p \Big) \\
&+ V_0^\delta \left(N(d_+) - \left(\frac{s}{K} \right)^p N(d_-) - \frac{N'(d_+)}{\sigma^* \sqrt{T-t}} \right. \\
&- \left. \frac{2N(d_-)}{(\sigma^* \sqrt{T-t})^2} \log \left(\frac{s}{K} \right) \left(\frac{s}{K} \right)^p + \frac{N'(d_-)}{\sigma^* \sqrt{T-t}} \left(\frac{s}{K} \right)^p \right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{s \downarrow K} v_1(t, s) &= e^{-(r+\lambda(x))(T-t)}(T-t)^2 V_1^\delta \frac{\partial \lambda}{\partial x} \left[\left(p \right. \right. \\
&+ \left. \left. \frac{2}{(\sigma^* \sqrt{T-t})^2} \right) N(d) - \frac{(2+p)N'(d)}{\sigma^* \sqrt{T-t}} \right]
\end{aligned} \tag{D.19}$$

Similarly, we obtain

$$\begin{aligned}
v_3(t, s) &= -\frac{1}{2} e^{-(r+\lambda(x))(T-t)}(T-t)^2 \frac{\partial \lambda}{\partial x} \left(N(d_+) - \left(\frac{s}{K} \right)^p N(d_-) \right. \\
&- \left. \left[\frac{N'(d_+)}{\sigma^* \sqrt{T-t}} - p \left(\frac{s}{K} \right)^p N(d_-) + \left(\frac{s}{K} \right)^p \frac{N'(d_-)}{\sigma^* \sqrt{T-t}} \right] \right. \\
&+ \left. \left[\frac{-d_+ N'(d_+)}{(\sigma^* \sqrt{T-t})^2} - \frac{N'(d_+)}{\sigma^* \sqrt{T-t}} \right] + \left(\frac{s}{K} \right)^p \left[(p+p^2-1)N(d_-) + \frac{d_- N'(d_-)}{\sigma^* \sqrt{T-t}} \right] \right)
\end{aligned}$$

Hence,

$$\begin{aligned}
\lim_{s \downarrow K} v_3(t, s) &= -\frac{1}{2} e^{-(r+\lambda(x))(T-t)}(T-t)^2 \frac{\partial \lambda}{\partial x} \left[(p^2 + 2p - 1)N(d) \right. \\
&+ \left. \frac{dN'(d)}{\sigma^* \sqrt{T-t}} - \frac{3N'(d)}{\sigma^* \sqrt{T-t}} - \frac{dN'(d)}{(\sigma^* \sqrt{T-t})^2} \right]
\end{aligned} \tag{D.20}$$

Last step is the calculation of $v_4(t, s)$. It is done exactly the same way as in section E, only $g(t)$ replaced by $n(t)$. Hence,

$$v_4(t, s) = \frac{\left(\frac{s}{K} \right)^{p/2}}{\sigma^* \sqrt{2\pi}} \int_t^T \frac{\log \frac{s}{K}}{(z-t)^{3/2}} e^{-\frac{\log(\frac{s}{K})^2}{2\sigma^{*2}(z-t)}} e^{(\frac{(\sigma^* p)^2}{8} + r + \lambda^*)(z-t)} n(z) dz \tag{D.21}$$

Appendix E

Fast Scale Correction Formulas without Default Boundary

For a detailed derivation and explicit form of the functions $\bar{P}_0^*$ and $\bar{P}_1^*$ see section (8.1) of [28] in the context of European binary option pricing.

Fast Scale Correction Formulas with Default Boundary

In this appendix we derive the integral formulas, in particular the functions $g(t)$ in section 2.7.3. Using straight forward calculus we compute the following four quantities which are needed. Keeping the notation $\tau = T - t$

$$\begin{aligned}
e^{(r+\lambda(x))\tau} u_0 &= N(d_+) - \left(\frac{s}{K}\right)^p N(d_-) \\
e^{(r+\lambda(x))\tau} \frac{\partial u_0}{\partial s} &= \frac{1}{s} \left[\frac{N'(d_+)}{\sigma^* \sqrt{\tau}} - p \left(\frac{s}{K}\right)^p N(d_-) + \left(\frac{s}{K}\right)^p \frac{N'(d_-)}{\sigma^* \sqrt{\tau}} \right] \\
e^{(r+\lambda(x))\tau} \frac{\partial^2 u_0}{\partial s^2} &= \frac{1}{s^2} \left[\frac{-d_+ N'(d_+)}{(\sigma^* \sqrt{\tau})^2} - \frac{N'(d_+)}{\sigma^* \sqrt{\tau}} \right] + \frac{1}{s^2} \left(\frac{s}{K}\right)^p [(p - p^2) N(d_-) \\
&\quad + \frac{(2p-1)N'(d_-)}{\sigma^* \sqrt{\tau}} + \frac{d_- N'(d_-)}{(\sigma^* \sqrt{\tau})^2}] \\
e^{(r+\lambda(x))\tau} \frac{\partial}{\partial s} \left(\frac{\partial^2 u_0}{\partial s^2} \right) &= N'(d_+) \left[\frac{d_+^2 - 1}{(\sigma^* \sqrt{\tau})^3} + \frac{d_+}{(\sigma^* \sqrt{\tau})^2} \right] \\
&\quad + N'(d_-) \left[\frac{d_-^2 - 1}{(\sigma^* \sqrt{\tau})^3} + \frac{(3p-1)d_-}{(\sigma^* \sqrt{\tau})^2} + \frac{p(3p-2)}{(\sigma^* \sqrt{\tau})} \right] \left(\frac{s}{K}\right)^p \\
&\quad + N(d_-) [(1-p)p^2] \left(\frac{s}{K}\right)^p
\end{aligned} \tag{E.1}$$

We first show how we convert the system (2.53) to the system (2.54) and by the help of the above set of formulas, in particular the last one, we calculate the resulting $g(t)$.

Then we write out the solution of the homogeneous equation as an integral formula and that is as close as we can get to a closed form solution. First define

$$y_1(t, s) = \tau V_3^\epsilon u_0^* = \tau V_3^\epsilon s \frac{\partial}{\partial s^2} \left(\frac{\partial^2 u_0^*}{\partial s^2} \right) \quad (\text{E.2})$$

and then consider

$$y_2(t, s) = u_1^*(t, s) - y_1(t, s) \quad (\text{E.3})$$

Note that,

$$\mathcal{L}_{BS}(\sigma^*, r + \lambda) y_1(t, s) = -V_3^\epsilon s \frac{\partial}{\partial s^2} \left(\frac{\partial^2 u_0^*}{\partial s^2} \right) \quad (\text{E.4})$$

Therefore y_2 solves the system

$$\begin{cases} \mathcal{L}_{BS}(\sigma^*, r + \lambda) y_2 &= 0 \\ y_2(t, K) &= g(t) \quad \text{for } t \leq T \\ y_2(T, s) &= 0 \quad \text{for } s > K \end{cases} \quad (\text{E.5})$$

where $g(t) = \lim_{s \downarrow K} (u_1^*(t, s) - y_1(t, s)) = -\lim_{s \downarrow K} (y_1(t, s))$ since the first part of the limit is 0 by 2.53. Hence, the only part needs to be solved is the system E.5. Then

$$u_1^* = y_1 + y_2$$

Let us calculate y_1 first, it is given by the equations (E.2). Also using (E.1)

$$\begin{aligned} y_1(t, s) &= \tau V_3^\epsilon e^{-(r+\lambda(x))\tau} [N'(d_+) [\frac{d_+^2-1}{(\sigma^*\sqrt{\tau})^3} + \frac{d_+}{(\sigma^*\sqrt{\tau})^2}] \\ &+ N'(d_-) [\frac{d_-^2-1}{(\sigma^*\sqrt{\tau})^3} + \frac{(3p-1)d_-}{(\sigma^*\sqrt{\tau})^2} + \frac{p(3p-2)}{(\sigma^*\sqrt{\tau})}] (\frac{s}{K})^p \\ &+ N(d_-) [(1-p)p^2] (\frac{s}{K})^p] \end{aligned} \quad (\text{E.6})$$

Therefore,

$$\begin{aligned} g(t) &= -\lim_{s \downarrow K} y_1(t, s) \\ &= -e^{-(r+\lambda(x))\tau} \tau^2 V_3^\epsilon \{ N'(d) [\frac{2(d^2-1)}{(\sigma^*\sqrt{\tau})^3} + \frac{3pd}{(\sigma^*\sqrt{\tau})^2} + \frac{p(3p-2)}{\sigma^*\sqrt{\tau}}] + N(d)(1-p)p^2 \} \end{aligned} \quad (\text{E.7})$$

Last step is the calculation of $y_2(t, s)$. If you consider the system (E.3), we can write it as an expectation as follows

$$y_2(t, s) = \mathbb{E}\{g(\tau)1_{\{\tau < T\}}|S_t = s\} \quad (\text{E.8})$$

where the process S_t is a Geometric Brownian Motion with volatility mean $r + \lambda$ and σ^* . On the other hand, τ is the hitting time of the process S_t the level K . By a series of change of variables, first taking the logs and then applying Girsanov, the problem turns out to be simply the hitting time of a Standard Brownian Motion a level $\log(\frac{s}{K})$ and using the distribution of the hitting time of Brownian Motion (see section 2.8 in [48]) we obtain

$$y_2(t, s) = \frac{(\frac{s}{K})^{p/2}}{\sigma^* \sqrt{2\pi}} \int_t^T \frac{\log(\frac{s}{K})}{(z-t)^{3/2}} e^{-\frac{\log(\frac{s}{K})^2}{2\sigma^{*2}(z-t)}} e^{(\frac{(\sigma^* p)^2}{8} + r + \lambda^*)(z-t)} g(z) dz \quad (\text{E.9})$$

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