On the connectivity of extremal Ramsey graphs

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Abstract

An (r, b)-graph is a graph that contains no clique of size r and no independent set of size b. The set of extremal Ramsey graphs $ERG(r, b)$ consists of all (r, b)-graphs with $R(r, b) - 1$ vertices, where $R(r, b)$ is the classical Ramsey number. We show that any $G \in ERG(r, b)$ is $r - 1$ vertex connected and $2r - 4$ edge connected for $r, b \geq 3$. This settles a question posed by David Penman.

1 Introduction

Let $R(r, b)$ be the classical Ramsey number. An (r, b)*-graph* is a graph that contains no clique of size r and no independent set of size b. Let $ERG(r, b)$ (to abbreviate *extremal Ramsey graphs*) consist of all (r, b)-graphs with $R(r, b) - 1$ vertices. In this paper we identify $G \in ERG(r, b)$ with a red-blue coloring of the complete graph and we denote the graphs induced by the color classes as G_{red} and G_{blue}.

Penman [4] established various properties of extremal Ramsey graphs and conjectured that for $k \geq 3$ every graph $G \in ERG(k, k)$ is connected. Here we will observe that this conjecture is true. In fact, its validity follows fairly straightforwardly from a result of Xiaodong, Zheng, and Radziszowski [7] (see Section 2). For the sake of completeness we present a short self-contained proof, which follows by adopting the methods in [7]. (This conjecture was proven independently by Shane Malik and David Penman.)

Theorem 1 *Let $r \geq 3$, $b \geq 2$, and $G \in ERG(r, b)$. The red graph G_{red} is connected.*

Proof. Assume that $b \geq 3$, for otherwise $G_{\text{red}} = K_{r-1}$ is clearly connected.

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Suppose on the contrary that $V(G) = V_1 \cup V_2$ is a proper partition of the vertices with only blue edges across. Pick $x_i \in V_i, i = 1, 2$. Create a new graph G' by adding a new vertex x and coloring the incident edges as follows. Color xx_1 and xx_2 blue. Color xy with $y \in V_i - x_i$ by the color of $x_i y$. Finally, recolor x_1x_2 red.

Since we have strictly increased the number of vertices, G' must contain either a red K_r or a blue K_b. First suppose there is a red K_r on a set R. R must contain x: our only alteration to $G' - x = G$ was recoloring the edge x_1x_2 red and this solitary edge joining V_1 and V_2 cannot be in a red K_r for $r \geq 3$. Since xx_1 and xx_2 are blue, $x_1, x_2 \notin R$ and $R - x$ lies inside one part, say V_1. But then $R - x + x_1$ is a red K_r in G, a contradiction. Next suppose we have a blue K_b on a set B. We must have $x \in B$. At least one of x_1, x_2 is not in B since x_1x_2 is red now. Next suppose $x_1 \notin B$. But then $R - x + x_1$ is a blue K_b in G (because x_1x_2 is blue in the original graph G.) This contradiction proves the theorem. ∎

In the remainder of the paper, we explore how connected the red graph of $G \in ERG(r, b)$ must be. We show that for $r, b \geq 3$, the (vertex) connectivity $\kappa(G_{\text{red}}) \geq r - 1$ and the edge connectivity $\lambda(G_{\text{red}}) \geq 2r - 4$. It is no doubt that these bounds are very far from best possible, which may be even exponential in $\min(r, b)$. However, it is not clear how to get any essential improvement.

2 Vertex Connectivity

We will use the following result of Xiaodong, Zheng, and Radziszowski [7, Theorem 3] which builds upon the ideas from Burr, Erdős, Faudree, and Schelp [1].

Theorem 2 (Xiaodong et al [7]) *If $2 \leq p \leq q$ and $3 \leq r$, then*

$$R(r, p + q - 1) \geq R(r, p) + R(r, q) + \begin{cases} r - 3, & \text{if } p = 2, \\ r - 2, & \text{if } p \geq 3. \end{cases}$$

In particular, the case $p = 2$ and $q = b - 1$ gives the original result of Burr et al [1, Theorem 1] (see also [7, Theorem 1] for a small correction, namely that (1) is not true for $b = 2$): for any $r \geq 2$ and $b \geq 3$,

$$R(r, b) \geq R(r, b - 1) + 2r - 3. \tag{1}$$

It follows that for any $G \in ERG(r, b)$, with $b \geq 3$, the minimum red degree

$$\delta(G_{\text{red}}) \geq 2r - 4. \tag{2}$$

Indeed, take any vertex x and let $d_{G_{\text{red}}}(x)$ denote its red degree. The blue neighborhood of x has $R(r, b) - 2 - d_{G_{\text{red}}}(x)$ vertices and induces an $(r, b - 1)$-graph. Hence,

$$R(r, b) - 2 - d_{G_{\text{red}}}(x) \leq R(r, b - 1) - 1,$$

and $d_{G_{\mathrm{red}}}(x) \geq 2r-4$ by (1), as required.

Theorem 3 *Let $r, b \geq 3$ and $G \in ERG(r,b)$. The (vertex) connectivity of the red graph G_{red} satisfies $\kappa(G_{\mathrm{red}}) \geq r-1$.*

Proof. Suppose on the contrary that we can find a partition $V(G) = V_1 \cup V_2 \cup X$ such that all edges between V_1 and V_2 are blue and $0 < |X| \leq r-2$ (X is nonempty by Theorem 1). Let $p-1$ and $q-1$ be the sizes of a maximum blue clique inside V_1 and V_2 respectively. Assume without loss of generality that $p \leq q$.

If $p = 2$, then V_1 spans a red clique, so $|V_1| \leq r-1$. The red degree of each vertex of V_1 is at most $|V_1| - 1 + |X| \leq 2r-4$. By (2), all these inequalities are equalities; in particular, $|V_1| = r-1$, $|X| = r-2$, and all edges between V_1 and X are red. But then V_1 plus any vertex from $X \neq \emptyset$ spans a red K_r, a contradiction.

So assume $p \geq 3$. We have $(p-1)+(q-1) \leq b-1$, and

$$R(r,p) - 1 + R(r,q) - 1 \geq |V_1| + |V_2| = V(G) - |X| = R(r,b) - 1 - |X|.$$

It follows from Theorem 2 that $|X| \geq r-1$, a contradiction. ∎

Corollary 4 *Let $r \geq b \geq 3$ and $G \in ERG(r,b)$. The red graph G_{red} is Hamiltonian.*

Proof. The Chvátal-Erdős condition [2] states that a graph has a Hamiltonian cycle if it is k-connected and does not contain a set of $k+1$ independent points. By Theorem 3, this condition is satisfied for G_{red} when $r \geq b \geq 3$. ∎

3 Edge Connectivity

Theorem 5 *Let $r, b \geq 3$ and $G \in ERG(r,b)$. The edge connectivity of the red graph G_{red} satisfies*

$$\lambda(G_{\mathrm{red}}) \geq \min\{\delta(G_{\mathrm{red}}), \kappa(G_{\mathrm{red}}) + r - 3\}. \tag{3}$$

Note that we have the trivial upper bound $\lambda(G_{\mathrm{red}}) \leq \delta(G_{\mathrm{red}})$. Our lower bounds for $\delta(G_{\mathrm{red}})$ and $\kappa(G_{\mathrm{red}})$ imply $\lambda(G_{\mathrm{red}}) \geq 2r-4$.

Proof. For $r = 3$, the statement $\lambda(G_{\mathrm{red}}) \geq \min\{\delta(G_{\mathrm{red}}), \kappa(G_{\mathrm{red}})\} = \kappa(G_{\mathrm{red}})$ is simply the trivial lower bound.

Consider $r \geq 4$. Suppose that the claim is not true, that is, we can find a proper partition $V(G) = V_1 \cup V_2$ such that we have at most $k = \min\{\delta(G_{\mathrm{red}})-1,\ \kappa(G_{\mathrm{red}})+r-4\}$ red edges across.

Call these red edges F. Let $v_i = |V_i|$. We claim $v_i \geq \delta(G_{\text{red}}) + 1$, $i = 1, 2$. Indeed, there is a vertex in V_i whose F-degree is at most $\lfloor \frac{k}{v_i} \rfloor \leq \lfloor \frac{\delta(G_{\text{red}})-1}{v_i} \rfloor$. We have $v_i - 1 + \lfloor \frac{\delta(G_{\text{red}})-1}{v_i} \rfloor \geq \delta(G_{\text{red}})$, which routinely implies that $v_i \geq \delta(G_{\text{red}}) + 1$. Take $x_i \in V_i$ not incident to any edge of M, which is possible since $v_i \geq \delta(G_{\text{red}}) + 1 > |F|$.

Let X be a minimal set of vertices that cover the edge set F. X is a cutset of vertices for G_{red} ($x_i \notin X$ so $V_i - X \neq \emptyset$ for $i = 1, 2$) and therefore $|X| \geq \kappa(G_{\text{red}})$. By Kőnig's theorem for bipartite graphs, there is a matching $M \subset F$ of size $|X|$. Hence $|F| - |M| \leq k - \kappa(G_{\text{red}}) \leq r - 4$. As F consists of a matching along with no more than $r - 4$ additional edges, there is no red K_{r-1} that intersects both V_1 and V_2.

Construct G' from G as in the proof of Theorem 1. Namely, add a new vertex x. Color xx_1 and xx_2 blue. For any vertex $y \in V_i \setminus \{x_i\}$, color xy with the color of $x_i y$, $i = 1, 2$. Finally, recolor $x_1 x_2$ red. We have strictly increased the number of vertices and $G \in ERG(r, b)$ so we must have a large monochromatic clique in G'. First, suppose we have a red K_r in G', say on a set R. This set R must contain x because the only red edge added to G, namely $x_1 x_2$, cannot appear in a red K_r for $r \geq 3$. But then $x_1, x_2 \notin R$. Moreover, $R - x$ lies entirely inside either V_1 or V_2. Suppose that $R - x \subset V_1$. But then $R - x + x_1$ is a red r-clique in G, a contradiction.

Next suppose that we have a blue $K_b \subset G'$, on a set B. We have $x \in B$ and at least one of x_1, x_2 does not belong to B. Suppose $x_1 \notin B$. But then $B - x + x_1$ spans a blue K_b in G (note that all edges between x_1 and V_2 are blue in G by the definition of x_1), a contradiction. ∎

4 Some Small Extremal Ramsey Graphs

In conclusion, we compare our lower bounds on connectivity with the actual connectivities for some small extremal Ramsey graphs.

(r,b)	$(3,3)$	$(3,4)$	$(3,5)$	$(3,6)$	$(3,7)$	$(4,4)$
$R(r,b)$	6	9	14	18	23	18
$\|ERG(r,b)\|$	1	3	1	7	191	1
$\kappa(G_{\text{red}})$ lower bound	2	2	2	2	2	3
$\kappa(G_{\text{red}})$ actual	2	2	4	4	4,5,6	8
$\lambda(G_{\text{red}})$ lower bound	2	2	2	2	2	4
$\lambda(G_{\text{red}})$ actual	2	2	4	4	4,5,6	8
$\kappa(G_{\text{blue}})$ lower bound	2	3	4	5	6	3
$\kappa(G_{\text{blue}})$ actual	2	4	8	11	15	8
$\lambda(G_{\text{blue}})$ lower bound	2	4	4	8	10	4
$\lambda(G_{\text{blue}})$ actual	2	4	8	11	15	8

Table 1: Lower bounds for connectivity from Theorems 3 and 5
and the actual connectivity for small extremal Ramsey graphs

Let us explain how this table is constructed. Currently, there are 6 pairs (r, b) such that $3 \leq r \leq b$ and all graphs in $ERG(r, b)$ have been enumerated, see Brendan McKay's combinatorial data website [3]. The extremal Ramsey graph catalog for $r = 3$ and $4 \leq b \leq 7$ and $r = b = 4$ (in addition to the pair $(r, b) = (3, 3)$) was obtained from [3]. This data (in the graph6 format) was processed by McKay's `showg` executable and subsequently checked and analyzed by the Combinatorica package for Mathematica.

For every such pair we checked all extremal (r, b)-graphs and wrote down the observed edge and vertex connectivities into the table rows marked 'actual'. As it turns out, the pair $(r, b) = (3, 7)$ is the only pair from our list where different extremal (r, b)-graphs may have different connectivities. For example, of the 191 graphs in $ERG(3, 7)$, 3 are red 4-connected, 178 are red 5-connected and 10 are red 6-connected.

For the ease of reference, we have also included our lower bounds, $\kappa(G_{\text{red}}) \geq r - 1$ and $\lambda(G_{\text{red}}) \geq 2r - 4$ of Theorem 3 and the comment immediately after Theorem 5, respectively. As expected, there is certainly room for improvement in the lower bounds for connectivity, even for these small graphs.

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