Existence of solutions to degenerate parabolic equations via the Monge-Kantorovich theory.

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Summary

We obtain solutions of the nonlinear degenerate parabolic equation

$$\frac{\partial s}{\partial t} = \mathrm{div}\left\{ s\,\nabla c^\star\ \Big[\,\nabla \Big(F'(s)+V\Big)\,\Big]\,\right\}$$

as a steepest descent of an energy with respect to a convex cost functional. The method used here is variational. It requires less uniform convexity assumption than that imposed by Alt and Luckhaus in their pioneering work [2]. In fact, their assumption may fail in our equation. This class of problems includes the Fokker-Planck equation, the Porous-medium equation, and the p-Laplacian equation.

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Chapter 1

Preliminaries on Elementary Analysis

1.1 Introduction

We consider a class of parabolic evolution equations, so-called doubly degenerate parabolic equations. These equations arise in many applications in physics and biology [11], [19], [20], [22]. They are used to model a variety of physical problems: the evolution of a fluid in a certain domain: Porous-medium equation [18], Fokker-Planck equation [12], etc. In this work, we focus on these parabolic equations of the form

$$\begin{cases} \frac{\partial b(u)}{\partial t} = \text{div}\ (a\,(b(u), \nabla u)) & \text{on} \quad (0,\infty)\times\Omega \\ \\ u(t=0) = u_0 & \text{on} \quad \Omega \\ \\ a\,(b(u), \nabla u)\cdot\nu = 0 & \text{on} \quad (0,\infty)\times\partial\,\Omega. \end{cases} \tag{1.1}$$

where

$$a\,(b(u), \nabla u) := f\,(b(u))\,\nabla c^{\star}\,[\,\nabla(u+V)\,]\,,$$

and $c^{\star}$ denotes the Legendre transform of $c : \mathbb{R}^d \to [0,\infty),$ that is,

$$c^{\star}(z) = \sup_{x\in R^d}\{x\cdot z - c(x)\},$$

for $z \in \mathbb{R}^d$. Here, Ω is a bounded domain of $\mathbb{R}^d$, ν is the outward unit normal to $\partial\Omega$, $b : \mathbb{R} \to \mathbb{R}$ is a monotone non-decreasing function, $V : \bar{\Omega} \to \mathbb{R}$ is a potential, $c : \mathbb{R}^d \to [0,\infty)$ is a cost function, f is a non-negative real-valued function, and $u_0 : \Omega \to \mathbb{R}$ is a measurable function. The unknown is $u : [0,\infty)\times\Omega \to \mathbb{R}, \quad u = u(t,x).$

In a previous work, Alt and Luckhauss [2] proved existence of weak solutions to (1.1), when $V = 0,$ under the following "p-uniform convexity" assumption on $a(t,z) := f(t)\nabla c^\star(z)$,

$$\langle a(t,z_1) - a(t,z_2), z_1 - z_2\rangle \geq \lambda \,|\, z_1 - z_2 \,|^p, \tag{1.2}$$

for $z_1,\ z_2 \in I\!\!R^d$ and for some $\lambda > 0$ and $p \geq 2$. This amounts to imposing that f is bounded below, and the cost function c satisfies the ellipticity condition

$$\langle \nabla c^\star(z_1) - \nabla c^\star(z_2), z_1 - z_2\rangle \geq \lambda \,|\, z_1 - z_2 \,|^p. \tag{1.3}$$

They approximated (1.1) by a time discretization, and they used a Galerkin type argument to solve the resulting elliptic problems. Then, they obtained a sequence of functions which converges to a solution of (1.1).
In the same paper, they proved uniqueness of solutions to (1.1) when $V = 0$, assuming that (1.2) holds, and the distributional derivative $\frac{\partial b(u)}{\partial t}$ of a solution u of (1.1) is an integrable function. The last condition was removed by Otto in [16]. The technique used in [16] is called the "doubling of variables", and goes as follows: given two solutions $u = u(t,x)$ and $v = v(\tau, y)$ of a PDE, one doubles the variables of u and v, that is, $u \equiv u(t,x,\tau,y)$ and $v \equiv v(\tau,y,t,x)$, and then, treats each solution as a constant with respect to the differential equation satisfied by the other solution. This technique was introduced by Kružkov in [13], and helped to overcome a lack of regularity in solutions. In [16], Otto doubled only the time variable of solutions u_1 and u_2 of (1.1), and then established the L^1-contraction principle

$$\int_\Omega [\, b\,(u_1(t)) - b\,(u_2(t))\,]_+ \leq \int_\Omega [\, b\,(u_1(0)) - b\,(u_2(0))\,]_+$$

from which he concludes uniqueness of the solution to (1.1).
Carillo, Jüngel, Markowich, Toscani, and Unterreiter studied the large-time behavior of solutions to (1.1), under assumption (1.2) [5]. They proved that, in the absence of a potential V, solutions of (1.1) decay algebraically to the steady state zero. But, in the case of the Fokker-Planck equation, they obtained an exponential convergence in relative entropy.

In this work, we eliminate assumption (1.3), and we impose instead, the following growth condition on the cost function c :

$$\beta\,|z|^q \leq c(z) \leq \alpha\,\left(|z|^q + 1\right), \tag{1.4}$$

for $z \in I\!\!R^d$ and for some $\alpha,\, \beta > 0$ and $q > 1$. Notice that (1.4) is much weaker than the ellipticity condition (1.3) imposed by Alt and Luckhaus in [2]; for example, $c(z) = \frac{|\,z\,|^3}{3}$ or $c^\star(z) = \frac{2}{3}|\,z\,|^{3/2}$ satisfies (1.4), but not (1.3). We interpret (1.1) as a dissipative system, and then, we introduce the internal energy density function $F : [0,\infty) \to I\!\!R$, satisfying $F' = b^{-1}$.

Setting $s := b(u)$, $s_0 := b(u_0)$, and $f(x) = \max(x, 0)$, we rewrite (1.1) as

$$\begin{cases} \frac{\partial s}{\partial t} + \operatorname{div}(s\,U_s) = 0 & \text{on} \quad (0,\infty)\times\Omega \\ s(t=0) = s_0 & \text{on} \quad \Omega \\ s\,U_s\cdot\nu = 0 & \text{on} \quad (0,\infty)\times\partial\Omega. \end{cases} \tag{1.5}$$

Here

$$U_s := -\nabla c^\star\left[\,\nabla\left(F'(s)+V\right)\,\right]$$

denotes the vector field describing the average velocity of a fluid evolving with the continuity equation (1.5), $s_0 : \Omega \to [0,\infty)$ is the initial mass density of the fluid, and the unknown $s : [0,\infty)\times\Omega \to [0,\infty)$, $s = s(t,x)$, is the mass density of the fluid at time t and position x of Ω. The free energy associated with the fluid at time $t \in [0,\infty)$, is given by

$$E(s(t)) := \int_\Omega \left[\,F\left(s(t,x)\right) + s(t,x)V(x)\,\right] \mathrm{d}x.$$

Problem (1.5) includes the following well-known equations:

- *the Fokker-Planck equation*

$$\frac{\partial s}{\partial t} = \Delta s + \operatorname{div}(s\nabla V)$$

  where $c(z) := \dfrac{|\,z\,|^2}{2}$, $z \in \mathbb{R}^d$, and $F(x) := x\ln x$, $x \in (0,\infty)$,

- *the Porous-medium equation*

$$\frac{\partial s}{\partial t} = \Delta s^m, \quad m \ge 1$$

  where $c(z) := \dfrac{|\,z\,|^2}{2}$, $F(x) := \dfrac{x^m}{m-1}$, and $V = 0$,

- *the parabolic p-Laplacian*

$$\begin{cases} \frac{\partial s}{\partial t} & = & \operatorname{div}\left\{ |\nabla s^n|^{p-2}\nabla s^n \right\} \\ F(x) & = & \frac{n x^m}{m(m-1)}, \quad n := m - \frac{p-2}{p-1} > 0, \end{cases}$$

  where $c(z) := \dfrac{|\,z\,|^{p^\star}}{p^\star}$, $\dfrac{1}{p} + \dfrac{1}{p^\star} = 1$, $p > 1$, and $V = 0$.

  Note that when $n = 1$, we obtain the usual p-Laplacian

$$\frac{\partial s}{\partial t} = \operatorname{div}\left\{ |\,\nabla s\,|^{p-2}\nabla s \right\}.$$

We are interested in the following questions: under what conditions does (1.5) have solutions? Is the solution unique? What are the most relevant properties of c, F and V, which ensure that solutions converge asymptotically to an equilibrium?
In this work, we answer the first and the second questions. We prove the existence and the uniqueness of solutions to (1.5), when the initial mass density s_0 is bounded below and above, that is, $s_0 + \frac{1}{s_0} \in L^\infty(\Omega)$. This restriction was made to simplify the proofs, and not to bury fundamental facts into technical computations. More precisely, we construct a weak solution to (1.5) (see Theorems 3.4.1, 3.4.2) when c, F and V are sufficiently smooth convex functions satisfying

(HC) $c(0) = 0$, and (1.4),

(HF): $F(0) = 0$, F has a super-linear growth at $+\infty$, and $(0, \infty) \ni x \mapsto x^d F(x^{-d})$ is convex.

Furthermore, we anticipate the following asymptotic results on solutions to (1.5): if $c(z) := \frac{|z|^q}{q}$, $q > 1$, then

$$W_q\left(s(t), s_\infty\right) \leq \frac{1}{\lambda}\left[E\left(s(t)\right) - E(s_\infty)\right] \leq \frac{1}{\lambda} e^{-(q^\star \lambda^{q^\star - 1})t}\left[E(s_0) - E(s_\infty)\right], \tag{1.6}$$

and for general cost functions c,

$$W_c\left(s(t), s_\infty\right) \leq \frac{1}{\lambda}\left[E\left(s(t)\right) - E(s_\infty)\right] \leq \frac{1}{\lambda} e^{-t}\left[E(s_0) - E(s_\infty)\right]. \tag{1.7}$$

Here, λ is the constant of the uniform c-convexity of the potential V (see (4.5)), and s_∞ is the equilibrium solution of (1.5). We have not been able to establish (1.6) and (1.7) rigorously although we think that their proof are not out of reach. Notice that (1.6) extends the result stated in [5] on the asymptotic behavior of the Fokker-Planck equation, where $c(z) = \frac{|z|^2}{2}$, $F(x) = x \ln x$, and V is uniformly convex with $\mathrm{Hess}(V) \geq \lambda\, \mathrm{id}$.

Our approach in studying the existence of solutions to problem (1.5) was inspired by the work of Jordan, Kinderlehrer and Otto [12]. In [12], the authors observed that the Fokker-Planck equation $\left(c(z) = \frac{|z|^2}{2},\ F(x) = x \ln x\right)$, can be interpreted as the gradient flow of the entropy functional

$$H(s) := \int \left(s \ln s + s\, V\right) \mathrm{d}x,$$

with respect to the Wasserstein metric d_2, and then, they proved the existence of weak solutions to the Fokker-Planck equation. Recall that d_2 is a metric on the set of probability measures on $\mathbb{R}^d$, with finite second moments, defined by

$$d_2(\mu_0, \mu_1) := \left[\inf \left\{ \int_{R^d \times R^d} \frac{|x-y|^2}{2} \, d\gamma(x,y) : \quad \gamma \in \Gamma(\mu_0, \mu_1) \right\} \right]^{1/2},$$

where $\Gamma(\mu_0, \mu_1)$ denotes the set of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$, having μ_0 and μ_1 as their marginals (see the definition in Section 1.2.2 below). Here, the interpretation of our discrete scheme is that at each time, the system tries to minimize its free energy

$$E(s) := \int_\Omega [\, F \circ s + s\, V \,] \, dx$$

against a penalty $W_c^h(\cdot, s)$, viewed as the kinetic energy,

$$W_c^h(\rho_0, \rho_1) := \inf \left\{ \int_{R^d \times R^d} c\left(\frac{x-y}{h} \right) d\gamma(x,y) : \quad \gamma \in \Gamma(\rho_0, \rho_1) \right\},$$

where $h > 0$ is a time-step size, and $\frac{x-y}{h}$ is viewed as a velocity. In other words, if s_{k-1}^h is the state of the system at time $t_{k-1} = (k-1)h$, its state at time $t_k = kh$ is the unique minimizer of

$$E(s) + h\, W_c^h(s_{k-1}^h, s),$$

over all probability density functions s on Ω. Here, $W_c^h(s_{k-1}^h, s)$ is actually the minimum work required to move the system from the state s_{k-1}^h, to the state s at time $t = kh$.

Outline of methods

For the sake of illustration, we outline the proof of the existence theorem to problem (1.5), assuming that $V = 0$, and $s_0 + \frac{1}{s_0} \in L^\infty(\Omega)$. By standard approximation arguments (see Proposition 1.4.2), one can extend the result to the case where $\frac{1}{s_0}$ fails to be bounded. Once (1.5) is solved for special initial data s_0, one can look for a-priori estimates to extend the existence of solutions to a wider class of initial data, $s_0 \in L^p(\Omega)$, $p \geq q$. The proof of our existence theorem consists of three main parts:

1. We interpret (1.5) as a "steepest descent" of the internal energy functional

$$\mathcal{P}_a(\Omega) \ni s \mapsto E_i(s) := \int_\Omega F\left(s(x)\right) dx$$

against the Monge-Kantorovich work W_c^h, where $h > 0$ is the time-step size, and $\mathcal{P}_a(\Omega)$ denotes the set of all probability density functions $s : \Omega \to$

$[0, \infty)$. In other words, given a mass density s_{k-1}^h of the fluid at time $t_{k-1} = (k-1)h$, we define the mass density s_k^h at time $t_k = kh$, to be the unique minimizer of the variational problem

$$(P_k^h): \quad \inf_{s \in \mathcal{P}_a(\Omega)} \left\{ h\, W_c^h \left(s_{k-1}^h, s \right) + E_i(s) \right\} \tag{1.8}$$

(see Proposition 2.1.1). So, at each time t, the system tends to decrease its internal energy $E_i(s)$, while trying to minimize the work to move from state $s(t)$ to state $s(t+h)$.

2. We write the Euler-Lagrange equation of (P_k^h), and then, deduce that

$$\frac{s_k^h - s_{k-1}^h}{h} = \operatorname{div} \left\{ s_k^h \nabla c^\star \left[\nabla \left(F'(s_k^h) \right) \right] \right\} + A_k(h) \tag{1.9}$$

weakly, for $k \in \mathbb{N}$ (Proposition 2.3.1), where $A_k(h)$ tends to 0, as h goes to 0. The subsequent equality (1.9) shows clearly why (1.8) is a discretization of (1.5).

3. We define the approximate solution s^h to (1.5), as the time-discrete function

$$\begin{cases} s^h(t,x) &= s_k^h(x) \ \text{ if } \ t \in ((k-1)h, kh] \\ s^h(0,x) &= s_0(x), \end{cases}$$

and we deduce from (1.9) that, s^h satisfies

$$\begin{cases} \frac{\partial s^h}{\partial t} = \operatorname{div} \left\{ s^h \nabla c^\star \left[\nabla \left(F'(s^h) \right) \right] \right\} + A(h) \quad \text{on} \quad (0,\infty) \times \Omega \\ s^h(t=0) = s_0 \quad \text{on} \quad \Omega \end{cases} \tag{1.10}$$

in the weak sense (Proposition 2.5.1), where $A(h)$ is shown to be $0\left(h^{\epsilon(q)}\right)$, $\epsilon(q) := \min(1, q-1)$, as h goes to 0. (Proposition 3.1.2).

4. We let h go to 0 in (1.10), and we wish to show that the sequence $(s^h)_h$ converges to a function s, which solves (1.5) in the weak sense. Here, two convergence results are established:

4.1. the weak convergence of $(s^h)_h$ to s in $L^1((0,T) \times \Omega)$ (up to a subsequence) for $0 < T < \infty$, which clearly proves that $\left(\frac{\partial s^h}{\partial t}\right)_h$ converges weakly to $\frac{\partial s}{\partial t}$ in $\left[C_c^\infty(\mathbb{R} \times \mathbb{R}^d)\right]'$, and

4.2 the weak convergence of the nonlinear term $\left\{ s^h \nabla c^\star \left[\nabla \left(F'(s^h) \right) \right] \right\}_h$ to $s \nabla c^\star \left[\nabla \left(F'(s) \right) \right]$ in $L^1((0,T) \times \Omega)$, for a subsequence.

(4.1) follows from the fact that $E_i\left(s^h(t)\right) \leq E_i(s_0)$, and F has a superlinear growth at $+\infty$. But, since we require here that s_0 is bounded above, we use instead, the maximum principle stated in Proposition 2.2.1, to obtain (4.1) (see Lemma 3.2.1). Indeed, starting with an initial probability density function s_0 which is bounded above, that is, $s_0 \leq N$, we prove that, at any time $t_k = kh$, $k \in \mathbb{N}$, the probability density function s_k^h - solution of (P_k^h) - is bounded above, as well, that is, $s_k^h \leq N$. As a consequence, we obtain that $(s^h)_h$ is bounded in $L^\infty\left((0,\infty)\times\Omega\right)$, and then, we conclude (4.1).

(4.2) is one of the most difficult tasks in the proof of the existence theorem. Its proof requires elaborated intermediate results. We proceed as follows:

(i). First, we improve (4.1) by showing that, in fact, $(s^h)_h$ converges strongly to s, for a subsequence, in $L^1\left((0,T)\times\Omega\right)$ for $0 < T < \infty$ (Proposition 3.2.5).

(ii). Next, we show that $\left\{ \nabla c^\star\left[\nabla\left(F'(s^h)\right)\right] \right\}_h$ converges weakly to $\nabla c^\star\left[\nabla\left(F'(s)\right)\right]$ in $L^q\left((0,T)\times\Omega\right)$, for a subsequence (Theorem 3.3.3).

To prove (i) and (ii), one needs to have a good control on the spatial derivative of s^h, for example, to show that $\left\{ \nabla\left(F'(s^h)\right) \right\}_h$ is bounded in $L^{q^\star}\left((0,T)\times\Omega\right)$, for $T < \infty$. The main ingredient needed to establish this fact is the following *Monge-Kantorovich type energy inequality*:

$$E_i(\rho_0) - E_i(\rho_1) \geq \int_\Omega \langle \nabla\left(F'(\rho_1)\right), S(y) - y\rangle \rho_1(y)\,\mathrm{d}y, \tag{1.11}$$

for $\rho_0, \rho_1 \in \mathcal{P}_a(\Omega)$. Here, S denotes the c-optimal map that pushes ρ_1 forward to ρ_0 (see the definition in Proposition 1.2.1). A more general statement of the energy inequality can be found in Theorem 2.4.2. In fact, (1.11) is a direct consequence of the displacement convexity of the energy functional $\mathcal{P}_a(\Omega) \ni \rho \mapsto E_i(\rho)$, that is, the convexity of

$$[0,1] \ni t \mapsto E_i(\rho_{1-t}),$$

where,

$$\rho_{1-t} := \left((1-t)\,\mathrm{id} + tS\right)_\# \rho_1$$

is the shortest path joining ρ_1 and ρ_0 in $\mathcal{P}_a(\Omega)$. When $c(z) = \frac{|z|^2}{2}$, in which case S is the gradient of a convex function, the above interpolation in ρ_{1-t} was introduced by R.McCann in [14].
Indeed, setting $\rho_0 := s_{k-1}^h$ and $\rho_1 := s_k^h$, in (1.11), and using the Euler-Lagrange equation of (P_k^h), that is,

$$\frac{S_k^h - \mathrm{id}}{h} = \nabla c^\star\left[\nabla\left(F'(s_k^h)\right)\right], \tag{1.12}$$

where S_k^h is the c-optimal map pushing s_k^h forward to $s_{k-1}^h,$ we obtain that

$$h\int_\Omega \langle \nabla\left(F'(s_k^h)\right), \nabla c^\star\left[\nabla\left(F'(s_k^h)\right)\right]\rangle\, s_k^h \leq E_i(s_{k-1}^h) - E_i(s_k^h). \qquad (1.13)$$

We integrate (1.13) over $t \in [0,T],\ T < \infty,$ and we use Jensen's inequality, to deduce that

$$\int_0^T \int_\Omega \langle \nabla\left(F'(s^h)\right), \nabla c^\star\left[\nabla\left(F'(s^h)\right)\right]\rangle s^h \leq E_i(s_0) - |\,\Omega\,| F\left(\frac{1}{|\,\Omega\,|}\right). \qquad (1.14)$$

We combine (1.4), (1.14), and the fact that $(s^h)_h$ is bounded in $L^\infty\left((0,\infty)\times\Omega\right),$ to conclude that

$$\int_0^T \int_\Omega s^h \left|\nabla\left(F'(s^h)\right)\right|^{q^\star} \leq \text{cst}.$$

And since $\left(\frac{1}{s^h}\right)_h$ is bounded in $L^\infty\left((0,\infty)\times\Omega\right)$ - by the minimum principle of Proposition 2.2.1 -, we deduce that $\left\{\nabla\left(F'(s^h)\right)\right\}_h$ is bounded in $L^{q^\star}\left((0,T)\times\Omega\right),$ for $T < \infty.$

Organization of the work

This work consists of five chapters. In sections 1.1 and 1.2, we include the introduction, notations, assumptions and definitions used in the work. Some important results of previous authors are also collected in section 1.2. In sections 1.3 and 1.4, we recall two approximation results: the approximation of a convex cost function by regular cost functions, and the approximation of a probability density by probability densities which are bounded below and above. The first approximation is used to establish the energy inequality (2.58), and the second approximation is needed in the proof of the maximum/minimum principle of Proposition 2.2.1. In chapter 2, we study problem (1.5) by discretizing in time. In section 2.1, we prove the existence and uniqueness of the solution to the variational problem

$$(P): \quad \inf\left\{W_c^h(s_{k-1}^h, s) + \frac{1}{h}E(s): \ \ s \in \mathcal{P}_a(\Omega)\right\}.$$

In section 2.2, we establish a maximum/minimum principle for the minimizer s_k^h of (P), that is, s_k^h is bounded below and above, provided s_{k-1}^h is bounded below and above. We use the maximum principle in Lemma 3.2.1, to prove the weak convergence in $L^1\left((0,T)\times\infty\right),\ T < \infty,$ of the approximate sequence $(s^h)_h$ to a solution s of (1.5), and the minimum principle, in Lemma 3.2.2, to control the spatial derivative of $s^h,$ namely, $(\nabla s^h)_h$ is bounded in $L^{q^\star}((0,T)\times\Omega)$, $T<\infty$. Moreover, we show in section 2.2 that the interpolant probability density s_{1-t} between two probability densities s_0 and s_1 which are bounded above, is also bounded above. This result will be used in Lemma 3.2.3 to establish the time-compactness of the approximate sequence $(s^h)_h$, one of the ingredients needed to prove the strong convergence of a subsequence of $(s^h)_h$ to s in $L^1((0,T)\times\Omega)$, $T<\infty$. In section 2.3, we establish the Euler-Lagrange equation for (P) and some useful properties of s^h_k. In section 2.4, we prove the energy inequality for smooth cost functions, and then, extend it to general cost functions. In section 2.5, we define the approximate solution s^h to (1.5), and we show that s^h satisfies

$$\begin{cases} \frac{\partial s^h}{\partial t} = \operatorname{div}\Big\{ s^h\,\nabla c^\star\left[\nabla\left(F'(s^h)+V\right)\right]\Big\} + A(h) \\ \\ s^h(t=0) = s_0, \end{cases} \tag{1.15}$$

in the weak sense. Chapter 3 deals with the limit of (1.15), when s_0 is bounded below and above. In section 3.1, we show that $A(h)=0\left(h^{\epsilon(q)}\right)$, as h goes to 0, where $\epsilon(q):=\min(1,q-1)$. In section 3.2, we prove that the approximate sequence $(s^h)_h$ converges strongly to some s in $L^1((0,T)\times\Omega)$, for a subsequence. In section 3.3, we show that $\left\{\nabla c^\star\left[\nabla\left(F'(s^h)\right)\right]\right\}_h$ converges weakly to $\nabla c^\star\left[\nabla\left(F'(s)\right)\right]$ in $L^q((0,T)\times\Omega)$, for a subsequence. In section 3.4, we state and prove the existence and uniqueness theorem to problem (1.5), when s_0 is bounded below and above. In chapter 4, we comment on few open problems related to (1.5), namely, the asymptotic behavior of solutions of (1.5), and the contraction in the Wasserstein metric of two solutions of (1.5). The appendix is presented in chapter 5. In section 5.1, we collect some previous results of other authors which are used in this work. In section 5.2, we establish some intermediate results needed in the previous chapters.

1.2 Notations, Definitions, and Assumptions

1.2.1 Notations

The following notations will be used in this work.

- Ω denotes an open bounded convex and smooth subset of $I\!R^d$, where $d\geq 1$.
- $B_R(x)$ denotes the open ball of $I\!R^D$ of radius R, centered at $x\in I\!R^d$, and $B_R(x)^c := I\!R^D\setminus B_R(x)$, for some $D\geq 1$.
- $\mathcal{P}_a(\Omega)$ denotes the set of all probability measures on Ω, which are absolutely continuous with respect to the Lebesgue measure. We identify

these probability measures with their density functions;

$$\mathcal{P}_a(\Omega) := \Big\{ s : \Omega \to [0,\infty) \ \text{measurable}, \quad \int_\Omega s(x)\,\mathrm{d}x = 1 \Big\}$$

- $|A|$ denotes the Lebesgue measure of a Borel set A of $I\!R^d$.
- $\|\varphi\|_{L^q(\Omega)}$ denotes the L^q-norm of a function $\varphi : \Omega \to I\!R$.
- spt (φ) denotes the support of a real-valued function φ, that is, the closure of the set $\{x \in \Omega : \ \varphi(x) \neq 0\}$.
- $p^\star$ denotes the conjugate index of a positive real p, that is, $\frac{1}{p} + \frac{1}{p^\star} = 1$.
- If $x = (x_1, \cdots, x_d)$, and $y = (y_1, \cdots, y_d)$ are vectors in $I\!R^d$, then $\langle x, y\rangle$ denotes the scalar product of x and y, that is,

$$\langle x, y\rangle = \sum_{i=1}^{d} x_i y_i.$$

- If $x = (x_1, \cdots, x_d)$, then $|x|$ denotes the norm of x in $I\!R^d$, that is,

$$|x| = \sqrt{\langle x, x\rangle}\,.$$

- If $G : I\!R^d \to I\!R$ is convex, then $G^\star : I\!R^d \to I\!R$ denotes the Legendre transform of G, that is,

$$G^\star(y) := \sup_{x \in R^d} \{\langle x, y\rangle - G(x)\}.$$

- If A is a Borel subset of $I\!R^d$, $I\!I_A$ denotes the characteristic function of A, that is,

$$I\!I_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Throughout this work, M and N are positive reals, a.e. refers to the d-dimensional Lebesgue measure, and

$$c_h(z) := c\left(\frac{z}{h}\right).$$

The following definitions are needed in the work.

1.2.2 Definitions

1.2.2.1 Probability measures with marginals

Let μ_0 and μ_1 be probability measures on $I\!R^d$. A Borel probability measure γ on the product space $I\!R^d \times I\!R^d$ is said to have μ_0 and μ_1 as its marginals, if one of the following equivalent conditions holds:

(i). for Borel $A \subset I\!R^d$,

$$\gamma[A \times I\!R^d] = \mu_0[A],$$

and

$$\gamma[I\!R^d \times A] = \mu_1[A].$$

(ii).

$$\int_{R^d \times R^d} [\varphi(x) + \psi(y)] \, d\gamma(x,y) = \int_{R^d} \varphi(x) \, d\mu_0(x) + \int_{R^d} \psi(y) \, d\mu_1(y),$$

for all $(\varphi, \psi) \in L^1_{\mu_0}(I\!R^d) \times L^1_{\mu_1}(I\!R^d)$, where $L^1_{\mu_i}(I\!R^d)$ denotes the space of μ_i-integrable functions on $I\!R^d$ $(i = 1, 2)$.
We denote by $\Gamma(\mu_0, \mu_1)$, the set of all probability measures satisfying (i) or (ii). If μ_0 and μ_1 are absolutely continuous with respect to Lebesgue, and ρ_0, ρ_1 denote their respective density functions, we simply write $\Gamma(\rho_0, \rho_1)$.

1.2.2.2 Push-forward mapping

Let μ_0 and μ_1 be probability measures on $I\!R^d$. A Borel map $T: I\!R^d \to I\!R^d$ is said to push μ_0 forward to μ_1, if

(i). $\mu_1[A] = \mu_0[T^{-1}(A)]$ for Borel $A \subset I\!R^d$, or equivalently

(ii). $\int_{R^d} \varphi(y) \, d\mu_1(y) = \int_{R^d} \varphi(T(x)) \, d\mu_0(x)$ for all $\varphi \in L^1_{\mu_1}(I\!R^d)$.

Whenever (i) or (ii) holds, we write that $\mu_1 = T_\# \mu_0$, and we say that T pushes μ_0 forward to μ_1.

The next proposition is due to Caffarelli [3], and Gangbo-McCann [10]. It asserts the existence and uniqueness of the minimizer for the Monge-Kantorovich problem.

Proposition 1.2.1 *(Existence of optimal maps.)*
Let $c : I\!R^d \to [0, \infty)$ be strictly convex, and $\rho_0, \rho_1 \in \mathcal{P}_a(\Omega)$. Then

(i). there is a function $v : \overline{\Omega} \to I\!R$ such that, $T := id - (\nabla c^\star) \circ \nabla u$ pushes ρ_0 forward to ρ_1, where $u(x) = \inf_{y \in \overline{\Omega}} \left\{ c(x-y) - v(y) \right\}$ for $x \in \overline{\Omega}$.

(ii). *T is the unique minimizer (a.e. with respect to ρ_0) of the Monge problem*

$$(\mathcal{M}): \quad \inf \Big\{ \int_\Omega c\,(x - Tx)\ \rho_0(x)\, dx, \quad T_\# \rho_0 = \rho_1 \Big\}.$$

(iii). *The joint measure $\gamma := (id \times T)_\# \rho_0$ uniquely solves the Kantorovich problem*

$$(\mathcal{K}): \quad \inf \Big\{ \int_{R^d \times R^d} c\,(x - y)\ d\gamma(x,y), \quad \gamma \in \Gamma(\rho_0, \rho_1) \Big\}.$$

(iv). *T is one-to-one, that is, there exits a map S pushing ρ_1 forward to ρ_0, such that $T\,(S(y)) = y$ a.e. with respect to ρ_1, while $S\,(T(x)) = x$ a.e. with respect to ρ_0.*
Moreover, $S = id + \nabla c^\star(-\nabla v)$, where $v(y) = \inf_{x \in \overline{\Omega}} \Big\{ c(x-y) - u(x) \Big\}$ for $y \in \overline{\Omega}$.
v is called the c-transform of u, and it is denoted by $v := u^c$.

We will refer to T (respectively S) as the c-optimal map that pushes ρ_0 (respectively ρ_1) forward to ρ_1 (respectively ρ_0), and γ will be called the c-optimal measure in $\Gamma(\rho_0, \rho_1)$.

1.2.2.3 Wasserstein metric

Let $c : I\!R^d \to [0, \infty)$ be strictly convex, $h > 0$, and $\rho_0,\ \rho_1 \in \mathcal{P}_a(\Omega)$. We define

$$W_c^h\,(\rho_0, \rho_1) := \inf \Big\{ \int_{R^d \times R^d} c \left(\frac{x - y}{h} \right) \mathrm{d}\gamma(x,y) : \quad \gamma \in \Gamma(\rho_0, \rho_1) \Big\}.$$

If $c(z) = \frac{|z|^q}{q}$, we denote W_c^h by W_q^h. When $c(z) = \frac{|z|^2}{2}$ and $h = 1$, $d_2 := \sqrt{W_2^h}$ is called the Wasserstein metric.

We deduce from Proposition 1.2.1 that, there exist a unique probability measure $\gamma \in \Gamma(\rho_0, \rho_1)$, and a unique mapping T pushing ρ_0 forward to ρ_1, whose inverse S pushes ρ_1 forward to ρ_0, such that

$$\begin{aligned} W_c^h(\rho_0, \rho_1) &= \int_{R^d \times R^d} c \left(\frac{x - y}{h} \right) \mathrm{d}\gamma(x,y) \\ &= \int_\Omega c \left(\frac{x - Tx}{h} \right) \rho_0(x)\, \mathrm{d}x \\ &= \int_\Omega c \left(\frac{Sy - y}{h} \right) \rho_1(y)\, \mathrm{d}y. \end{aligned}$$

The following assumptions will be needed in this work.

1.2.3 List of assumptions

(HC1) : $c : \mathbb{R}^d \to [0, \infty)$ is strictly convex.

(HC2) : $0 = c(0) < c(z)$, for $z \neq 0$.

(HC3) : $\lim_{|x|\to\infty} \frac{c(x)}{|x|} = \infty$, i.e. c is coercive.

(HC4) : $\beta \,|\, z \,|^q \leq c(z) \leq \alpha \,(|\, z \,|^q + 1)$, for $z \in \mathbb{R}^d$, where α, $\beta > 0$ and $q > 1$.

(HF1) : $\lim_{x\to+\infty} \frac{F(x)}{x} = +\infty$, i.e. $F : [0, \infty) \to \mathbb{R}$ has a super-linear growth at infinity.

(HF2) : $(0, \infty) \ni x \mapsto x^d F(x^{-d})$ is convex.

Examples of cost and energy density functions

- Any finite, positive, linear combination of cost functions $|\, z \,|^{q_i}$, $q_i > 1$, $i \in \mathbb{N}$, satisfies (HC1) - (HC4). For example, if $c(z) = \sum_{i=1}^n A_i \,|\, z \,|^{q_i}$, where $n \in \mathbb{N}$, $q_i > 1$, and the $A_i \geq 0$ are not all zeros, then $q = \max_{\{i=1,\cdots,n\}}(q_i) = q_{i_0}$, $\beta = A_{i_0}$, and $\alpha = \sum_{i=1}^n A_i$.
- The following energy density functions satisfy (HF1) - (HF2): $F(x) = x \ln x$, $F(x) = x^m$, where $m > 1$, and $F(x) = \sum_{i=1}^n A_i \, F_i(x)$, where $n \in \mathbb{N}$, the $A_i \geq 0$ are not all zeros, and the F_i are like the previous F.

1.3 Approximation of convex costs by smooth costs

Throughout this section, $c : \mathbb{R}^d \to [0, \infty)$ denotes a convex function. We prove that c can be approximated by a sequence $(g_\epsilon)_{\epsilon\downarrow 0}$ of strictly convex and smooth functions, such that the Legendre transforms $g_\epsilon^\star$ are twice continuously differentiable. This approximation will be used in section 2.4 to extend the energy inequality (2.21) to general cost functions (see Theorem 2.4.2).

Proposition 1.3.1 *Assume that c is of class C^1 and satisfies $c(0) = 0$ and (HC4). Then, there exists a sequence $(g_\epsilon)_{\epsilon\downarrow 0}$ of non-negative, strictly convex functions on $\mathbb{R}^d$, satisfying:*

(i). $g_\epsilon \in C^\infty(\mathbb{R}^d)$, *and* $g_\epsilon^\star \in C^2(\mathbb{R}^d)$.

(ii). $(g_\epsilon)_{\epsilon\downarrow 0}$ *converges to* c*, locally in* $C^1(\mathbb{R}^d)$.

(iii). $0 = g_\epsilon(0) < g_\epsilon(z)$, *for* $z \neq 0$.

`Proof.` Because $c \in C^1(\mathbb{R}^d)$ is convex and has a super-linear growth as $|\,x\,| \to \infty$, we have that $c^\star \in C(\mathbb{R}^d)$. Let $\rho \in C_c^\infty(\mathbb{R}^d)$ be such that $\rho \geq 0$, $\operatorname{spt}(\rho) \subset \overline{B_1(0)}$, and $\int_{R^d} \rho(x)\,\mathrm{d}x = 1$. For $\epsilon > 0$, consider

$$c_\epsilon(z) := \rho_\epsilon \star c(z) + \frac{\epsilon}{2}\,|\,z\,|^2, \quad \forall\, z \in \mathbb{R}^d,$$

where $\rho_\epsilon(z) := \frac{1}{\epsilon^d}\,\rho(\frac{z}{\epsilon})$ is the standard mollifier, and $\rho_\epsilon \star c$ denotes the convolution of ρ_ϵ and c. It is clear that $c_\epsilon \in C^\infty(\mathbb{R}^d)$ is non-negative and strictly convex, as the sum of the convex function $\rho_\epsilon \star c$ and the strictly convex function $z \mapsto \frac{\epsilon}{2}\,|z|^2$. Also, $(c_\epsilon)_{\epsilon\downarrow 0}$ converges to c locally in $C^1(\mathbb{R}^d)$. Furthermore, standard arguments show that $c_\epsilon^\star \in C^1(\mathbb{R}^d)$, and

$$\nabla c_\epsilon[\,\nabla c_\epsilon^\star(z)\,] = z, \quad \forall\, z \in \mathbb{R}^d.$$

As a consequence, the function

$$z \mapsto \frac{\operatorname{cof}\left(D^2 c_\epsilon[\,\nabla c_\epsilon^\star(z)\,]\right)}{\det\left(D^2 c_\epsilon[\,\nabla c_\epsilon^\star(z)\,]\right)} := A_\epsilon(z),$$

is well-defined, continuous on $\mathbb{R}^d$, and $D^2 c_\epsilon^\star = A_\epsilon$; here, $\operatorname{cof}(A)$ denotes the matrix of the cofactors of a matrix A. Therefore, $c_\epsilon^\star \in C^2(\mathbb{R}^d)$. Now, let R and ϵ be such that $R^q > \frac{\alpha}{\beta}$, and $0 < \epsilon^q \leq \min(R^q, \frac{\beta}{\alpha}R^q - 1)$. Because of (HC4), we have that $c(z) \geq \beta R^q$, for $z \in \mathbb{R}^d \setminus B_R(0)$. So, for $|\,z\,| \geq 2R$, we deduce that

$$\begin{aligned} c_\epsilon(z) > \rho_\epsilon \star c(z) &= \int_{B_\epsilon(z)} \rho_\epsilon(z-y)c(y)\,\mathrm{d}y \\ &\geq \beta R^q \int_{B_\epsilon(z)} \rho_\epsilon(z-y)\,\mathrm{d}y = \beta R^q, \end{aligned}$$

where, the last inequality holds because $B_\epsilon(z) \subset \mathbb{R}^d \setminus B_R(0)$. Hence,

$$\inf_{|\,z\,| \geq 2R} c_\epsilon(z) \geq \beta R^q. \tag{1.16}$$

Furthermore, due to (HC4), we have that

$$c_\epsilon(0) = \int_{B_\epsilon(0)} \rho_\epsilon(y)c(y)\,\mathrm{d}y \leq \alpha(\epsilon^q + 1) \leq \beta R^q. \tag{1.17}$$

Combining (1.16) and (1.17), we conclude that

$$\inf_{z \in B_{2R}(0)} c_\epsilon(z) \leq c_\epsilon(0) \leq \inf_{|\,z\,| \geq 2R} c_\epsilon(z).$$

Therefore, c_ϵ attains its minimum over $\mathbb{R}^d$ at some $z_\epsilon \in B_{2R}(0)$. Let $\bar{z}$ denote the limit of $(z_\epsilon)_{\epsilon\downarrow 0}$, which exists because $(z_\epsilon)_\epsilon$ is bounded. We have that

$$c(\bar{z}) = \lim_{\epsilon\downarrow 0} c_\epsilon(z_\epsilon) \leq \lim_{\epsilon\downarrow 0} c_\epsilon(z) = c(z), \quad \forall\, z \in \mathbb{R}^d,$$

and then, $\bar{z} = 0$, because $0 = c(0) < c(z)$ for all $z \neq 0$. Now, define g_ϵ as follows:

$$g_\epsilon(z) := \bar{g}_\epsilon(z) - \bar{g}_\epsilon(0), \quad \text{where } \bar{g}_\epsilon(z) := c_\epsilon(z + z_\epsilon).$$

Clearly, g_ϵ is strictly convex and satisfies (i). Moreover, the non-negativity of g_ϵ and (iii) follow easily from the following inequality:

$$\bar{g}_\epsilon(0) < \bar{g}_\epsilon(z), \quad \forall\, z \neq 0, \tag{1.18}$$

which we prove next. For all $z \in \mathbb{R}^d$, we have that

$$\bar{g}_\epsilon(0) = c_\epsilon(z_\epsilon) \leq c_\epsilon(z + z_\epsilon) = \bar{g}_\epsilon(z),$$

with strict inequality, unless $z = 0$. Indeed, if $\bar{g}_\epsilon(z) = \bar{g}_\epsilon(0)$, we have that $\nabla\bar{g}_\epsilon(z) = 0 = \nabla\bar{g}_\epsilon(0)$, which implies that $z = 0$, by the strict-convexity of $\bar{g}_\epsilon$.

To prove (ii), let $r > 0$. It is straightforward to check that

$$\begin{aligned}\sup_{z\in B_r(0)} &|\, \bar{g}_\epsilon(z) - c(z)\,| \\ &\leq \sup_{B_{r+2R}(0)} |\, c_\epsilon(z) - c(z)\,| + \sup_{B_r(0)} |\, c(z + z_\epsilon) - c(z)\,|.\end{aligned}$$

The limit as ϵ goes to 0 of the above inequality shows that $(\bar{g}_\epsilon)_{\epsilon\downarrow 0}$ converges to c, uniformly on $B_r(0)$. And since $c(0) = 0$, we conclude that (g_ϵ) converges to c, locally in $C(\mathbb{R}^d)$. Furthermore, because $\nabla g_\epsilon(z) = \nabla c_\epsilon(z + z_\epsilon)$, $(z_\epsilon)_{\epsilon\downarrow 0}$ converges to 0, and ∇c_ϵ converges to ∇c locally in $C(\mathbb{R}^d)$, we have that $(\nabla g_\epsilon)_{\epsilon\downarrow 0}$ converges to ∇c locally in $C(\mathbb{R}^d)$. This proves (ii), and then completes the proof of the proposition □

1.4 Approximation of probability densities

In 1.4.1, we prove that a probability density function s on Ω, which belongs to $L^p(\Omega)$ for some $1 \leq p \leq \infty$, can be approximated in $L^p(\Omega)$, by a sequence $(s_R)_{R\uparrow\infty} \subset L^\infty(\Omega)$ of probability densities. Furthermore, if $F : [0,\infty) \to \mathbb{R}$ is convex, and $\int_\Omega F(s(x))\,\mathrm{d}x$ is finite, we show that $\int_\Omega F(s_R(x))\,\mathrm{d}x$ is bounded, uniformly in R. In 1.4.2, we prove that s_R can be approximated by probability densities which are bounded below and above.

1.4.1 Approximation of a probability density by probability densities that are bounded above

Proposition 1.4.1 *Let* $s \in \mathcal{P}_a(\Omega) \cap L^p(\Omega)$, $1 \le p \le \infty$, *and* $F : [0,\infty) \to I\!R$ *be convex. Assume that* $\int_\Omega F(s(x))\,dx < \infty$. *Then, there exists a sequence* $(s_R)_{R\uparrow\infty}$ *in* $\mathcal{P}_a(\Omega) \cap L^\infty(\Omega)$, *such that*

(i). $(s_R)_{R\uparrow\infty}$ *converges to* s, *in* $L^p(\Omega)$, *and*

(ii). $\int_\Omega F(s_R(x))\,dx \le \int_\Omega F(s(x))\,dx$.

`Proof.` To avoid trivialities, we assume that $s \notin L^\infty(\Omega)$. Since $\int_\Omega s(x)\,\mathrm{d}x = \int_{[s>0]} s(x)\,\mathrm{d}x$, we assume without loss of generality that s is positive a.e. on Ω; otherwise, we use the restriction of s to the subset $[s>0]$ of Ω. For $R > 0$, we define $A_R := \{\, x \in \Omega : \ s(x) > R \,\}$, and

$$s_R(x) := \begin{cases} s(x) & \text{if} \quad x \in \Omega \setminus A_R \\ \\ \frac{1}{|\,A_R\,|} \int_{A_R} s(y)\,\mathrm{d}y & \text{if} \quad x \in A_R. \end{cases}$$

Clearly, $s_R \in \mathcal{P}_a(\Omega)$ (because $s \in \mathcal{P}_a(\Omega)$), and

$$0 \le s_R(x) \le \max\left(R, \frac{1}{|\,A_R\,|} \int_{A_R} s(x)\,\mathrm{d}x \right) := \epsilon_R \ge R.$$

Since $|\,a+b\,|^p \le 2^p(\,|\,a\,|^p + |\,b\,|^p)$ for all $a, b \in I\!R$, we have, because of Jensen's inequality, that

$$\begin{aligned} &\int_\Omega \Big|\, s(x) - s_R(x) \Big|^p \,\mathrm{d}x \\ &\quad \le \ 2^p \left(\int_{A_R} s(x)^p \,\mathrm{d}x + \int_{A_R} \left[\frac{1}{|\,A_R\,|} \int_{A_R} s(y)^p \,\mathrm{d}y \right] \mathrm{d}x \right) \\ &\quad = \ 2^{p+1} \int_{A_R} s(x)^p \,\mathrm{d}x. \end{aligned}$$

And, because $s \in L^p(\Omega)$ and $\lim_{R\uparrow\infty} |\,A_R\,| = 0$, we conclude that $(s_R)_{R\uparrow\infty}$ converges to s in $L^p(\Omega)$. Furthermore, by Jensen's inequality, we have that

$$\begin{aligned} \int_\Omega F(s_R(x))\,\mathrm{d}x \ &\le \ \int_{\Omega\setminus A_R} F(s(x))\,\mathrm{d}x + \int_{A_R} \left[\frac{1}{|\,A_R\,|} \int_{A_R} F(s(y))\,\mathrm{d}y \right] \mathrm{d}x \\ &= \ \int_\Omega F(s(x))\,\mathrm{d}x. \end{aligned}$$

This completes the proof of the proposition □

1.4.2 Approximation of a probability density by probability densities that are bounded below and above

Proposition 1.4.2 *Let $s \in \mathcal{P}_a(\Omega)$ be such that $0 \le s \le R$, for some $R > 0$. Let $F : [0,\infty) \to I\!R$ be convex, and let $1 \le p \le \infty$. Then, there exists a sequence $(s_\delta)_{\delta\downarrow 0}$ in $\mathcal{P}_a(\Omega) \cap L^\infty(\Omega)$ satisfying the following properties:*

(i). $\eta_\delta \le s_\delta \le R$, *for some* $0 < \eta_\delta \le \delta$.

(ii). $(s_\delta)_{\delta\downarrow 0}$ *converges to* s *in* $L^p(\Omega)$.

(iii). $\int_\Omega F(s_\delta(x))\,dx \le \int_\Omega F(s(x))\,dx$.

`Proof.` The proof is analogue to that of Proposition 1.4.1. Indeed, without loss of generality, we assume, as before, that s is positive a.e. For $\delta > 0$, we define $B_\delta := \{x \in \Omega : \; s(x) < \delta\}$, and

$$s_\delta(x) := \begin{cases} s(x) & \text{if} \quad x \in \Omega \setminus B_\delta \\ \\ \frac{1}{|B_\delta|}\int_{B_\delta} s(x)\,\mathrm{d}x & \text{if} \quad x \in B_\delta. \end{cases}$$

We have that, $s_\delta \in \mathcal{P}_a(\Omega)$, and

$$s_\delta(x) \ge \min\left(\delta, \frac{1}{|B_\delta|}\int_{B_\delta} s(x)\,\mathrm{d}x\right) := \eta_\delta \in (0,\delta].$$

Moreover, $s_\delta \le R$, because $s \le R$. Hence, (i) holds. Next, we observe that $\lim_{\delta\downarrow 0} |B_\delta| = 0$ (because $s > 0$ a.e.), and then, we follow the lines of the proof of Proposition 1.4.1 to conclude (ii) and (iii) □

Combining Proposition 1.4.1 and Proposition 1.4.2, we deduce the following corollary:

Corollary 1.4.3 *Let $s \in \mathcal{P}_a(\Omega) \cap L^p(\Omega)$, $1 \le p \le \infty$, and $F : [0,\infty) \to I\!R$ be convex. Assume that $\int_\Omega F(s(x))\,dx < \infty$. Then, there exists a sequence $(s_R)_{R\uparrow\infty}$ in $\mathcal{P}_a(\Omega)$, such that*

(i). $\eta_R \le s_R \le \epsilon_R$, *for some* $0 < \eta_R \le \frac{1}{R}$, *and* $\epsilon_R \ge R$.

(ii). $(s_R)_{R\uparrow\infty}$ *converges to* s, *in* $L^p(\Omega)$, *and*

(iii). $\int_\Omega F(s_R(x))\,dx \le \int_\Omega F(s(x))\,dx$.

Chapter 2

Calculus of Variations on $\mathcal{P}_a(\Omega)$

In this chapter, we discretize (1.5), and prove in section 2.1 that the problem

$$(P): \quad \inf\Big\{I(s) := W_c^h(s_0, s) + \frac{1}{h}E(s): \quad s \in \mathcal{P}_a(\Omega)\Big\}, \tag{2.1}$$

admits a minimizer s_1; here

$$E(s) = \int_\Omega [\, F(s(x)) + s(x)V(x)\,]\ \mathrm{d}x,$$

for $s \in \mathcal{P}_a(\Omega)$. The reason why we minimize such a functional will be clear in section 2.3, where we find the Euler-Lagrange equation of (P). In fact, we shall see that the Euler-Lagrange equation is nothing but the discretization of (1.5). In section 2.4, we show that

$$E(s_0) - E(s_1) \geq \frac{d\,E(s_{1-t})}{dt}\Big|_{t=0}, \tag{2.2}$$

where s_{1-t} denotes the probability density obtained by interpolating s_0 and s_1 along "geodesics" joining them in $\mathcal{P}_a(\Omega)$ [14]. We refer to (2.2) as the energy inequality. We shall see later on, that (2.2) is an essential ingredient in the proof of the convergence of the approximate sequence $(s^h)_h$ (See the definition in section 2.5) to solutions of problem (1.5).

2.1 Existence of solution to a minimization problem (P)

Proposition 2.1.1 *Let $F : [0, \infty) \to I\!R$ be strictly convex and satisfy (HF1). Let $h > 0$, and $s_0 \in \mathcal{P}_a(\Omega)$ be such that $E(s_0) < \infty$. Let*

$V : \bar{\Omega} \to [0,\infty)$ be convex, and assume that $c : \mathbb{R}^d \to [0,\infty)$ satisfies (HC1) - (HC2). Then (P) has a unique minimizer s_1. Moreover

$$|\,\Omega\,|\, F\left(\frac{1}{|\,\Omega\,|}\right) \leq E(s_1) \leq E(s_0). \tag{2.3}$$

`Proof.` Let I_{inf} denote the infimum of $I(s)$ over $s \in \mathcal{P}_a(\Omega)$. Since $s_0 \in \mathcal{P}_a(\Omega)$, $E(s_0) < \infty$ and $c(0) = 0$, we have that

$$\textbf{(i)} \qquad\qquad I_{inf} \leq \tfrac{1}{h} E(s_0).$$

Moreover, Jensen's inequality gives that

$$\frac{1}{|\,\Omega\,|}\int_\Omega F(s(x))\,\mathrm{d}x \;\geq\; F\left(\frac{1}{|\,\Omega\,|}\int_\Omega s(x)\,\mathrm{d}x\right) \;=\; F\left(\frac{1}{|\,\Omega\,|}\right), \tag{2.4}$$

for $s \in \mathcal{P}_a(\Omega)$, and then, we use that c and V are non-negative, to deduce that

$$\textbf{(ii)} \qquad\qquad I_{inf} \geq \tfrac{|\,\Omega\,|}{h} F\left(\tfrac{1}{|\,\Omega\,|}\right).$$

We combine (i) and (ii), to conclude that I_{inf} is finite. Now, let $\left(s^{(n)}\right)_n$ be a minimizing sequence for (P), that is, $\lim_{n\to\infty} I(s^{(n)}) = I_{inf}$. It is clear that $\left(I(s^{(n)})\right)_n$ is bounded in $\mathbb{R}$. As a consequence,

$$\sup_n \left[\int_\Omega F(s^{(n)})(x)\,\mathrm{d}x\right] < \infty.$$

The above inequality, together with (HF1) imply that $\left(s^{(n)}\right)_n$ converges weakly to a function s_1 in $L^1(\Omega)$ (up to a subsequence). Clearly $s_1 \in \mathcal{P}_a(\Omega)$. Furthermore, because of Proposition 5.3.1 and the continuity of V, we have that $\mathcal{P}_a(\Omega) \ni s \mapsto I(s)$ is weakly lower semi-continuous on $L^1(\Omega)$, as the sum of weakly lower semi-continuous functions. Therefore

$$I(s_1) \leq \liminf_{n\to\infty} I(s^{(n)}) = I_{inf} \leq I(s_1).$$

This proves that s_1 is a minimizer of (P). The uniqueness of s_1 follows from the convexity of $\mathcal{P}_a(\Omega) \ni s \mapsto W_c^h(s_0, s)$ and $\mathcal{P}_a(\Omega) \ni s \mapsto \int_\Omega s(x)V(x)\,\mathrm{d}x$, and the strict-convexity of $\mathcal{P}_a(\Omega) \ni s \mapsto \int_\Omega F(s(x))\,\mathrm{d}x$ (See Proposition 5.3.1).

Next, we have that $I(s_1) \leq I(s_0)$, and since $W_c^h(s_0, s_0) = 0$ and $W_c^h(s_0, s_1) \geq 0$ (because of (HC2)), we deduce that $E(s_1) \leq E(s_0)$. Finally, we use (2.4) and the fact that $s_1, V \geq 0$ to conclude (2.3). This finishes the proof of the proposition. □

2.2 Maximum and Minimum principles

In this section, we state three propositions. In the first proposition, we show that, if $s_0 \in \mathcal{P}_a(\Omega)$ is bounded below and above, then the probability density s_1, minimizing the variational problem (P) (2.1), is bounded below and above, as well. In the second proposition, we prove that the interpolant probability densities $s_{1-t} = (S_t)_{\#} s_1,\ t \in [0,1]$, between two probability densities $s_0,\ s_1 \in \mathcal{P}_a(\Omega)$, which are bounded above, are also bounded above; here $S_t = (1-t)\,\mathrm{id} + tS$, and S is the c-optimal map, that pushes s_1 forward to s_0. The third proposition will be used in the next chapter, to prove the strong convergence of the approximate sequence $(s^h)_{h\downarrow 0}$ in $L^1\left((0,T)\times\Omega\right)$, for $0 < T < \infty$.

Proposition 2.2.1 *(Maximum/minimum principle)*
Let $h > 0$, and $s_0 \in \mathcal{P}_a(\Omega)$ be such that $N \le s_0 \le M$. Let $F : [0,\infty) \to I\!R$ be strictly convex, and satisfy $F \in C^2\left((0,\infty)\right)$ and (HF1). Let $V : \bar{\Omega} \to [0,\infty)$ be convex, of class C^1, and assume that $c : I\!R^d \to [0,\infty)$ satisfies (HC1) - (HC2). Denote by s_1 the minimizer for (P) (2.1).

- *If $\nabla V = 0$, then $N \le s_1 \le M$ a.e.*
- *If $\nabla V \neq 0$, and in addition to the assumptions, c satisfies $c(z) \ge \beta |\, z\,|^q$ for some $q > 1$ and $\beta > 0$, then $N \le s_1 \le M$ a.e.*

`Proof.` Let $R > 2M$, and set $\mathcal{P}_a^{(R)}(\Omega) := \{ s \in \mathcal{P}_a(\Omega) : \ s \le R \text{ a.e.} \}$. Clearly, $\mathcal{P}_a^{(R)}(\Omega)$ is precompact for the weak topology in $L^1(\Omega)$. So, as in Proposition 2.1.1, I admits a unique minimizer s_{1R} over $\mathcal{P}_a^{(R)}(\Omega)$, and

$$\int_\Omega F(s_{1R}) \le E(s_0). \tag{2.5}$$

Claim 1 If $N \le s_0 \le M$, then $N \le s_{1R} \le M$ a.e.

`Proof.` Since the proof of "$s_{1R} \ge N$ a.e." is analogue to that of "$s_{1R} \le M$ a.e.", we only prove that $s_{1R} \le M$ a.e. Suppose by contradiction that $E := \{ y \in \Omega : \ s_{1R}(y) > M \}$ has a positive Lebesgue measure. The idea is to come up with $s_{1R}^{(\epsilon)} \in \mathcal{P}_a^{(R)}(\Omega)$, such that $I(s_{1R}) > I(s_{1R}^{(\epsilon)})$. This will contradict that s_{1R} is the minimizer of I over $\mathcal{P}_a^{(R)}(\Omega)$.
Let γ_R be the c_h-optimal measure in $\Gamma(s_0, s_{1R})$. We have that

$$\textbf{(i)} \qquad \gamma_R(E^c \times E) > 0,$$

where $E^c := I\!R^d \setminus E$; otherwise

$$\begin{aligned} M|\,E\,| < \int_E s_{1R}(y)\,\mathrm{d}y &= \gamma_R(I\!R^d \times E) = \gamma_R(E \times E) \le \gamma_R(E \times I\!R^d) \\ &= \int_E s_0(x)\,\mathrm{d}x \le M\,|\,E\,|, \end{aligned}$$

which yields a contradiction. Consider the measure $\nu := \gamma_R \, \mathbb{I}_{E^c \times E}$ defined by

$$\int_{R^d \times R^d} \xi(x,y)\,\mathrm{d}\nu(x,y) = \int_{E^c \times E} \xi(x,y)\,\mathrm{d}\gamma_R(x,y),$$

for $\xi \in C_0(\mathbb{R}^d \times \mathbb{R}^d)$, or equivalently

$$\nu(F) = \gamma_R\,[\,F \cap (E^c \times E)\,]\,,$$

for Borel sets $F \subset \mathbb{R}^d \times \mathbb{R}^d$. Denote by ν_0 and ν_1 its marginals, that is,

$$\int_{R^d \times R^d} [\,\varphi(x) + \psi(y)\,]\;\mathrm{d}\nu(x,y) = \int_{R^d} \varphi(x)\,\mathrm{d}\nu_0(x) + \int_{R^d} \psi(y)\,\mathrm{d}\nu_1(y),$$

for $\varphi, \psi \in C_0(\mathbb{R}^d)$. Since $\nu \le \gamma_R$ and $\gamma_R \in \Gamma(s_0, s_{1R})$, we have that $\nu_0 \le s_0(x)\,\mathrm{d}x$ and $\nu_1 \le s_{1R}(y)\,\mathrm{d}y$. As a consequence,

(ii) ν_0 and ν_1 are absolutely continuous with respect to Lebesgue.

Let v_0 and v_1 denote their respective density functions. Clearly

(iii) $0 \le v_0 \le M$ a.e., and $0 \le v_1 \le R$ a.e.

Moreover,

$$\begin{aligned} \nu_0(E) &= \gamma_R\left[(E \cap E^c) \times (\mathbb{R}^d \cap E)\right] = 0 \\ &= \gamma_R\left[(\mathbb{R}^d \cap E^c) \times (E \cap E^c)\right] = \nu_1(E^c). \end{aligned} \tag{2.6}$$

Hence

(iv) $v_0 = 0$ a.e. on E, and $v_1 = 0$ a.e. on E^c.

Now, let $\epsilon \in (0,1)$, and define $s_{1R}^{(\epsilon)} := s_{1R} + \epsilon(v_0 - v_1)$, and the probability measure $\gamma_R^{(\epsilon)}$ by

$$\begin{aligned} \int_{R^d \times R^d} \xi(x,y)\,\mathrm{d}\gamma_R^{(\epsilon)}(x,y) &= \int_{R^d \times R^d} \xi(x,y)\,\mathrm{d}\gamma_R(x,y) \\ &\quad + \epsilon \int_{E^c \times E} [\xi(x,x) - \xi(x,y)]\;\mathrm{d}\gamma_R(x,y), \end{aligned}$$

for $\xi \in C_0(\mathbb{R}^d \times \mathbb{R}^d)$. Because of (ii), (iv) and the fact that $2M < R$, we have that $0 \le s_{1R}^{(\epsilon)} \le R$, and

$$\int_\Omega s_{1R}^{(\epsilon)}(y)\,\mathrm{d}y = 1 + \epsilon\,[\gamma_R(E^c \times E) - \gamma_R(E^c \times E)] = 1.$$

Hence, $s_{1R}^{(\epsilon)} \in \mathcal{P}_a^{(R)}(\Omega)$. Moreover, since $\gamma_R \in \Gamma(s_0, s_{1R})$, and ν has marginals $v_0(x)\,\mathrm{d}x$ and $v_1(y)\mathrm{d}y$, we have that

$$\int_{R^d\times R^d} \varphi(x)\, d\gamma_R^{(\epsilon)}(x,y) = \int_{R^d} \varphi(x) s_0(x)\,\mathrm{d}x,$$

and

$$\begin{aligned}\int_{R^d\times R^d} \varphi(y)\, d\gamma_R^{(\epsilon)}(x,y) &= \int_\Omega \varphi(y) s_{1R}(y)\,\mathrm{d}y \\ &\quad +\epsilon\left[\int_{R^d} \varphi(x) v_0(x)\,\mathrm{d}x - \int_{R^d}\varphi(y)v_1(y)\,\mathrm{d}y\right] \\ &= \int_\Omega \varphi(y) s_{1R}^{(\epsilon)}(y)\,\mathrm{d}y.\end{aligned}$$

Hence $\gamma_R^{(\epsilon)} \in \Gamma(s_0, s_{1R}^{(\epsilon)})$. Next, we show that $I(s_{1R}^{(\epsilon)}) < I(s_{1R})$, for ϵ small enough. Indeed,

$$\begin{aligned}I(s_{1R}^{(\epsilon)}) - I(s_{1R}) &= \left[W_c^h(s_0, s_{1R}^{(\epsilon)}) - W_c^h(s_0, s_{1R})\right] + \frac{1}{h}\int_\Omega \left[F(s_{1R}^{(\epsilon)}) - F(s_{1R})\right] \\ &\quad + \frac{1}{h}\int_\Omega \left(s_{1R}^{(\epsilon)} - s_{1R}\right) V. \end{aligned} \tag{2.7}$$

Because $\gamma_R^{(\epsilon)} \in \Gamma(s_0, s_{1R}^{(\epsilon)})$ and $c(0) = 0$, we have that

$$\begin{aligned}W_c^h(s_0, s_{1R}^{(\epsilon)}) - W_c^h(s_0, s_{1R}) &\leq \int_{R^d\times R^d} c\left(\frac{x-y}{h}\right) \mathrm{d}\gamma_R^{(\epsilon)}(x,y) \\ &\quad - \int_{R^d\times R^d} c\left(\frac{x-y}{h}\right) \mathrm{d}\gamma_R(x,y) \\ &= -\epsilon \int_{E^c\times E} c\left(\frac{x-y}{h}\right) \mathrm{d}\gamma_R(x,y).\end{aligned} \tag{2.8}$$

On the other hand, according to (iv), we have, for ϵ small enough, that

$$s_{1R}^{(\epsilon)} = s_{1R} - \epsilon v_1 \geq M - \epsilon v_1 > 0 \quad \text{on} \quad E, \tag{2.9}$$

and

$$s_{1R}^{(\epsilon)} = s_{1R} + \epsilon v_0 \geq \epsilon v_0 > 0 \quad \text{on} \quad E^c \cap [\, v_0 > 0\,]. \tag{2.10}$$

We combine (iii), (iv), (2.9), (2.10), and the fact that $F \in C^1((0,\infty))$ is convex, and $\nu = \gamma_R \mathbb{I}_{E^c\times E}$ has marginals $v_0(x)\,\mathrm{d}x$ and $v_1(y)\,\mathrm{d}y$, to obtain that

$$\begin{aligned}&\int_\Omega \left[F(s_{1R}^{(\epsilon)}) - F(s_{1R})\right] \\ &\quad = \int_{E^c} [\,F(s_{1R} + \epsilon v_0) - F(s_{1R})\,] + \int_E [\,F(s_{1R} - \epsilon v_1) - F(s_{1R})\,]\end{aligned}$$

$$
\begin{aligned}
&\leq \epsilon \left[\int_{E^c \cap [v_0 > 0]} F'(s_{1R} + \epsilon v_0) v_0 - \int_E F'(s_{1R} - \epsilon v_1) v_1 \right] \\
&\leq \epsilon \left[\int_{E^c} F'(M + \epsilon v_0) v_0 - \int_E F'(M - \epsilon v_1) v_1 \right] \\
&= \epsilon \left[\int_{E^c \times E} \big(F'(M + \epsilon v_0(x)) - F'(M - \epsilon v_1(y)) \big) \, \mathrm{d}\gamma_R(x,y) \right].
\end{aligned}
$$

And since $F \in C^2((0,\infty))$, (iii) and the above estimate give that

$$
\int_\Omega \Big[F(s_{1R}^{(\epsilon)} - F(s_{1R}) \Big] = 0(\epsilon^2). \tag{2.11}
$$

Furthermore, because of (ii), (iv), and the fact that $V \in C^1(\Omega)$ is convex, we have that

$$
\begin{aligned}
\int_\Omega \left(s_{1R}^{(\epsilon)} - s_{1R} \right) V &= \epsilon \int_\Omega [\, v_0(x) - v_1(x) \,] \, V(x) \, \mathrm{d}x \\
&= \epsilon \int_{E^c \times E} [\, V(x) - V(y) \,] \, \mathrm{d}\gamma_R(x,y) \\
&\leq \epsilon \int_{E^c \times E} \langle \nabla V(x), x - y \rangle \, \mathrm{d}\gamma_R(x,y).
\end{aligned} \tag{2.12}
$$

Combining (2.7) - (2.8) and (2.11) - (2.12), we conclude that

$$
\begin{aligned}
I(s_{1R}^{(\epsilon)}) - I(s_{1R}) \leq & -\epsilon \int_{E^c \times E} c\left(\frac{x-y}{h} \right) \mathrm{d}\gamma_R(x,y) \\
& + \frac{\epsilon}{h} \int_{E^c \times E} \langle \, \nabla V(x), x - y \, \rangle \, \mathrm{d}\gamma_R(x,y).
\end{aligned}
$$

If $\nabla V = 0$, we deduce, because of (HC2) and (i), that

$$
I\left(s_{1R}^{(\epsilon)} \right) - I(s_{1R}) \leq -\epsilon \int_{E^c \times E} c\left(\frac{x-y}{h} \right) \mathrm{d}\gamma_R(x,y) < 0.
$$

If $\nabla V \neq 0$, using that $c(z) \geq \beta \, |z|^q$, and $V \in C^1(\overline{\Omega})$, we have that

$$
\frac{\epsilon}{h} \int_{E^c \times E} \langle \, \nabla V(x), x - y \, \rangle \, \mathrm{d}\gamma_R(x,y) = 0\left(\frac{1}{h} \right),
$$

and

$$
-\epsilon \int_{E^c \times E} c\left(\frac{x-y}{h} \right) \mathrm{d}\gamma_R(x,y) = 0\left(\frac{1}{h^q} \right).
$$

We deduce that

$$
I\left(s_{1R}^{(\epsilon)} \right) - I(s_{1R}) \leq -\epsilon \int_{E^c \times E} c\left(\frac{x-y}{h} \right) \mathrm{d}\gamma_R(x,y) < 0.
$$

This completes the proof of Claim 1.

Claim 2 $(s_{1R})_{R\uparrow\infty}$ converges weakly to s_1, in $L^1(\Omega)$. Therefore, $N \le s_1 \le M$ a.e.

`Proof.` According to (2.5), we have that

$$\sup_{R>2M} \int_\Omega F(s_{1R}) \le E(s_0).$$

And since $E(s_0) < \infty$ and F satisfies (HF1), we deduce that $(s_{1R})_{R\uparrow\infty}$ converges weakly to a function $\bar{s}_1$ in $L^1(\Omega)$, for a subsequence. Clearly, $\bar{s}_1 \in \mathcal{P}_a(\Omega)$, and $N \le \bar{s}_1 \le M$ a.e., because $N \le s_{1R} \le M$ a.e. for $R > 2M$. To complete the proof, we show that $\bar{s}_1 = s_1$. Let $\left(s_1^{(R)}\right)_{R\uparrow\infty}$ be a sequence in $\mathcal{P}_a^{(R)}(\Omega)$, converging to s_1 in $L^1(\Omega)$, and such that

$$\int_\Omega F\left(s_1^{(R)}\right) \le \int_\Omega F(s_1), \tag{2.13}$$

as in Proposition 1.4.1. Since s_{1R} is the minimizer of I over $\mathcal{P}_a^{(R)}$, we have that

$$I(s_{1R}) \le I\left(s_1^{(R)}\right), \tag{2.14}$$

and since $\left(s_1^{(R)}\right)_R$ converges to s_1 in $L^1(\Omega)$, Proposition 5.3.2 gives that

$$\lim_{R\uparrow\infty} W_c^h\left(s_0, s_1^{(R)}\right) = W_c^h(s_0, s_1). \tag{2.15}$$

We let R go to ∞ in (2.14), and we use (2.13), (2.15), and the fact that $V \in C^1(\bar{\Omega})$, $\left(s_1^{(R)}\right)_R$ converges strongly to s_1 in $L^1(\Omega)$, and $s \mapsto I(s)$ is weakly lower-continuous on $L^1(\Omega)$, to deduce that $I(\bar{s}_1) \le I(s_1)$. And, since s_1 is the unique minimizer of (P), we conclude that $\bar{s}_1 = s_1$. This completes the proofs of Claim 2 and Proposition 2.2.1 $\square$

We state a lemma needed to establish the second proposition of this section. Let $(c_k)_k$ denote a sequence of strictly convex cost functions satisfying

$$\begin{cases} c_k,\, c_k^\star \in C^2(\mathbb{R}^d), \\ c_k \to c \ \text{ locally in } \ C^1(\mathbb{R}^d), \\ 0 = c_k(0) < c_k(z) \ \text{ for } \ z \neq 0 \end{cases} \tag{2.16}$$

where $c : \mathbb{R}^d \to [0,\infty)$ is convex, of class $C^1(\mathbb{R}^d)$, and satisfies $c(0) = 0$, and (HC4) (see Proposition 1.3.1).

Lemma 2.2.2 *Let $s_0,\, s_1 \in \mathcal{P}_a(\Omega)$, and let $c : I\!R^d \to [0,\infty)$ be strictly convex, of class $C^1(I\!R^d)$, and satisfy $c(0) = 0$ and (HC4). Denote by γ_k (respectively γ_∞) the c_k (respectively c) - optimal measure in $\Gamma(s_0, s_1)$. Then, there exists a subsequence $(\gamma_{kj})_j$ of $(\gamma_k)_k$, which converges weakly $\star$ to γ_∞. Therefore, $(S_{kj})_j$ converges to S, in $L^2_{s_1}(\Omega, I\!R^d)$; here, S_{kj} and S denote respectively the c_{kj} and c - optimal maps that push s_1 forward to s_0, and $L^2_{s_1}(\Omega, I\!R^d)$ is the set of functions $\Omega \to I\!R^d$, whose square are summable with respect to the measure $\mu_1 := s_1(y)\, dy$.*

`Proof.` We set $W_{c_k} := W^h_{c_k}$ and $W_c := W^h_c$ when $h = 1$. Because $\mathrm{spt}\,(s_0), \mathrm{spt}\,(s_1) \subset \bar\Omega$, we have that

$$\mathrm{spt}\,(\gamma) \subset \bar\Omega \times \bar\Omega, \quad \forall\, \gamma \in \Gamma(s_0, s_1).$$

Then, $(\gamma_k)_k$ is a tight sequence of probability measures on $I\!R^d \times I\!R^d$. Therefore, there is a subsequence $(\gamma_{kj})_j$ of $(\gamma_k)_k$, which converges weakly $\star$ to a probability measure $\tilde\gamma$. Clearly, $\tilde\gamma \in \Gamma(s_0, s_1)$. Now, we prove that $\tilde\gamma = \gamma_\infty$. Let $\gamma \in \Gamma(s_0, s_1)$. Because $\mathrm{spt}\,(\gamma) \subset \bar\Omega \times \bar\Omega$, $\gamma\left[I\!R^d \times I\!R^d\right] = 1$, and $(c_{kj})_j$ converges to c, uniformly on compact sets, we have that

$$\lim_{j\to\infty} \int_{R^d \times R^d} c_{kj}(x-y)\,\mathrm{d}\gamma(x,y) = \int_{R^d\times R^d} c(x-y)\,\mathrm{d}\gamma(x,y). \tag{2.17}$$

And since γ is arbitrarily chosen in (2.17), we deduce that

$$\lim_{j\to\infty} W_{c_{kj}}(s_0, s_1) = W_c(s_0, s_1). \tag{2.18}$$

On the other hand, because $\mathrm{spt}\,(\gamma_{kj}) \subset \bar\Omega \times \bar\Omega$, we have that

$$\begin{aligned} W_{c_{kj}}(s_0, s_1) \;&=\; \int_{\bar\Omega\times\bar\Omega} \left[c_{kj}(x-y) - c(x-y)\right]\,\mathrm{d}\gamma_{kj}(x,y) \\ &\quad + \int_{\bar\Omega\times\bar\Omega} c(x-y)\,\mathrm{d}\gamma_{kj}(x,y). \end{aligned} \tag{2.19}$$

We let j go to ∞ in (2.19), and we use that $(\gamma_{kj})_j$ converges weakly $\star$ to $\tilde\gamma$, $\gamma_{kj}(I\!R^d \times I\!R^d) = 1$, $(c_{kj})_j$ converges to c, uniformly on compact sets, and $\mathrm{spt}\,(\tilde\gamma) \subset \bar\Omega \times \bar\Omega$, to deduce that

$$\lim_{j\to\infty} W_{c_{kj}}(s_0, s_1) = \int_{\bar\Omega\times\bar\Omega} c(x-y)\,\mathrm{d}\tilde\gamma(x,y) = \int_{R^d\times R^d} c(x-y)\,\mathrm{d}\tilde\gamma(x,y). \tag{2.20}$$

We combine (2.18), (2.20), and we use the uniqueness of γ_∞ in $W_c(s_0, s_1)$, to conclude that $\tilde\gamma = \gamma_\infty$.

Now, set $F_1(x,y) = |\,x\,|^2$ and $F_2(x,y) = \langle x, Sy\rangle$, $\forall\,(x,y) \in \bar\Omega\times\bar\Omega$. Because $Sy = y + \nabla c^\star\left(-\nabla v(y)\right)$, and v is Lipschitz, there exists a sequence $\left(v^{(n)}\right)_n$ in $C^1(\bar\Omega)$, such that $F_2^{(n)}(x,y) := \langle x, S^{(n)}y\rangle$ converges to $F_2(x,y)$ for a.e.

$y \in \bar{\Omega}$; here $S^{(n)}(y) := y + \nabla c^\star\left(-\nabla v^{(n)}(y)\right)$. Clearly, $F_2^{(n)} \in C(\bar{\Omega}\times\bar{\Omega})$ and $\{F_2^{(n)},\, F_2\}_n$ are uniformly bounded. We use the dominated convergence theorem, and the fact that $\gamma_{k_j}(\bar{\Omega}\times\bar{\Omega}) = 1$, to obtain that

$$
\begin{aligned}
&\int_{\bar{\Omega}\times\bar{\Omega}} F_2(x,y)\,\mathrm{d}\gamma_{kj}(x,y)\\
&\qquad = \lim_{n\to\infty}\int_{\bar{\Omega}\times\bar{\Omega}} \left[\,F_2(x,y) - F_2^{(n)}(x,y)\,\right]\,\mathrm{d}\gamma_{kj}(x,y)\\
&\qquad\quad + \lim_{n\to\infty}\int_{\bar{\Omega}\times\bar{\Omega}} F_2^{(n)}(x,y)\,\mathrm{d}\gamma_{kj}(x,y)\\
&\qquad = \lim_{n\to\infty}\int_{\bar{\Omega}\times\bar{\Omega}} F_2^{(n)}(x,y)\,\mathrm{d}\gamma_{kj}(x,y). \qquad (2.21)
\end{aligned}
$$

We let j go to ∞ in (2.21), and we use that $F_2^{(n)} \in C(\bar{\Omega}\times\bar{\Omega})$, $(\gamma_{kj})_j$ converges weakly $\star$ to γ_∞, and the dominated convergence theorem, to deduce that

$$
\begin{aligned}
\lim_{j\to\infty}\int_{\bar{\Omega}\times\bar{\Omega}} F_2(x,y)\,\mathrm{d}\gamma_{kj}(x,y) &= \lim_{n\to\infty}\lim_{j\to\infty}\int_{\bar{\Omega}\times\bar{\Omega}} F_2^{(n)}(x,y)\,\mathrm{d}\gamma_{kj}(x,y)\\
&= \int_{\bar{\Omega}\times\bar{\Omega}} F_2(x,y)\,\mathrm{d}\gamma_\infty(x,y). \qquad (2.22)
\end{aligned}
$$

Furthermore, since $\gamma_{kj} = (\mathrm{id}\times S_{kj})_{\#}s_1$ and $\gamma_\infty = (\mathrm{id}\times S)_{\#}s_1$, we have that

$$
\begin{aligned}
&\int_\Omega |\,S_{kj}y - Sy\,|^2 s_1(y)\,\mathrm{d}y\\
&\qquad = \int_\Omega \left[\,F_1(S_{kj}y,y) + F_1(Sy,y) - 2F_2(S_{kj}y,y)\,\right] s_1(y)\,\mathrm{d}y\\
&\qquad = \int_{\bar{\Omega}\times\bar{\Omega}} F_1(x,y)\,\mathrm{d}\gamma_{kj}(x,y) + \int_{\bar{\Omega}\times\bar{\Omega}} F_1(x,y)\,\mathrm{d}\gamma_\infty(x,y)\\
&\qquad\quad -2\int_{\bar{\Omega}\times\bar{\Omega}} F_2(x,y)\,\mathrm{d}\gamma_{kj}(x,y).
\end{aligned}
$$

We let j go to ∞ in the above equality, and we use (2.22), and the fact that $(\gamma_{kj})_j$ converges weakly $\star$ to γ_∞, to conclude that

$$
\begin{aligned}
\lim_{j\to\infty}\int_\Omega |\,S_{kj}y - Sy\,|^2 s_1(y)\,\mathrm{d}y &= 2\int_{\bar{\Omega}\times\bar{\Omega}} \left[\,F_1(x,y) - F_2(x,y)\,\right]\,\mathrm{d}\gamma_\infty(x,y)\\
&= 2\int_\Omega \left(|\,Sy\,|^2 - |\,Sy\,|^2\right) s_1(y)\,\mathrm{d}y = 0.
\end{aligned}
$$

This completes the proof of Lemma 2.2.2 □

Proposition 2.2.3 *Let $s_0,\, s_1 \in \mathcal{P}_a(\Omega)$ be such that $s_0,\, s_1 \leq M$ a.e. Let $c : I\!R^d \to [0,\infty)$ be strictly convex, of class C^1, and satisfy $c(0) = 0$ and (HC4). Let S be the c-optimal map that pushes s_1 forward to s_0, and define the interpolant map*

$$S_t := (1-t)\, id + tS,$$

for $t \in [0,1]$. Then,

$$\int_\Omega \xi\,(S_t(y))\; s_1(y)\, dy \;\leq\; M \int_{R^d} \xi\,(x)\, dx, \tag{2.23}$$

for $\xi \in C_c^0(I\!R^d),\ \xi \geq 0.$

`Proof.` The proof will be done in two steps. In step 1, we prove (2.23) for sufficiently regular cost functions. The key ideas in this proof is the fact that ∇S is diagonalizable with positive eigenvalues (see (ii)), and $A \mapsto (\det)^{1/d}$ is concave on the set of $d \times d$ diagonalizable matrices with positive eigenvalues (see 2.26). In step 2, we approximate a general cost function c, by regular cost functions c_k, and we obtain (2.23) in the limit as k goes to ∞.

Step 1. *Case where c is strictly convex, and $c,\, c^\star \in C^2(I\!R^d)$.*
Theorem 5.2.2 gives that $\mu_{1-t} := (S_t)_\# s_1$ is absolutely continuous with respect to Lebesgue, for all $t \in [0,1]$. Let s_{1-t} denote the density function of μ_{1-t}. Then, (2.23) reads as

$$\int_\Omega \xi\,(x)\, s_{1-t}\,(x)\,\mathrm{d}x \;\leq\; M \int_{R^d} \xi\,(x)\,\mathrm{d}x.$$

Thus, it suffices to show that

$$s_{1-t} \leq M.$$

Because of Theorem 5.2.1, Theorem 5.2.2 and Corollary 5.2.3, there exists a set $K \subset \Omega$ of full measure for $\mu_1 := s_1(y)\,\mathrm{d}y,$ such that

(i) S_t is injective on K,

and for $y \in K$,

(ii) $\nabla S(y)$ is diagonalizable with positive eigenvalues,

and

$$0 \neq s_1(y) = s_{1-t}\,(S_t(y))\, \det\,[\,\nabla S_t(y)\,]\,,\ \ \forall\, t \in [0,1], \tag{2.24}$$

where

$$\nabla S_t(y) = (1-t)\mathrm{id} + t\nabla S(y).$$

Moreover, since $s_0,\, s_1 \le M$ a.e., and $S_\# s_1 = s_0,$ then K can be chosen so that

$$s_1(y),\ s_0\left(S(y)\right) \le M, \quad \forall\, y \in K.$$

We use $t = 1$ in (2.24), and the fact that $s_0\left(S(y)\right) \le M,$ to deduce that

$$\det\left[\nabla S(y)\right] \ge \frac{s_1(y)}{M}. \tag{2.25}$$

Because $A \mapsto (\det A)^{1/d}$ is concave on the set of $d \times d$ diagonalizable matrices with positives eigenvalues, we have that

$$\left[\det \nabla S_t(y)\right]^{1/d} \ge (1-t) + t\left(\det\left[\nabla S(y)\right]\right)^{1/d}. \tag{2.26}$$

We use (2.25), (2.26), and the fact that $s_1(y) \le M,$ to obtain that

$$\det\left[\nabla S_t(y)\right] \ge \frac{s_1(y)}{M}. \tag{2.27}$$

Combining (2.24), (2.27) and (i), we conclude that

$$s_{1-t} \le M, \quad \text{on } S_t(K). \tag{2.28}$$

But, since $\mu_1(K^c) = 0,$ and $\mu_{1-t} = (S_t)_\# \mu_1,$ we have that $\mu_{1-t}\left[\left(S_t(K)\right)^c\right] = 0,$ and then

$$s_{1-t} = 0 \quad \text{on } \left[S_t(K)\right]^c. \tag{2.29}$$

We combine (2.28) and (2.29), to conclude that $s_{1-t} \le M.$

Step 2. *c satisfies the assumptions of the Proposition.*
Let $(c_k)_k$ be a sequence of strictly convex cost functions satisfying (2.16), and $(S_{kj})_j$ be as in Lemma 2.2.2. Set

$$S_{k_j}^{(t)} := (1-t)\,\mathrm{id} + t S_{k_j}, \quad \forall\, t \in [0,1],\ j \in \mathbb{N}.$$

Because of Lemma 2.2.2 and Step 1, we have that

(iii) $\left(S_{k_j}^{(t)}\right)_j$ converges to $S_t,$ a.e., on $[s_1 \ne 0],$ a.e.,

for a subsequence, and

$$\int_\Omega \xi\left(S_{k_j}^{(t)}(y)\right) s_1(y)\,\mathrm{d}y \le M \int_{R^d} \xi(x)\,\mathrm{d}x. \tag{2.30}$$

We let j go to ∞ in (2.30), and we use (iii), $0 \le \xi \in C_c^0(\mathbb{R}^d),$ and Fatou's lemma, to deduce (2.23) □

Next, we state a proposition needed in the next chapter, to prove the strong convergence of the approximate sequence $(s^h)_{h\downarrow 0}$ in $L^1\left((0,T)\times\Omega\right),$ for $0 < T < \infty.$

Proposition 2.2.4 *Let X be a bounded open subset of $\mathbb{R}^d$, $d > 1$. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be strictly convex, of class $C^1(\mathbb{R})$, and assume that $\lim_{t\to\infty} \frac{g(t)}{t} = \infty$. Given M, $\delta > 0$, and p, $q > 1$, define*

$$A_{M,\delta} := \Big\{ (u_1, u_2) \in L^q(X)^2 : \; \| u_j \|_{L^q(X)} \le M, \; \| g'(u_j) \|_{W^{1,p}(X)} \le M, \\ \int_X [\, f'(u_2) - f'(u_1)\,]\,[\, u_2 - u_1\,] \le \delta, \; (j = 1, 2) \Big\},$$

and set

$$\Lambda_M(\delta) := \sup_{(u_1,u_2)\in A_{M,\delta}} \| u_2 - u_1 \|_{L^1(X)}.$$

Then

$$\lim_{\delta\downarrow 0} \Lambda_M(\delta) = 0.$$

`Proof:` Suppose by contradiction that there exist $\kappa > 0$ and $(u_j^\delta)_{\delta\downarrow 0}$ $(j = 1, 2)$, such that $(u_1^\delta, u_2^\delta) \in A_{M,\delta}$, and

$$\| u_2^\delta - u_1^\delta \|_{L^1(X)} > \kappa. \tag{2.31}$$

By the Sobolev embedding theorem, $\Big(g'(u_j^\delta)\Big)_\delta$ converges strongly in $L^p(X)$, and then, a.e., for a subsequence (not relabeled). Since $g \in C^1(\mathbb{R})$ is strictly convex and has a super-linear growth at ∞, we have that $(g')^{-1}$ is continuous. We deduce that

(i) $(u_j^\delta)_\delta$ converges to some function u_j, a.e., for $j = 1, 2$.

We use (i), $\| u_j^\delta \|_{L^q(X)} \le M$, and the fact that $q > 1$ to conclude that

(ii) $(u_j^\delta)_\delta$ converges strongly to u_j in $L^1(X)$.

We combine (ii) and (2.31) to obtain that

$$\| u_2 - u_1 \|_{L^1(X)} > \kappa. \tag{2.32}$$

Now, we use (i), the convexity of f, and the fact that

$$\int_X \Big[f'(u_2^\delta) - f'(u_1^\delta) \Big] \Big[u_2^\delta - u_1^\delta \Big] \le \delta,$$

to have that

$$\begin{aligned} 0 &\le \int_X \big[f'(u_2) - f'(u_1)\big]\,[u_2 - u_1] \\ &\le \liminf_{\delta\downarrow 0} \int_X \Big[f'(u_2^\delta) - f'(u_1^\delta)\Big]\,[u_{2,\delta} - u_{1,\delta}] \le 0. \end{aligned}$$

This implies that

$$\big[f'(u_2) - f'(u_1)\big]\,[u_2 - u_1] = 0 \;\text{ a.e.} \tag{2.33}$$

By (2.33), we either have

$$u_2(x) = u_1(x) \text{ for } x \in X \text{ a.e.},$$

or

$$f'(u_2(x)) = f'(u_1(x)) \text{ for } x \in X \text{ a.e.} \tag{2.34}$$

But, because $f \in C^1(I\!R)$ is strictly convex, we have that f' is one-to-one, and then, (2.34) implies that

$$u_1(x) = u_2(x) \text{ for } x \in X \text{ a.e.}$$

This yields a contradiction to (2.32) □

2.3 Properties of the minimizer for (P)

In this section, we establish the Euler-Lagrange equation for the variational problem (P) studied in section 2.1, and we derive some properties of the minimizer for this problem. We recall that

$$(P): \quad \inf\left\{W_c^h(s_0, s) + \frac{1}{h}E(s): \quad s \in \mathcal{P}_a(\Omega)\right\},$$

where

$$E(s) := \int_\Omega [\, F(s) + sV\,],$$

and $s_0 \in \mathcal{P}_a(\Omega)$. The next proposition is the first step towards showing that (P) is a discretization of (1.5), or in other words, (1.5) is the steepest descent of the energy functional E with respect to W_c^h.

Proposition 2.3.1 *Let $s_0 \in \mathcal{P}_a(\Omega)$ be such that $N \le s_0 \le M$. Let $F : [0,\infty) \to I\!R$ be strictly convex, and satisfy $F \in C^2((0,\infty))$, and (HF1). Let $V : \bar\Omega \to [0,\infty)$ be convex, and of class C^1. Let $c : I\!R^d \to [0,\infty)$ be strictly convex, of class C^1, and satisfy $c(0) = 0$, (HC3), and $c(z) \ge \beta|z|^q$, for some $q > 1$ and $\beta > 0$. If s_1 denotes the minimizer for (P), then the followings hold:*

$$\int_{\Omega\times\Omega} \langle \nabla c\left(\frac{x-y}{h}\right), \psi(y)\,\rangle\, d\gamma(x,y) + \int_\Omega P(s_1(y))\, div\, \psi(y)\, dy \\ - \int_\Omega s_1(y) \langle\, \nabla V(y), \psi(y)\,\rangle\, dy = 0 \tag{2.35}$$

for $\psi \in C_c^\infty(\Omega, I\!R^d)$. Here $P(x) := P_F(x) := x\,F'(x) - F(x)$ for $x \in (0,\infty)$, and γ is the c_h-optimal measure in $\Gamma(s_0, s_1)$. Moreover,

(i). $P(s_1) \in W^{1,\infty}(\Omega)$.

(ii). If S is the c_h-optimal map that pushes s_1 forward to s_0, then

$$\frac{Sy - y}{h} = \nabla c^\star \left[\nabla(F'(s_1(y)) + V(y))\right], \tag{2.36}$$

for a.e. $y \in \Omega$, and

$$\Big| \int_\Omega \frac{s_1(y) - s_0(y)}{h} \varphi(y)\, dy + \int_\Omega s_1(y) \langle \nabla c^\star \left[\nabla \left(F'(s_1(y)) + V(y) \right) \right], \nabla\varphi(y) \rangle \, dy \Big| \leq \frac{1}{2h} \sup_{x \in \bar\Omega} \Big| D^2\varphi(x) \Big| \int_{\Omega\times\Omega} |x - y|^2 \, d\gamma(x,y), \tag{2.37}$$

for $\varphi \in C^2(\bar\Omega)$.

`Proof.` Since $c \in C^1(\mathbb{R}^d)$ is strictly convex and has a super-linear growth as $|x| \to \infty$, we have that $c^\star \in C^1(\mathbb{R}^d)$, and $(\nabla c)^{-1} = \nabla c^\star$. We consider the flow map $(\phi_\epsilon)_{\epsilon \in R}$ in $C^\infty(\Omega, \Omega)$, defined by

$$\begin{cases} \frac{\partial \phi_\epsilon}{\partial \epsilon} = \psi \circ \phi_\epsilon \\ \phi_0 = \text{id}. \end{cases} \tag{2.38}$$

where $\psi \in C_c^\infty(\Omega, \mathbb{R}^d)$. We have that $det\,(\nabla\phi_\epsilon) \neq 0$, and

$$\frac{\partial (det\, \nabla\phi_\epsilon)}{\partial\epsilon}\Big|_{\epsilon=0} = \operatorname{div}\psi. \tag{2.39}$$

We define on Ω, the probability measure $\mu_\epsilon := (\phi_\epsilon)_\# s_1$. Since ϕ_ϵ is a C^1-diffeomorphism, then μ_ϵ is absolutely continuous with respect to Lebesgue. Let s_ϵ denote its density function. Clearly, $s_\epsilon \in \mathcal{P}_a(\Omega)$, and

$$(s_\epsilon \circ \phi_\epsilon)\, det\,(\nabla\phi_\epsilon) = s_1 \quad \text{a.e.} \tag{2.40}$$

Next, we define on $\Omega \times \Omega$, the probability measure $\gamma_\epsilon := (id \times \phi_\epsilon)_\# \gamma$, i.e.

$$\int_{\Omega\times\Omega} \xi(x,y)\, \mathrm{d}\gamma_\epsilon(x,y) = \int_{\Omega\times\Omega} \xi\left[(x, \phi_\epsilon(y))\right] \mathrm{d}\gamma(x,y), \qquad \forall\, \xi \in C(\Omega\times\Omega).$$

We have that $\gamma_\epsilon \in \Gamma(s_0, s_\epsilon)$, and then, the mean-value theorem gives that

$$\begin{aligned} &\frac{W_c^h(s_0, s_\epsilon) - W_c^h(s_0, s_1)}{\epsilon} \\ &\leq \int \frac{1}{\epsilon} \left[c_h(x - \phi_\epsilon(y)) - c_h(x - y) \right] \mathrm{d}\gamma(x,y) \\ &= -\int \langle \nabla c_h \left[x - y + \theta(y - \phi_\epsilon(y)) \right], \frac{\phi_\epsilon - \phi_0}{\epsilon}(y) \rangle \, \mathrm{d}\gamma(x,y), \end{aligned}$$

where $\theta \in [0,1]$. Because of (2.38), we have that $|\frac{\phi_\epsilon - \phi_0}{\epsilon}| \leq \| \psi \|_{L^\infty}$, for $\epsilon > 0$. Then, we use that $c \in C^1(\mathbb{R}^d)$, the Lebesgue dominated convergence theorem, and (2.38), to obtain that

$$\limsup_{\epsilon \downarrow 0} \frac{W_c^h(s_0, s_\epsilon) - W_c^h(s_0, s_1)}{\epsilon} \leq -\int \langle \nabla c_h(x-y), \psi(y) \rangle \, \mathrm{d}\gamma(x,y). \quad (2.41)$$

On the other hand, because of (2.40), we have that

$$\begin{aligned} \int_\Omega F\left(s_\epsilon(x)\right) \mathrm{d}x &= \int_\Omega F(s_\epsilon \circ \phi_\epsilon(y)) \det \nabla \phi_\epsilon(y) \, \mathrm{d}y \\ &= \int_\Omega F\left(\frac{s_1(y)}{\det \nabla \phi_\epsilon(y)}\right) \det \nabla \phi_\epsilon(y) \, \mathrm{d}y. \end{aligned}$$

And since, $F \in C^1\left((0,\infty)\right)$, we deduce by the mean-value theorem that

$$\begin{aligned} &\int_\Omega \frac{F(s_\epsilon(x)) - F(s_1(x))}{\epsilon} \mathrm{d}x \\ &= \frac{1}{\epsilon} \int_\Omega \left[\left(F\left(\frac{s_1}{det \nabla \phi_\epsilon}\right) - F(s_1) \right) det \nabla \phi_\epsilon + F(s_1)(det \nabla \phi_\epsilon - 1) \right] \\ &= \int_\Omega \left[-F'\left(s_1 + \theta\left(\frac{s_1}{det \nabla \phi_\epsilon} - s_1\right)\right) s_1 \frac{det \nabla \phi_\epsilon - 1}{\epsilon} \right] \\ &\quad + \int_\Omega \left[F(s_1) \frac{det \nabla \phi_\epsilon - 1}{\epsilon} \right], \end{aligned} \quad (2.42)$$

where $\theta \in [0,1]$. We combine (2.38), (2.39) and (2.42) to conclude that

$$\lim_{\epsilon \downarrow 0} \int_\Omega \frac{F(s_\epsilon(y)) - F(s_1(y))}{\epsilon} \mathrm{d}y = -\int_\Omega P(s_1(y)) \, div \, \psi(y) \, \mathrm{d}y. \quad (2.43)$$

Furthermore, because of (2.40), we have that

$$\begin{aligned} \int_\Omega s_\epsilon(x) V(x) \, \mathrm{d}x &= \int_\Omega [\, s_\epsilon \circ \phi_\epsilon(y) \,] \, V\left(\phi_\epsilon(y)\right) \det \nabla \phi_\epsilon(y) \, \mathrm{d}y \\ &= \int_\Omega s_1(y) V\left(\phi_\epsilon(y)\right) \mathrm{d}y. \end{aligned}$$

And since $V \in C^1(\Omega)$, we deduce by the mean-value theorem, that

$$\begin{aligned} &\int_\Omega \frac{s_\epsilon(x) - s_1(x)}{\epsilon} V(x) \, \mathrm{d}x \\ &= \frac{1}{\epsilon} \int_\Omega \left[V\left(\phi_\epsilon(y)\right) - V(y) \right] s_1(y) \, \mathrm{d}y \\ &= \int_\Omega s_1(y) \left\langle \nabla V\left(y + \theta(\phi_\epsilon(y) - y)\right), \frac{\phi_\epsilon - \phi_0}{\epsilon}(y) \right\rangle \mathrm{d}y, \end{aligned}$$

where $\theta \in [0,1]$. We let ϵ go to 0 in both sides of the above equality, and we use (2.38) and the Lebesgue dominated convergence theorem, to conclude that

$$\lim_{\epsilon\downarrow 0}\int_\Omega \frac{s_\epsilon(x)-s_1(x)}{\epsilon}V(x)\,\mathrm{d}x = \int_\Omega s_1(y)\,\langle\,\nabla V(y),\psi(y)\,\rangle > \mathrm{d}y. \tag{2.44}$$

We combine (2.41), (2.43) and (2.44) to obtain that

$$\begin{aligned}&\int_{\Omega\times\Omega}\langle\,\nabla c_h(x-y),\psi(y)\,\rangle\,\mathrm{d}\gamma(x,y) \\ &\quad+\frac{1}{h}\left[\int_\Omega P(s_1(y))\,\mathrm{div}\,\psi(y)\,\mathrm{d}y - \int_\Omega s_1(y)\langle\,\nabla V(y),\psi(y)\,\rangle \mathrm{d}y\right]\le 0.\end{aligned} \tag{2.45}$$

Since $\nabla c_h(z)=\frac{1}{h}\nabla c\left(\frac{z}{h}\right)$, and ψ is arbitrarily chosen in $C_c^\infty(\Omega,\mathbb{R}^d)$, then (2.45) implies (2.35).

(i). Proposition 2.2.1 gives that $N\le s_1\le M$ a.e., and since $F\in C^1((0,\infty))$, we have that $P(s_1)\in L^\infty(\Omega)$.

Now, let $\varphi\in C_c^\infty(\Omega)$, and for an arbitrary $i\in\mathbb{N}$, define $\psi=(\psi_j)_{j=1,\cdots,d}\in C_c^\infty(\Omega,\mathbb{R}^d)$ by $\psi_j:=\delta_{ij}\,\varphi$, where δ_{ij} denotes the Kronecker symbol. Because of (2.35), we have that

$$\begin{aligned}&\Big|\int_\Omega P(s_1(y))\frac{\partial\varphi}{\partial z_i}(y)\Big| \\ &\quad=\Big|\int_{\Omega\times\Omega}\frac{\partial c}{\partial z_i}\left(\frac{x-y}{h}\right)\varphi(y)\,\mathrm{d}\gamma(x,y)-\int_\Omega s_1(y)\frac{\partial V(y)}{\partial z_i}\varphi(y)\,\mathrm{d}y\Big| \\ &\quad\le\left[\sup_{x,y\in\Omega}\Big|\frac{\partial c}{\partial z_i}\left(\frac{x-y}{h}\right)\Big|+\|\,\nabla V\,\|_{L^\infty(\Omega)}\right]\int_\Omega|\,\varphi(y)\,|s_1(y)\,\mathrm{d}y \\ &\quad\le M\left[\sup_{x,y\in\Omega}\Big|\frac{\partial c}{\partial z_i}\left(\frac{x-y}{h}\right)\Big|+\|\,\nabla V\,\|_{L^\infty(\Omega)}\right]\|\,\varphi\,\|_{L^1(\Omega)}.\end{aligned}$$

And since $V\in C^1(\bar\Omega)$ and $c\in C^1(\mathbb{R}^d)$, we deduce (i).

(ii). Because $P(s_1)\in W^{1,\infty}(\Omega)$, we can integrate by parts in (2.35). We use that $\gamma\in\Gamma(s_0,s_1)$ and $S_\# s_1=s_0$, to obtain that

$$\begin{aligned}&\int_\Omega\langle\,\nabla c\left(\frac{Sy-y}{h}\right),\psi(y)\,\rangle\,s_1(y)\,\mathrm{d}y \\ &\quad=\int_\Omega\langle\,\nabla\left[\,P(s_1(y))\,\right]+s_1(y)\nabla V(y),\psi(y)\,\rangle\,\mathrm{d}y \\ &\quad=\int_\Omega s_1(y)\,\langle\,\nabla\left[F'(s_1(y))+V(y)\right],\psi(y)\,\rangle\,\mathrm{d}y,\end{aligned}$$

for $\psi\in C_c^\infty(\Omega,\mathbb{R}^d)$. And since ψ is arbitrarily chosen, we deduce that

$$\nabla c\left(\frac{Sy-y}{h}\right)s_1(y)=\nabla\left[\,F'(s_1(y))+V(y)\,\right]\,s_1(y), \tag{2.46}$$

for a.e. $y \in \Omega$. We combine (2.46), and the fact that $(\nabla c)^{-1} = \nabla c^\star$ and $s_1 \neq 0$ a.e., to conclude (2.36).
Next, we consider $\varphi \in C^2(\bar\Omega)$, we take the scalar product of both sides of (2.36) with $s_1(y)\nabla\varphi(y)$, and we use that $\gamma = (\mathrm{id} \times S)_\# s_1$, to obtain that

$$\frac{1}{h}\int_{\Omega\times\Omega} \langle\, y - x, \nabla\varphi(y)\,\rangle \, d\gamma(x,y)$$
$$= -\int_\Omega \langle\, \nabla c^\star \left[\, \nabla \left(F'\left(s_1(y)\right) + V(y)\right)\right], \nabla\,\varphi(y)\,\rangle\, s_1(y)\,\mathrm{d}y. \quad (2.47)$$

Now, we express $\frac{1}{h}\int_{\Omega\times\Omega}\langle\, y-x, \nabla\varphi(y)\,\rangle\, d\gamma(x,y)$ in terms of $\int_\Omega \frac{s_1(y)-s_0(y)}{h}\,\varphi(y)\,\mathrm{d}y$. Since $\gamma \in \Gamma(s_0, s_1)$, we have that

$$\int_\Omega \frac{s_1(y)-s_0(y)}{h}\,\varphi(y)\,\mathrm{d}y = \frac{1}{h}\int_{\Omega\times\Omega} \left[\,\varphi(y)-\varphi(x)\,\right]\,\mathrm{d}\gamma(x,y).$$

Combining the above equality with the first order Taylor expansion of φ around y, we obtain that

$$\Big|\, \frac{1}{h}\int_{\Omega\times\Omega}\langle\, y-x, \nabla\varphi(y)\,\rangle d\gamma(x,y) - \frac{1}{h}\int_\Omega \left(s_1(y)-s_0(y)\right)\varphi(y)\,\Big|$$
$$\leq \frac{1}{2h}\sup_{x\in\bar\Omega}|\, D^2\varphi(x)\,| \int_{\Omega\times\Omega} |\, x-y\,|^2\,\mathrm{d}\gamma(x,y). \quad (2.48)$$

We substitute (2.47) into (2.48) to conclude (2.37). This completes the proof of the proposition. □

2.4 Energy inequality

In this section, we establish an inequality relating the energy $E(s_0)$ and $E(s_1)$ of two probability density functions s_0 and s_1. This inequality will be called *energy inequality*, and will be used later on, to improve compactness properties of the approximate sequence s^h (see the definition in Section 2.5), to solutions of problem (1.5). First, we prove this inequality for smooth cost functions c, whose Legendre transform $c^\star$ are C^2. Instead of using the density function F, we consider a more general function G, which satisfies some assumptions to be specified later on. The (internal) energy inequality reads as

$$\int_\Omega G(s_0(y))\,\mathrm{d}y - \int_\Omega G(s_1(y))\,\mathrm{d}y \geq -\int_\Omega P_G(s_1(y))\,\mathrm{div}\,(Sy - y)\,\mathrm{d}y, \quad (2.49)$$

where S is the c-optimal map pushing s_1 forward to s_0, and $P_G(x) := xG'(x) - G(x)$. For smooth cost functions c, this inequality is simply a consequence of the displacement convexity of $\mathcal{P}_a(\Omega) \ni s \mapsto \int_\Omega G(s(x))\,\mathrm{d}x$, that is, the convexity of $[0,1] \ni t \mapsto \int_\Omega G(s_{1-t}(x))\,\mathrm{d}x$; here s_{1-t} is the probability density obtained by interpolating s_0 and s_1, along the "geodesics" joining them in $\mathcal{P}_a(\Omega)$ (see Theorem 5.2.2 - (iii)). To prove (2.49), we rather follow a more direct procedure, using the following result of Cordero and Otto (Theorem 5.2.1): *if $s_0,\, s_1 \in \mathcal{P}_a(\Omega)$, $c,\, c^\star \in C^2(\mathbb{R}^d)$, and S is the c-optimal map that pushes s_1 forward to s_0, then $\nabla S(y)$ is diagonalizable with positive eigenvalues for $\mu_1 := s_1(y)\,dy$ - a.e. $y \in \Omega$. Moreover, the pointwise Jacobian, $\det \nabla S$, satisfies*

$$0 \neq s_1(y) = \det \nabla S(y)\, s_0(Sy), \tag{2.50}$$

for μ_1 - a.e. $y \in \Omega$.

Proposition 2.4.1 *(Energy inequality for regular cost functions)*
Let $s_0,\, s_1 \in \mathcal{P}_a(\Omega)$ be density functions of two Borel probability measures μ_0 and μ_1 on $\mathbb{R}^d$, respectively. Let $\bar{c} : \mathbb{R}^d \to [0,\infty)$ be strictly convex, such that $\bar{c},\, \bar{c}^\star \in C^2(\mathbb{R}^d)$. Let $G : [0,\infty) \to \mathbb{R}$ be differentiable on $(0,\infty)$, such that $G(0) = 0$, and $(0,\infty) \ni x \mapsto x^d G(x^{-d})$ be convex and non-increasing. Let $V : \bar{\Omega} \to [0,\infty)$ be convex, of class $C^1(\Omega)$, and denote by S the $\bar{c}$-optimal map, such that $S_\# s_1 = s_0$. Then, the internal energy inequality (2.49), and the following potential energy inequality hold:

$$\int_\Omega s_0(y)V(y)\,dy - \int_\Omega s_1(y)V(y)\,dy \geq \int_\Omega \langle \nabla V(y), Sy - y \rangle s_1(y)\,dy. \tag{2.51}$$

In addition, if $P_G(s_1) \in W^{1,\infty}(\Omega)$, and $s_1 > 0$ a.e., then

$$\int_\Omega G(s_0(y))\,dy - \int_\Omega G(s_1(y))\,dy \geq \int_\Omega \langle \nabla[\, G'\,(s_1(y))\,], Sy - y\rangle s_1(y)\,dy. \tag{2.52}$$

`Proof.` Set

$$A(x) := x^d G(x^{-d}), \quad \forall\, x \in (0,\infty).$$

We observe that

$$A'(x) = -dx^{d-1} P_G(x^{-d}). \tag{2.53}$$

Since A is non-increasing, we have that $P_G \geq 0$, and then

(i) $\qquad (0,\infty) \ni x \mapsto \frac{G(x)}{x}$ is non-decreasing.

Theorem 5.2.1 gives that $\nabla S(y)$ is diagonalizable with positive eigenvalues, and that (2.50) holds for μ_1 - a.e. $y \in \Omega$. So, $s_0(Sy) \neq 0$ for μ_1 - a.e. $y \in \Omega$. We use that $G(0) = 0$, $S_\# s_1 = s_0$, and (2.50), to deduce that

$$\begin{aligned} \int_\Omega G(s_0(x))\,\mathrm{d}x &= \int_{[\,s_0 \neq 0\,]} G\left(s_0(x)\right)\,\mathrm{d}x \\ &= \int_\Omega \frac{G(s_0(Sy))}{s_0(Sy)} s_1(y)\,\mathrm{d}y \\ &= \int_\Omega G\left(\frac{s_1(y)}{\det \nabla S(y)}\right) \det \nabla S(y)\,\mathrm{d}y. \end{aligned} \tag{2.54}$$

Comparing the geometric mean $(\det \nabla S(y))^{1/d}$ to the arithmetic mean $\frac{\operatorname{tr}\nabla S(y)}{d}$, we have that

$$\frac{s_1(y)}{\det \nabla S(y)} \geq s_1(y)\left(\frac{d}{\operatorname{tr}\nabla S(y)}\right)^d.$$

Then, we deduce from (i) and the above inequality, that

$$G\left(\frac{s_1(y)}{det\,\nabla S(y)}\right) det\,\nabla S(y) \geq \Lambda^d\, G\left(\frac{s_1(y)}{\Lambda^d}\right) = s_1(y) A\left(\frac{\Lambda}{s_1(y)^{1/d}}\right) \quad (2.55)$$

where

$$\Lambda := \frac{tr\,\nabla S(y)}{d}.$$

Now, we use (2.53) and the convexity of A, to obtain that

$$\begin{aligned} s_1(y) A\left(\frac{\Lambda}{s_1(y)^{1/d}}\right) &\geq s_1(y)\left[A\left(\frac{1}{s_1(y)^{1/d}}\right) + A'\left(\frac{1}{s_1(y)^{1/d}}\right)\left(\frac{\Lambda - 1}{s_1(y)^{1/d}}\right)\right] \\ &= s_1(y)\left[\frac{G_1\left(s_1(y)\right)}{s_1(y)} - d(\Lambda - 1)\frac{P_G\left(s_1(y)\right)}{s_1(y)}\right] \\ &= G_1\left(s_1(y)\right) - P_G\left(s_1(y)\right)\, tr\left(\nabla S(y) - \mathrm{id}\right). \qquad (2.56) \end{aligned}$$

Combining (2.54) - (2.56), we conclude that

$$\begin{aligned} \int_\Omega G\left(s_0(y)\right) \mathrm{d}y - \int_\Omega G\left(s_1(y)\right) \mathrm{d}y &\geq -\int_\Omega P_G\left(s_1(y)\right)\, tr\left(\nabla S(y) - \mathrm{id}\right) \mathrm{d}y \\ &= -\int_\Omega P_G\left(s_1(y)\right) \operatorname{div}\left(Sy - y\right) \mathrm{d}y. \end{aligned}$$

This proves (2.49).
Now, because $S_\# s_1 = s_0$, we have that

$$\int_\Omega s_0(x) V(x)\,\mathrm{d}x - \int_\Omega s_1(y) V(y)\,\mathrm{d}y = \int_\Omega \left[V(Sy) - V(y)\right] s_1(y)\,\mathrm{d}y.$$

And since $V(Sy) - V(y) \geq \langle \nabla V(y), Sy - y\rangle$ (because $V \in C^1(\Omega)$ is convex), we deduce (2.51).
Next, assume that $P_G(s_1) \in W^{1,\infty}(\Omega)$ and $s_1 > 0$ a.e.. Because $P_G \geq 0$, we can approximate $P_G(s_1)$ by non-negative functions in $C_c^\infty(\mathbb{R}^d)$. We use Theorem 5.2.1 - (iii), to obtain that

$$\begin{aligned} -\int_\Omega P_G(s_1(y))\, div\,(Sy - y)\,\mathrm{d}y &\geq \int_\Omega \langle \nabla[P_G(s_1(y))], Sy - y\rangle\,\mathrm{d}y \qquad (2.57) \\ &= \int_\Omega \langle \nabla[G'(s_1(y))], Sy - y\rangle\, s_1(y)\,\mathrm{d}y. \end{aligned}$$

We combine (2.49) and (2.57) to conclude (2.52) □

The next theorem extends the energy inequalities (2.51) and (2.52) to general cost functions c.

Theorem 2.4.2 *(Energy inequality for general cost functions).*
Let $s_0,\, s_1 \in \mathcal{P}_a(\Omega)$ be such that $s_1 > 0$ a.e., and $c : \mathbb{R}^d \to [0,\infty)$ be strictly convex, of class C^1 and satisfy $c(0) = 0$ and (HC4). Let $G : [0,\infty) \to \mathbb{R}$ be differentiable on $(0,\infty)$, such that $G(0) = 0$, $(0,\infty) \ni x \mapsto x^d\, G(x^{-d})$ be convex and non-increasing, $\nabla\,(G'(s_1)) \in L^\infty(\Omega)$, and $P_G(s_1) \in W^{1,\infty}(\Omega)$. Let $V : \bar{\Omega} \to [0,\infty)$ be convex, of class C^1, and denote by S, the c - optimal map, such that $S_\# s_1 = s_0$. Then, the following energy inequalities hold

$$\int_\Omega G(s_0(y))\, dy - \int_\Omega G(s_1(y))\, dy \geq \int_\Omega \langle \nabla[\, G'\,(s_1(y))\,], Sy - y\rangle s_1(y)\, dy, \quad (2.58)$$

and

$$\int_\Omega s_0(y) V(y)\, dy - \int_\Omega s_1(y) V(y)\, dy \geq \int_\Omega \langle\, \nabla V(y), Sy - y\,\rangle s_1(y)\, dy. \quad (2.59)$$

Proof. Let $(c_k)_k$ be a sequence of regular cost functions satisfying (2.16). By Proposition 2.4.1, we have that

$$\int_\Omega G(s_0(y))\, \mathrm{d}y - \int_\Omega G(s_1(y))\, \mathrm{d}y \geq \int_\Omega \langle\, \nabla\left(G'\,(s_1(y))\right), S_{kj}(y) - y\,\rangle s_1(y)\, \mathrm{d}y, \quad (2.60)$$

and

$$\int_\Omega s_0(y) V(y)\, \mathrm{d}y - \int_\Omega s_1(y) V(y)\, \mathrm{d}y \geq \int_\Omega \langle\, \nabla V(y), S_{kj}(y) - y\,\rangle s_1(y)\, \mathrm{d}y, \quad (2.61)$$

for all $j \in \mathbb{N}$, where S_{kj} is defined as in Lemma 2.2.2. We let j go to ∞ in (2.60) and (2.61), and we use that $\nabla\,(G'(s_1)) \in L^\infty(\Omega)$, $\nabla V \in L^\infty(\Omega)$, and $(S_{kj})_j$ converges to S in $L^2_{s_1}(\Omega, \mathbb{R}^d)$ (see Lemma 2.2.2), to conclude (2.58) and (2.59) □

2.5 Approximate solutions to the parabolic equation

Throughout this section, s_0 denotes a probability density in $\mathcal{P}_a(\Omega)$ which is bounded below and above, that is, $s_0 + \frac{1}{s_0} \in L^\infty(\Omega)$. For fixed $h > 0$ and $i \in \mathbb{N}$, we denote by s_i^h the minimizer of

$$(P_i^h): \quad \inf \left\{ I(s) := W_c^h(s_{i-1}^h, s) + \frac{1}{h}\, E(s) : \quad s \in \mathcal{P}_a(\Omega) \right\},$$

where $s_0^h := s_0$ and

$$E(s) := \int_\Omega [\, F(s) + sV\,]$$

(see Proposition 2.1.1). We define the approximate solution s^h to (1.5), as

$$s^h(t,x) := \begin{cases} s_0(x) & \text{if} \quad t=0 \\ \\ s_i^h(x) & \text{if} \quad t \in (t_{i-1}, t_i], \end{cases} \tag{2.62}$$

where $t_i = ih,$ for all $i \in I\!N$. The next proposition shows that

$$\frac{\partial s^h}{\partial t} = \operatorname{div}\left\{ s^h \, \nabla c^\star \left[\nabla \left(F'(s^h) + V \right) \right] \right\} + \Lambda(h),$$

in the weak sense. We show in the next section, that

$$\| \Lambda(h) \|_{(W^{2,\infty}(\Omega))^\star} = 0\left(h^{\epsilon(q)} \right),$$

where $\epsilon(q) = \min(1, q-1)$.

Proposition 2.5.1 *Let $F : [0,\infty) \to I\!R$ be strictly convex and satisfy $F \in C^2((0,\infty))$ and (HF1). Let $V : \bar{\Omega} \to [0,\infty)$ be convex, of class C^1, and assume that $c : I\!R^d \to [0,\infty)$ is strictly convex, of class C^1, and satisfies $c(0) = 0$ and $c(z) \geq \beta\, |z|^q$, for some $q > 1$ and $\beta > 0$. Then*

$$\begin{aligned} \Big| \int_0^T \int_\Omega (s_0 - s^h)\, \partial_t^h \xi \; dx\; dt \\ + \int_0^T \int_\Omega \langle\, s^h \nabla c^\star \left[\nabla \left(F'(s^h) + V \right) \right], \nabla \xi \,\rangle \; dx\; dt \Big| \\ \leq \frac{1}{2} \sup_{[0,T]\times\bar{\Omega}} \left| D^2 \xi(t,x) \right| \sum_{i=1}^{T/h} \int_{\Omega\times\Omega} |x-y|^2 \, d\gamma_i^h(x,y), \end{aligned} \tag{2.63}$$

where, $\xi : I\!R \times \Omega \to I\!R$ is such that $\xi(t,.) \in C^2(\bar{\Omega})$ for $t \in I\!R$, and $spt \xi(.,x) \subset [-T,T]$ for $x \in \Omega$, and for some $T > 0$. Here,

$$\partial_t^h \xi(t,x) := \frac{\xi(t+h,x) - \xi(t,x)}{h},$$

and γ_i^h is the c_h-optimal measure in $\Gamma(s_{i-1}^h, s_i^h)$, for $i \in I\!N$.

`Proof.` Without loss of generality, we assume that $\frac{T}{h} \in I\!N$. Because of (2.37), we have that

$$\Big| \int_\Omega A_i^h(t,x) \, \mathrm{d}x \Big| \leq B_i^h,$$

for $t \in (0,T)$, where

$$\begin{aligned} A_i^h(t,x) \;:=\; & \frac{s_i^h(x) - s_{i-1}^h(x)}{h} \, \xi(t,x) \\ & + \left\langle s_i^h(x) \nabla c^\star \left[\nabla \left(F'\left(s_i^h(x) \right) + V(x) \right) \right], \nabla \xi(t,x) \right\rangle, \end{aligned}$$

and

$$B_i^h := \frac{1}{2h} \sup_{[0,T]\times\bar{\Omega}} \Big| D^2\xi(t,x) \Big| \int_{\Omega\times\Omega} |x-y|^2 \, d\gamma_i^h(x,y).$$

We integrate the above inequality over $t \in (0,T)$, to obtain that

$$\Big| \sum_{i=1}^{T/h} \int_{t_{i-1}}^{t_i} \mathrm{d}t \int_\Omega A_i^h(t,x)\,\mathrm{d}x \Big| \le h \sum_{i=1}^{T/h} B_i^h. \tag{2.64}$$

The right hand side of (2.64) gives that

$$h \sum_{i=1}^{T/h} B_i^h = \frac{1}{2} \sup_{[0,T]\times\bar{\Omega}} \Big| D^2\xi(t,x) \Big| \sum_{i=1}^{T/h} \int_{\Omega\times\Omega} |x-y|^2 d\gamma_i^h(x,y), \tag{2.65}$$

while, on the left hand side, we have that

$$\begin{aligned} &\sum_{i=1}^{T/h} \int_{t_{i-1}}^{t_i} \int_\Omega A_i^h(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \sum_{i=1}^{T/h} \int_{t_{i-1}}^{t_i} \int_\Omega \frac{s_i^h(x) - s_{i-1}^h(x)}{h}\, \xi(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_0^T \int_\Omega \Big\langle s^h \nabla c^\star \Big[\nabla \Big(F'(s^h) + V \Big) \Big], \nabla\xi \Big\rangle \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{2.66}$$

By a direct computation, the first term on the right hand side of (2.66) gives that

$$\begin{aligned} &\sum_{i=1}^{T/h} \int_{t_{i-1}}^{t_i} \int_\Omega \frac{s_i^h(x) - s_{i-1}^h(x)}{h}\, \xi(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \frac{1}{h} \int_0^T \int_\Omega s^h(t,x)\,\xi(t,x)\,\mathrm{d}x\,dt \\ &\qquad - \frac{1}{h} \sum_{i=2}^{T/h} \int_{t_{i-1}}^{t_i} \int_\Omega s^h(\tau - h, x)\,\xi(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau \\ &\qquad - \frac{1}{h} \int_0^h \int_\Omega s_0(x)\,\xi(t,x)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

We use the substitution $\tau = t + h$ in the above expression to obtain that

$$\begin{aligned} &\sum_{i=1}^{T/h} \int_{t_{i-1}}^{t_i} \int_\Omega \frac{s_i^h(x) - s_{i-1}^h(x)}{h}\, \xi(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \frac{1}{h} \int_0^T \int_\Omega s^h(t,x)\,\xi(t,x)\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{h} \int_0^{T-h} s^h(t,x)\,\xi(t+h,x)\,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{h}\int_0^h \int_\Omega s_0(x)\,\xi(t,x)\,\mathrm{d}x\,\mathrm{d}t \\
=&-\int_0^T \int_\Omega s^h(t,x)\,\partial_t^h\,\xi(t,x)\,\mathrm{d}x\,\mathrm{d}t \,+\, \frac{1}{h}\int_{T-h}^T s^h(t,x)\,\xi(t+h,x) \\
&-\frac{1}{h}\int_0^h \int_\Omega s_0(x)\,\xi(t,x)\,dt\,\mathrm{d}x.
\end{aligned}
$$

Noting that

$$
-\frac{1}{h}\int_0^h \int_\Omega s_0(x)\,\xi(t,x)\,\mathrm{d}t\,\mathrm{d}x = \int_0^T \int_\Omega s_0(x)\,\partial_t^h\xi(t,x)\,\mathrm{d}x\,\mathrm{d}t,
$$

and $\xi(t+h)=0$ for $t\in(T-h,T)$, we deduce that

$$
\begin{aligned}
&\sum_{i=1}^{T/h}\int_{t_{i-1}}^{t_i}\int_\Omega \frac{s_i - s_{i-1}^h}{h}(x)\,\xi(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_0^T \int_\Omega \Big(s_0(x) - s^h(t,x)\Big)\;\partial_t^h\xi(t,x)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{2.67}
$$

We combine (2.64) - (2.67) to conclude (2.63) □

Chapter 3

Existence of weak solutions

Below, we study the limit of (2.63), as h goes to 0. This chapter is divided into four sections. The first three sections deal with the limits of the three terms of inequality (2.63), and the last section proves the existence theorem to problem (1.5), when the probability density s_0 is bounded below and above. The energy inequalities (2.58) and (2.59) will be crucial in the proofs of these limits.
Throughout this chapter, s^h is defined as (2.62), and $\Omega_T := (0,T)\times\Omega$, for $0 < T \leq \infty$.

3.1 Second moments of the optimal measures

In this section, we show that, if $c(z) \geq \beta\,|\,z\,|^q$, for some $\beta > 0$ and $q > 1$, then

$$\sum_{i=1}^{T/h} \int_{\Omega\times\Omega} |x-y|^2 \, \mathrm{d}\gamma_i^h(x,y) = 0(h^{\epsilon(q)}), \tag{3.1}$$

where $\epsilon(q) = \min(1, q-1)$, γ_i^h denotes the c_h-optimal measure in $\Gamma(s_{i-1}^h, s_i^h)$, and s_i^h is the unique minimizer of

$$(P_i^h): \quad \inf\left\{W_c^h(s_{i-1}^h, s) + \frac{1}{h}E(s): \quad s \in \mathcal{P}_a(\Omega)\right\},$$

with $s_0^h := s_0$ and

$$E(s) := \int_\Omega [\,F(s) + sV\,].$$

The first step toward proving (3.1) is the next lemma, which states that $\sum_{i=1}^\infty W_c^h(s_{i-1}^h, s_i^h)$ is bounded, uniformly in h.

Lemma 3.1.1 *Let* $F : [0,\infty) \to I\!R$ *be convex and satisfy (HF1). Let* $s_0 \in \mathcal{P}_a(\Omega)$ *be such that* $E(s_0) < \infty$. *Let* $V : \bar{\Omega} \to [0,\infty)$ *be continuous on*

Ω, and assume that $c : \mathbb{R}^d \to [0,\infty)$ satisfies (HC1) - (HC2). Then

$$\sum_{i=1}^{\infty} h\, W_c^h(s_{i-1}^h, s_i^h) \le E(s_0) - |\,\Omega\,| F\left(\frac{1}{|\,\Omega\,|}\right). \tag{3.2}$$

`Proof.` Let $T > 0$, be such that $\frac{T}{h} \in \mathbb{N}$. Since $c(0) = 0$, Proposition 2.1.1 gives that

$$h\, W_c^h(s_{i-1}^h, s_i^h) \le E(s_{i-1}^h) - E(s_i^h),$$

for $i \in \mathbb{N}$. We sum both sides of the above inequality over i, and we use that V and s_i^h are non-negative, to obtain that

$$\sum_{i=1}^{T/h} h\, W_c^h(s_{i-1}^h, s_i^h) \le E(s_0) - \int_\Omega F(s_{T/h}^h(x))\, \mathrm{d}x.$$

We apply Jensen's inequality to the integral term above, and we let T go to ∞, to deduce (3.2) □

Proposition 3.1.2 *Assume that $F : [0,\infty) \to \mathbb{R}$ is convex and satisfies (HF1), and $s_0 \in \mathcal{P}_a(\Omega)$ is such that $E(s_0) < \infty$. Assume that $V : \bar{\Omega} \to [0,\infty)$ is continuous on Ω, and $c : \mathbb{R}^d \to [0,\infty)$ is strictly convex and satisfies $c(0) = 0$ and $c(z) \ge \beta\,|\,z\,|^q$, for some $\beta > 0$ and $q > 1$. Then, for $T > 0$ and $h \in (0,1)$, such that $\frac{T}{h} \in \mathbb{N}$,*

$$\sum_{i=1}^{T/h} \int_{\Omega\times\Omega} |\,x-y\,|^2\, d\gamma_i^h(x,y) \le M(\Omega, T, F, s_0, q, \beta)\, h^{\epsilon(q)}, \tag{3.3}$$

where $\epsilon(q) = \min(1, q-1)$.

`Proof.` Since $c(z) \ge \beta\,|z|^q$, we have that

$$\int_{\Omega\times\Omega} |\,x-y\,|^q\, \mathrm{d}\gamma_i^h(x,y) \;\le\; \frac{h^q}{\beta}\, W_c^h(s_{i-1}^h, s_i^h), \tag{3.4}$$

for $i \in \mathbb{N}$. We distinguish two cases, based on the values of q.
<u>*Case 1:*</u> $1 < q \le 2$.
Because of (3.4), we have, for $i \in \mathbb{N}$, that

$$\begin{aligned}\int_{\Omega\times\Omega} |\,x-y\,|^2\, \mathrm{d}\gamma_i^h(x,y) &\le \sup_{x,y\in\Omega} |\,x-y\,|^{(2-q)} \int_{\Omega\times\Omega} |\,x-y\,|^q\, \mathrm{d}\gamma_i^h(x,y) \\ &\le \frac{(\operatorname{diam}\Omega)^{(2-q)}}{\beta}\, h^q\, W_c^h(s_{i-1}^h, s_i^h),\end{aligned}$$

where, $\operatorname{diam}\Omega$ denotes the diameter of Ω. We sum both sides of the above inequality over i, and we use (3.2), to conclude that

$$\sum_{i=1}^{T/h} \int_{\Omega\times\Omega} |\,x-y\,|^2\, \mathrm{d}\gamma_i^h(x,y) \;\le M(\Omega, F, s_0, q, \beta)\, h^{q-1}.$$

Case 2: $q > 2$.
Because of Jensen's inequality and (3.4), we have that

$$\begin{aligned}\int_{\Omega\times\Omega} |\, x-y\,|^2 \, \mathrm{d}\gamma_i^h(x,y) &\le \left(\int_{\Omega\times\Omega} |\, x-y\,|^q \, \mathrm{d}\gamma_i^h(x,y)\right)^{2/q} \\ &\le \frac{h^2}{\beta^{2/q}} \left[W_c^h(s_{i-1}^h, s_i^h)\right]^{2/q}.\end{aligned}$$

We sum both sides of the above inequality over i, and we use Hölder's inequality on the right hand side term, to obtain that

$$\begin{aligned}\sum_{i=1}^{T/h} \int_{\Omega\times\Omega} |\, x-y\,|^2 \, \mathrm{d}\gamma_i^h(x,y) &\le \frac{h^2}{\beta^{2/q}} \left(\frac{T}{h}\right)^{1-\frac{2}{q}} \left[\sum_{i=1}^{T/h} W_c^h(s_{i-1}^h, s_i^h)\right]^{2/q} \\ &= T^{1-\frac{2}{q}} \frac{h}{\beta^{2/q}} \left[\sum_{i=1}^{T/h} h\, W_c^h(s_{i-1}^h, s_i^h)\right]^{2/q} \quad (3.5)\end{aligned}$$

We combine (3.2) and (3.5), to conclude that

$$\sum_{i=1}^{T/h} \int_{\Omega\times\Omega} |\, x-y\,|^2 \, \mathrm{d}\gamma_i^h(x,y) \le M(\Omega, T, F, s_0, q, \beta)\, h.$$

This completes the proof of the proposition □

In the following two sections, we assume that $V = 0$, and we denote the internal energy by

$$E_i(\rho) := \int_\Omega F\left(\rho(x)\right) \, \mathrm{d}x,$$

for $\rho \in \mathcal{P}_a(\Omega)$.

3.2 Strong convergence of the approximate solutions

In this section, we prove that $(s^h)_h$ is compact in $L^1\left(0,T\right) \times \Omega)$, for $0 < T < \infty$. The main ingredient in the proof is the energy inequality (2.58). It allows us to obtain a uniform bound in h, of the $L^{q^\star}$- norm of $\nabla\left(F'(s^h)\right)$, which leads to the compactness of $(s^h)_h$ in $L^1\left(0,T\right) \times \Omega)$.
We first show that $(s^h)_h$ converges weakly in $L^1\left((0,T) \times \Omega\right)$, for a subsequence, and for $0 < T < \infty$. We introduce the following constant needed in the next lemma:

$$\begin{aligned}&\bar{M}(\Omega, T, F, s_0, q, \alpha) \\ &\quad := M(\alpha, q) \left(E_i(s_0) - |\,\Omega\,| F\left(\frac{1}{|\,\Omega\,|}\right) + \alpha T |\,\Omega\,|\, \|\, s_0 \,\|_{L^\infty(\Omega)}\right),\end{aligned}$$

where $M(\alpha, q)$ is a constant which depends on α and q.

Lemma 3.2.1 *Assume that $c : I\!\!R^d \to [0, \infty)$ is strictly convex, of class C^1 and satisfies (HC2), and $F : [0, \infty) \to I\!\!R$ is strictly convex, of class $C^2((0, \infty))$, and satisfies (HF1). If $s_0 \in \mathcal{P}_a(\Omega) \cap L^\infty(\Omega)$, then,*

$$\| s^h \|_{L^\infty(R; L^\infty(\Omega))} \leq \| s_0 \|_{L^\infty(\Omega)}. \tag{3.6}$$

Therefore, there exists $s : I\!\!R \times \Omega \to I\!\!R$ and a subsequence of $(s^h)_{h\downarrow 0}$, which converges to s, weakly in $L^1((0, T) \times \Omega)$, for $0 < T < \infty$.
In addition, if c satisfies (HC4), F satisfies $F(0) = 0$ and (HF2), then

$$\int_{\Omega_T} s^h \Big| \nabla \Big(F'(s^h) \Big) \Big|^{q^\star} \leq \bar{M}(\Omega, T, F, s_0, q, \alpha). \tag{3.7}$$

`Proof.` Because of the maximum principle of Proposition 2.2.1, we have that

$$s_i^h \leq \| s_0 \|_{L^\infty(\Omega)}, \quad \forall\, i \in I\!\!N,$$

which reads as

$$\| s^h(t) \|_{L^\infty(\Omega)} \leq \| s_0 \|_{L^\infty(\Omega)}, \quad \forall\, t \in I\!\!R.$$

We take the supremum of the subsequent inequality over $t \in I\!\!R$, to deduce (3.6).
Due to (3.6), we have that $(s^h)_h$ is precompact in $L^1((0, T) \times \Omega)$, for $0 < T < \infty$. We use the standard diagonal argument, to conclude that $(s^h)_{h\downarrow 0}$ converges weakly to some function $s : (0, \infty) \times \Omega \to I\!\!R$ in $L^1((0, T) \times \Omega)$, for a subsequence.
Because of Proposition 2.3.1, the maximum/minimum principle of Proposition 2.2.1, and the fact that $\nabla \left(P(s_i^h) \right) = s_i^h \nabla \left(F'(s_i^h) \right)$, we have that $P(s_i^h) \in W^{1,\infty}(\Omega)$ and $\nabla \left(F'(s_i^h) \right) \in L^\infty(\Omega)$, for $i \in I\!\!N$. Then, we choose $G := F$ in the energy inequality (2.58), and we use (2.36), to obtain that

$$h \int_\Omega \Big\langle \nabla \Big(F'(s_i^h) \Big), \nabla c^\star \Big[\nabla \Big(F'(s_i^h) \Big) \Big] \Big\rangle s_i^h \leq \int_\Omega F(s_{i-1}^h) - \int_\Omega F(s_i^h),$$

for $i \in I\!\!N$. We sum both sides of the subsequent inequality over i, and we use Jensen's inequality, to deduce that

$$\int_{\Omega_T} \Big\langle \nabla \Big(F'(s^h) \Big), \nabla c^\star \Big[\nabla \Big(F'(s^h) \Big) \Big] \Big\rangle s^h \leq \int_\Omega F(s_0) - |\,\Omega\,| \, F\left(\frac{1}{|\,\Omega\,|} \right). \tag{3.8}$$

Because of (5.13), and the fact that $c(z) \leq \alpha\,(|\,z\,|^q + 1),$ we have that

$$\langle\, z, \nabla c^\star(z)\,\rangle \geq c^\star(z) \geq M(\alpha,q)\,|z\,|^{q^\star} \;-\; \alpha,$$

and then, (3.8) implies that

$$\begin{aligned} M(\alpha,q) \int_{\Omega_T} s^h \left| \nabla \left(F'(s^h) \right) \right|^{q^\star} \\ \leq \int_{\Omega} F(s_0) - |\,\Omega\,|\, F\left(\frac{1}{|\,\Omega\,|} \right) + \alpha \int_{\Omega_T} s^h. \end{aligned} \tag{3.9}$$

We combine (3.6) and (3.9), to obtain that

$$\begin{aligned} M(\alpha,q) \int_{\Omega_T} s^h \left| \nabla \left(F'(s^h) \right) \right|^{q^\star} \\ \leq \int_{\Omega} F(s_0) - |\,\Omega\,|\, F\left(\frac{1}{|\,\Omega\,|} \right) + \alpha T |\,\Omega\,|\, \|\, s_0\, \|_{L^\infty(\Omega)}. \end{aligned}$$

We divide both sides of the above inequality by $M(\alpha,q),$ to conclude (3.7)□

Next, we establish the space compactness of $(s^h)_h$ on $(0,T) \times \Omega,$ for $0 < T < \infty.$

Lemma 3.2.2 *(Space-compactness)*
Assume that $c : \mathbb{R}^d \to [0,\infty)$ *is strictly convex, of class* C^1 *and satisfies* $c(0) = 0$ *and (HC4), and* $F : [0,\infty) \to \mathbb{R}$ *is strictly convex, of class* $C^2\left((0,\infty)\right),$ *and satisfies* $F(0) = 0$ *and (HF1) - (HF2). If* $s_0 \in \mathcal{P}_a(\Omega)$ *is such that* $s_0 + \frac{1}{s_0} \in L^\infty(\Omega),$ *then, for all* $\eta \neq 0$ *and* $0 < T < \infty,$

$$\int_0^T \int_{\Omega^{(\eta)}} \left| s^h(t, x+\eta e) - s^h(t,x) \right| \leq M(\Omega, T, F, s_0, \alpha, q)\, |\eta|, \tag{3.10}$$

where e *is a unit vector of* $\mathbb{R}^d$*, and* $\Omega^{(\eta)}$ *is defined by*

$$\Omega^{(\eta)} := \{\, x \in \Omega : \quad dist\,(x, \partial\Omega) > |\,\eta\,|\,\}.$$

`Proof.` We set $\Omega_T^{(\eta)} := (0,T) \times \Omega^{(\eta)}.$ Since $s_0 + \frac{1}{s_0} \in L^\infty(\Omega),$ the maximum/minimum principle of Proposition 2.2.1 implies that $(s^h)_h$ is bounded below and above. Then, we use that $F \in C^2\left((0,\infty)\right),$ to obtain that

$$\begin{aligned} \left\| \nabla s^h \right\|^{q^\star}_{L^{q^\star}(\Omega_T)} &= \int_{\Omega_T} \frac{1}{s^h\, [F''(s^h)\,]^{q^\star}}\, s^h \left| \nabla \left(F'(s^h) \right) \right|^{q^\star} \\ &\leq M(\Omega, s_0, F) \int_{\Omega_T} s^h \left| \nabla \left(F'(s^h) \right) \right|^{q^\star}. \end{aligned} \tag{3.11}$$

We combine (3.7) and (3.11), to conclude that $\left(\nabla s^h\right)_h$ is bounded in $L^{q^\star}(\Omega_T).$ As a consequence, we have that $(s^h)_h$ is bounded in $W^{1,q^\star}(\Omega_T).$

Let $(\varphi_k)_k$ be a sequence in $C^\infty(\Omega_T)$ such that

(i) $\varphi_k \to s^h,$ in $L^{q^\star}(\Omega_T),$ as $k \to \infty$

and

(ii) $\nabla\varphi_k \to \nabla s^h$ in $L^{q^\star}(\Omega_T),$ as $k \to \infty.$

Fixing $t \in (0,T),$ and using the mean-value theorem on $\Omega \ni x \mapsto \varphi_k(t,x),$ we have that

$$\begin{aligned}\int_{\Omega^{(\eta)}} \Big|\, \varphi_k(t,x+\eta e) - \varphi_k(t,x)\,\Big|^{q^\star} &\leq \ |\,\eta\, e|^{q^\star} \int_{\Omega^{(\eta)}} \Big|\, \nabla\varphi_k(t,x+\theta_k\eta e\,\Big|^{q^\star} \\ &\leq \ |\,\eta\,|^{q^\star} \Big\|\, \nabla\varphi_k(t,.)\,\Big\|^{q^\star}_{L^{q^\star}(\Omega)},\end{aligned}$$

where $\theta_k \in [0,1].$ We integrate both sides of the subsequent inequality over $t \in (0,T),$ to have that

$$\int_{\Omega_T^{(\eta)}} \Big|\, \varphi_k(t,x+\eta e) - \varphi(t,x)\,\Big|^{q^\star} \leq |\,\eta\,|^{q^\star} \Big\|\, \nabla\varphi_k\,\Big\|^{q^\star}_{L^{q^\star}(\Omega_T)}. \tag{3.12}$$

We let k go to ∞ in (3.12), and we use (i), (ii), and the fact that $\left(\nabla s^h\right)_h$ is bounded in $L^{q^\star}(\Omega_T),$ to deduce that

$$\int_{\Omega_T^{(\eta)}} |\, s^h(t,x+\eta e) - s^h(t,x)\,|^{q^\star} \leq M(\Omega,T,F,s_0,\alpha,q)\,|\,\eta\,|^{q^\star}. \tag{3.13}$$

We combine (3.13) and Hölder's inequality, to conclude that

$$\begin{aligned}&\int_{\Omega_T^{(\eta)}} \Big|\, s^h(t,x+\eta e) - s^h(t,x)\,\Big| \\ &\quad \leq |\,\Omega_T\,|^{1/q} \left(\int_{\Omega_T^{(\eta)}} \Big|\, s^h(t,x+\eta e) - s^h(t,x)\,\Big|^{q^\star} \right)^{1/q^\star} \\ &\quad \leq M(\Omega,T,F,s_0,\alpha,q)\,|\,\eta\,|\end{aligned}$$

□

Now, we focus on the time compactness of $(s^h)_h$ on $(0,T)\times\Omega,$ for $0 < T < \infty.$ The following constant will be needed in the next lemma:

$$\begin{aligned}&\tilde{M}(\Omega,T,F,s_0,q,\alpha,\beta) \\ &\quad := \frac{\|\, s_0\|^{1/q^\star}_{L^\infty(\Omega)}}{\left\|\, \frac{1}{s_0}\,\right\|^{1/q^\star}_{L^\infty(\Omega)}}\, M(q,\alpha,\beta) \left(E_i(s_0) - |\,\Omega\,| F\left(\frac{1}{|\,\Omega\,|}\right) + \alpha T|\,\Omega\,|\,\|\, s_0\,\|_{L^\infty(\Omega)} \right),\end{aligned}$$

Lemma 3.2.3 *Assume that* $c : I\!R^d \to [0,\infty)$ *is strictly convex, of class* C^1 *and satisfies* $c(0)=0$ *and (HC4), and* $F:[0,\infty)\to I\!R$ *is strictly convex, of class* $C^2((0,\infty))$, *and satisfies* $F(0)=0$ *and (HF1) - (HF2). If* $s_0 \in \mathcal{P}_a(\Omega)$ *is such that* $s_0 + \frac{1}{s_0} \in L^\infty(\Omega)$, *then, for* $\tau > 0$ *and* $0 < T < \infty$,

$$\begin{aligned}&\textstyle\int_{\Omega_T} \left[F'\left(s^h(t+\tau,x)\right) - F'\left(s^h(t,x)\right)\right] \left[s^h(t+\tau,x) - s^h(t,x)\right] \\ &\quad \le \tilde{M}(\Omega,T,F,s_0,q,\alpha,\beta)\,\tau.\end{aligned} \tag{3.14}$$

`Proof.` Without loss of generality, we assume that $\frac{T}{h} \in I\!N$ and $\tau = Nh$, for some $N \in I\!N$. For simplicity, we set

$$L(h,\tau) := \int_{\Omega_T} \left[F'\left(s^h(t+\tau,x)\right) - F'\left(s^h(t,x)\right)\right] \left[s^h(t+\tau,x) - s^h(t,x)\right],$$

and

$$J(i,h,N) := \int_\Omega \left[F'\left(s^h_{i+N}(x)\right) - F'\left(s^h_i(x)\right)\right] \left[s^h_{i+N}(x) - s^h_i(x)\right].$$

It is straightforward to check that

$$L(h,\tau) = \sum_{i=1}^{T/h} h\, J(i,h,N). \tag{3.15}$$

Since W^h_c does not satisfy the triangle inequality, we introduce the q-Wasserstein metric $d^h_q := (W^h_q)^{1/q}$, defined by

$$d^h_q(s^h_i, s^h_{i+N}) := \left(\int_\Omega \left| \frac{y - S^h_q(y)}{h} \right|^q s^h_{i+N}(y)\,\mathrm{d}y \right)^{1/q}, \tag{3.16}$$

where S^h_q denotes the $|\frac{\cdot}{h}|^q$-optimal map that pushes s^h_{i+N} forward to s^h_i. Then, setting $\varphi^h_{i,N} := F'\left(s^h_{i+N}\right) - F'\left(s^h_i\right)$, we obtain that

$$J(i,h,N) = \int_\Omega \left[\varphi^h_{i,N}(y) - \varphi^h_{i,N}\left(S^h_q(y)\right)\right] s^h_{i+N}(y)\,\mathrm{d}y.$$

Since $s_0 + \frac{1}{s_0} \in L^\infty(\Omega)$, $F \in C^2((0,\infty))$, and $s^h_i \nabla\left(F'(s^h_i)\right) = \nabla\left(P(s^h_i)\right) \in L^\infty(\Omega)$ (see Proposition 2.3.1 - (i)), the maximum/minimum principle of Proposition 2.2.1 gives that $\varphi^h_{i,N} \in W^{1,\infty}(\Omega)$. So, approximating φ^h_{i+N} by $C^\infty(\Omega)$-functions, and using that $\left(S^h_q\right)_\# s^h_{i+N} = s^h_i$, and the mean-value theorem, we rewrite $J(i,h,N)$ as follows:

$$J(i,h,N) = \int_\Omega \int_0^1 \left\langle \nabla\varphi^h_{i,N}\left((1-t)y + tS^h_q(y)\right), y - S^h_q y \right\rangle s^h_{i+N}(y)\,\mathrm{d}t\,\mathrm{d}y.$$

We combine Hölder's inequality and (3.16), to deduce that

$$J(i,h,N) \tag{3.17}$$
$$\leq h\, d_q^h\left(s_i^h, s_{i+N}^h\right)\left[\int_\Omega \int_0^1 \left|\,\nabla\varphi_{i+N}^h\left((1-t)y + tS_q^h(y)\right)\,\right|^{q^\star} s_{i+N}^h(y)\,\mathrm{d}t\,\mathrm{d}y\right]^{1/q^\star}.$$

But, observe that $s_i^h,\ s_{i+N}^h \leq \|\, s_0\,\|_{L^\infty(\Omega)}$ because of Proposition 2.2.1, and $|\,\nabla\varphi_{i,N}^h\,|^{q^\star} \in L^\infty(\Omega)$. So, we approximate $|\,\nabla\varphi_{i,N}^h\,|^{q^\star}$ by non-negative functions in $C_c^\infty(I\!R^d)$, and we use (2.23) in Proposition 2.2.3, to deduce that

$$\int_\Omega \left|\,\nabla\varphi_{i,N}^h\left((1-t)y + tS_q^h(y)\right)\,\right|^{q^\star} s_{i+N}^h(y)\,\mathrm{d}y$$
$$\leq \|\, s_0\,\|_{L^\infty(\Omega)} \int_{R^d}\left|\,\nabla\varphi_{i,N}^h(y)\right|^{q^\star}\mathrm{d}y. \tag{3.18}$$

We combine (3.15), (3.17) and (3.18), to have that

$$L(h,\tau) \leq \|\, s_0\,\|_{L^\infty(\Omega)}^{1/q^\star}\, h^2 \sum_{i=1}^{T/h} d_q^h\left(s_i^h, s_{i+N}^h\right)\,\|\,\nabla\varphi_{i,N}^h\,\|_{L^{q^\star}(\Omega)}.$$

And since d_q^h is a metric, the triangle inequality gives that

$$L(h,\tau) \leq \|\, s_0\,\|_{L^\infty(\Omega)}^{\frac{1}{q^\star}}\, h^2 \sum_{k=1}^{N}\sum_{i=1}^{T/h} \|\,\nabla\varphi_{i,N}^h\,\|_{L^{q^\star}(\Omega)}\, d_q^h\left(s_{i+k-1}^h, s_{i+k}^h\right).$$

Then, we apply Hölder's inequality to the interior sum, to deduce that

$$L(h,\tau) \leq \tag{3.19}$$
$$\|\, s_0\,\|_{L^\infty(\Omega)}^{\frac{1}{q^\star}}\, h^{2-\frac{1}{q^\star}}\left(\sum_{i=1}^{T/h} h\,\|\,\nabla\varphi_{i,N}^h\,\|_{L^{q^\star}(\Omega)}^{q^\star}\right)^{1/q^\star}\sum_{k=1}^{N}\left[\sum_{i=1}^{T/h} d_q^h\left(s_{i+k-1}^h, s_{i+k}^h\right)^q\right]^{1/q}.$$

Because of (3.7) and the maximum/minimum principle of Proposition 2.2.1, the sequences $\left(h^{1/q^\star}\left\|\,\nabla\left(F'(s_i^h)\right)\,\right\|_{L^{q^\star}(\Omega)}\right)_{i=1,\cdots,\frac{T}{h}}$ and $\left(h^{1/q^\star}\left\|\,\nabla\left(F'(s_{i+N}^h)\right)\,\right\|_{L^{q^\star}(\Omega)}\right)_{i=1,\cdots,\frac{T}{h}}$ belong to $l_{q^\star}(\Omega)$. Then, we combine Hölder's inequality, Minkowski's inequality, (3.7), and the maximum/minimum principle of Proposition 2.2.1, to have that

$$\left(\sum_{i=1}^{T/h} h\left\|\nabla\varphi_{i,N}^h\,\right\|_{L^{q^\star}(\Omega)}^{q^\star}\right)^{1/q^\star}$$

$$
\begin{aligned}
&\leq \left(\sum_{i=1}^{T/h} \left(h^{1/q^\star} \left\| \nabla \left(F'(s^h_{i+N}) \right) \right\|_{L^{q^\star}(\Omega)} + h^{1/q^\star} \left\| \nabla \left(F'(s^h_{i}) \right) \right\|_{L^{q^\star}(\Omega)} \right)^{q^\star} \right)^{1/q^\star} \\
&\leq \left(\sum_{i=1}^{T/h} h \left\| \nabla \left(F'(s^h_{i+N}) \right) \right\|^{q^\star}_{L^{q^\star}(\Omega)} \right)^{1/q^\star} \\
&\qquad + \left(\sum_{i=1}^{T/h} h \left\| \nabla \left(F'(s^h_{i}) \right) \right\|^{q^\star}_{L^{q^\star}(\Omega)} \right)^{1/q^\star} \\
&\leq \frac{1}{\left\| \frac{1}{s_0} \right\|^{1/q^\star}_{L^\infty(\Omega)}} \left[\bar{M} \left(\Omega, T, F, s_0, q, \alpha \right) \right]^{1/q^\star}. \qquad (3.20)
\end{aligned}
$$

On the other hand, since $c(z) \geq \beta \,|\, z \,|^q$, we have that

$$\left(d^h_q \right)^q \leq \frac{1}{\beta} W^h_c,$$

and then,

$$
\begin{aligned}
&\sum_{k=1}^{N} \left[\sum_{i=1}^{T/h} d^h_q \left(s^h_{i+k-1}, s^h_{i+k} \right)^q \right]^{1/q} \\
&\qquad \leq \frac{1}{(\beta h)^{1/q}} \sum_{k=1}^{N} \left[\sum_{i=1}^{T/h} h \, W^h_c \left(s^h_{i+k-1}, s^h_{i+k} \right) \right]^{1/q}.
\end{aligned}
$$

We use (3.2) and the above inequality, to deduce that

$$
\begin{aligned}
&\sum_{k=1}^{N} \left[\sum_{i=1}^{T/h} d^h_q \left(s^h_{i+k-1}, s^h_{i+k} \right)^q \right]^{1/q} \\
&\qquad \leq \frac{1}{\beta} \left[E_i(s_0) - |\, \Omega \,| F \left(\frac{1}{|\, \Omega \,|} \right) \right]^{1/q} N h^{-1/q}. \qquad (3.21)
\end{aligned}
$$

We combine (3.19) - (3.21), and we use that $\tau = Nh$, to conclude that

$$L(h, \tau) \leq \tilde{M}(\Omega, T, F, s_0, q, \alpha, \beta)\, \tau.$$

This completes the proof of Lemma 3.2.3 □

Lemma 3.2.4 *(Time-compactness)*
Assume that the assumptions of Lemma 3.2.3 hold. Then, for $0 < T < \infty$, *and small* $\tau > 0$,

$$\int_0^T \int_\Omega \left| s^h(t+\tau, x) - s^h(t, x) \right| \leq M(R, \Omega, T, F, s_0, \alpha, q, \beta)\sqrt{\tau} + T\Lambda(\sqrt{\tau}),$$

where Λ *is such that*

$$\lim_{\tau \downarrow 0} \Lambda(\sqrt{\tau}) = 0.$$

`Proof.` Let $R > 0$, and for fixed h, T and τ, define

$$E_R := \Big\{ t \in (0,T): \quad \Delta_{h,\tau}(t) := \Big\| s^h(t) \Big\|_{L^q(\Omega)} + \Big\| s^h(t+\tau) \Big\|_{L^q(\Omega)}$$

$$+ \Big\| F'\left(s^h(t)\right) \Big\|_{W^{1,q^\star}(\Omega)} + \Big\| F'\left(s^h(t+\tau)\right) \Big\|_{W^{1,q^\star}(\Omega)}$$

$$+ \tfrac{1}{\tau} \int_\Omega \left[F'\left(s^h(t+\tau)\right) - F'\left(s^h(t)\right) \right] \left[s^h(t+\tau) - s^h(t) \right]$$

$$> R \Big\}.$$

Because of (3.7), (3.14), the maximum/minimum principle of Proposition 2.2.1, and the fact that $F \in C^2\left((0,\infty)\right)$, we have that $(0,T) \ni t \mapsto \Delta_{h,\tau}(t)$ belongs to $L^1\left((0,T)\right)$. Hence

$$|\, E_R \,| \leq \frac{M(\Omega, T, F, s_0, q, \alpha, \beta)}{R}. \tag{3.22}$$

We combine (3.6) and (3.22), to have that

$$\begin{aligned} \int_{E_R} \int_\Omega \Big| s^h(t+\tau, x) - s^h(t,x) \Big| &\leq 2 \, \| \, s_0 \, \|_{L^\infty(\Omega)} \, |\, \Omega \,| \, |\, E_R \,| \\ &\leq \frac{M(\Omega, T, F, s_0, q, \alpha, \beta)}{R}. \end{aligned} \tag{3.23}$$

On the other hand, if $t \in E_R^c := (0,T) \setminus E_R$, setting $s^h(t) := s_1$ and $s^h(t+\tau) := s_2$, we clearly have that $\| \, s_i \, \|_{L^q(\Omega)} \leq R$, $\| \, F'(s_i) \, \|_{W^{1,q^\star}(\Omega)} \leq R$ for all $i = 1, 2$, and $\int_\Omega \left[F'(s_2) - F'(s_1) \right] \left[s_2 - s_1 \right] \leq R\,\tau$. Then, Proposition 2.2.4 gives that

$$\int_{E_R^c} \int_\Omega |\, s^h(t+\tau, x) - s^h(t,x) \,| \leq \int_{E_R^c} \Lambda(R\,\tau) \leq T \Lambda(R\,\tau), \tag{3.24}$$

where $\Lambda(R\,\tau) := \Lambda_R(R\,\tau)$ is defined as in Proposition 2.2.4. We combine (3.23) - (3.24), and we choose $R = \frac{1}{\sqrt{\tau}}$, to conclude the proof of Lemma 3.2.4 □

Having proved the space-compactness and time-compactness of $(s^h)_h$, we are now ready to show that $(s^h)_h$ converges strongly to s, in $L^1\left((0,T) \times \Omega\right)$ (up to a subsequence), for $0 < T < \infty$; here s is defined as in Lemma 3.2.1.

Proposition 3.2.5 *Assume that* $c : \mathbb{R}^d \to [0,\infty)$ *is strictly convex, of class* C^1 *and satisfies* $c(0) = 0$ *and (HC4), and* $F : [0,\infty) \to \mathbb{R}$ *is strictly* *convex, of class* $C^2((0,\infty))$, *and satisfies* $F(0)=0$ *and (HF1) - (HF2). If* $s_0 \in \mathcal{P}_a(\Omega)$ *is such that* $s_0 + \frac{1}{s_0} \in L^\infty(\Omega)$, *then, for* $0 < T < \infty$, *there is a subsequence of* $(s^h)_{h\downarrow 0}$ *which converges strongly to* s *in* $L^r((0,T)\times\Omega)$, *for* $1 \le r < \infty$, *where* s *is defined as in Lemma 3.2.1.*

Proof. Fix $\delta > O$, and define $\Omega^{(\delta)} := \{ x \in \Omega : \ \operatorname{dist}(x, \partial\Omega) > \delta \}$, and $\Omega_T^{(\delta)} := (0,T)\times\Omega^{(\delta)}$, as in Lemma 3.2.2. Because of (3.6), we have that $(s_h)_h$ is bounded in $L^1\left(\Omega_T^{(\delta)}\right)$. Furthermore, for $\epsilon > 0$, and small $\tau > 0$ and $\eta \in (0,\delta)$, we have that $\Omega_T^{(\delta)} \subset \Omega_T^{(\eta)} \subset \Omega_T$, and then, Lemma 3.2.2 and Lemma 3.2.4 show that

$$\int_{\Omega_T^{(\delta)}} |\, s^h(t, x+\eta e) - s^h(t,x)\,| < \epsilon,$$

and

$$\int_{\Omega_T^{(\delta)}} |\, s^h(t+\tau, x) - s^h(t,x)\,| < \epsilon,$$

uniformly in h. We deduce that, $(s^h)_h$ is precompact in $L^1\left(\Omega_T^{(\delta)}\right)$ (See [1], Theorem 2.21). We observe that $\lim_{\delta\to 0} |\, \Omega \setminus \Omega^{(\delta)}\,| = 0$, and then, we use the diagonal argument, to obtain that, $(s^h)_h$ converges strongly to s, in $L^1(\Omega_T)$, for a subsequence. And since $(s^h)_h$ is bounded in $L^\infty(\Omega_T)$ (see (3.6)), we conclude that it converges to s, in $L^r(\Omega_T)$, for $1 \le r < \infty$ (up to a subsequence) □

3.3 Weak convergence of the nonlinear terms

We use the energy inequality (2.58), stated in Theorem 2.4.2, to show that $\left(\nabla c^\star\left[\nabla\left(F'(s^h)\right)\right]\right)_h$ converges weakly to $\nabla c^\star\left[\nabla\left(F'(s)\right)\right]$, in $L^q((0,T)\times\Omega)$, for a subsequence, and for all $0 < T < \infty$.
Throughout this section, $(s^h)_h$ will denote the subsequence of $(s^h)_h$ which converges to s, in $L^r((0,T)\times\Omega)$, for $1 \le r < \infty$, as in Proposition 3.2.5. For simplicity in the notations, we set

$$\sigma^h := \nabla c^\star\left[\,\nabla\left(F'(s^h)\right)\right].$$

The next lemma shows that $(\sigma^h)_h$ is bounded in $L^q((0,\infty)\times\Omega)$, and $\left(\nabla\left(F'(s^h)\right)\right)_h$ converges weakly to $\nabla\left(F'(s)\right)$ in $L^{q^\star}((0,T)\times\Omega)$, for a subsequence, and for all $0 < T < \infty$.

Lemma 3.3.1 *Assume that* $c : \mathbb{R}^d \to [0,\infty)$ *is strictly convex, of class* C^1 *and satisfies* $c(0)=0$ *and* $c(z) \ge \beta\,|\,z\,|^q$, *for some* $\beta > 0$ *and* $q > 1$.

Assume that $F : [0,\infty) \to I\!R$ *is strictly convex, of class* $C^2((0,\infty))$ *and satisfies (HF1). If* $s_0 \in \mathcal{P}_a(\Omega)$ *is such that* $s_0 + \frac{1}{s_0} \in L^\infty(\Omega)$, *then*

$$\| \sigma^h \|^q_{L^q(\Omega_\infty)} \leq \frac{1}{\beta \left\| \frac{1}{s_0} \right\|_{L^\infty(\Omega)}} \left[E_i(s_0) - |\,\Omega\,| F\left(\frac{1}{|\,\Omega\,|}\right)\right]. \tag{3.25}$$

Therefore,

(i) *there is a subsequence of* $\left(\sigma^h\right)_{h\downarrow 0}$, *which converges weakly to a function* σ *in* $L^q((0,T)\times\Omega)$, *for* $0 < T < \infty$.

In addition to the assumptions, if c satisfies (HC4), and F satisfies $F(0) = 0$ *and (HF2), then*

(ii) *there is a subsequence of* $\left\{\nabla\left(F'(s^h)\right)\right\}_{h\downarrow 0}$, *which converges weakly to* $\nabla(F'(s))$, *in* $L^{q^\star}((0,T)\times\Omega)$, *for* $0 < T < \infty$.

`Proof.` By (2.36), we have that

$$\frac{S_i^h(y) - y}{h} = \nabla c^\star \left[\nabla \left(F'(s_i^h(y))\right)\right], \tag{3.26}$$

for $i \in I\!N$, where S_i^h denotes the c_h-optimal map that pushes s_i^h forward to s_{i-1}^h. We use (3.26) and the maximum/minimum principle of Proposition 2.2.1, to deduce that

$$\begin{aligned} \| \sigma^h \|^q_{L^q(\Omega_\infty)} &= \sum_{i=1}^\infty h \int_\Omega \left| \nabla c^\star \left[\nabla \left(F'(s_i^h(y))\right)\right] \right|^q \mathrm{d}y \\ &= \sum_{i=1}^\infty h \int_\Omega \left| \frac{S_i^h(y) - y}{h} \right|^q \mathrm{d}y \\ &\leq \frac{1}{\left\| \frac{1}{s_0} \right\|_{L^\infty(\Omega)}} \sum_{i=1}^\infty h \int_\Omega \left| \frac{S_i^h(y) - y}{h} \right|^q s_i^h(y)\, \mathrm{d}y. \end{aligned}$$

Since $c(z) \geq \beta\,|\,z\,|^q$, we obtain that

$$\| \sigma_h \|^q_{L^q(\Omega_\infty)} \leq \frac{1}{\beta \left\| \frac{1}{s_0} \right\|_{L^\infty(\Omega)}} \sum_{i=1}^\infty h\, W_c^h(s_{i-1}^h, s_i^h). \tag{3.27}$$

We combine (3.2) and (3.27), to conclude (3.25).
Now, fix $0 < T < \infty$. By Proposition 3.2.5, $(s^h)_h$ converges strongly to s, in $L^1((0,T)\times\Omega)$, and by (3.6) and the fact that F' is continuous on $(0,\infty)$, $\left(F'(s^h)\right)_h$ is bounded in $L^\infty((0,T)\times\Omega)$. We deduce, due to the

continuity of F' on $(0,\infty)$, that $\left(F'(s^h)\right)_h$ converges weakly to $F'(s)$, in $L^{q^\star}((0,T)\times\Omega)$. And, since $\left\{ \nabla\left(F'(s^h)\right) \right\}_h$ is bounded in $L^{q^\star}((0,T)\times\Omega)$ (because of (3.7) and the maximum principle of Proposition 2.2.1), we conclude (ii) $\square$

The next lemma extends the energy inequality (2.58) to the time-space domain $(0,T)\times\Omega$.

Lemma 3.3.2 *(Energy inequality in time-space)*
Assume that $c : \mathbb{R}^d \to [0,\infty)$ *is strictly convex, of class* C^1*, and satisfies* $c(0)=0$ *and (HC4). Assume that* $F:[0,\infty)\to\mathbb{R}$ *is strictly convex, of class* $C^2((0,\infty))$*, and satisfies* $F(0)=0,\ F\in C^2((0,\infty))$ *and (HF1) - (HF2). If* $s_0\in\mathcal{P}_a(\Omega)$ *is such that* $s_0+\frac{1}{s_0}\in L^\infty(\Omega)$*, and* $t\mapsto u(t)$ *is a non-negative function in* $C_c^\infty(\mathbb{R})$*, then*

$$\int_0^\infty\int_\Omega \left\langle s^h\nabla\left(F'(s^h)\right), \nabla c^\star\left[\nabla\left(F'(s^h)\right)\right]\right\rangle u(t) \leq \frac{1}{h}\int_0^h\int_\Omega F\left(s_0(x)\right)u(t) + \int_0^\infty\int_\Omega F(s^h)\,\partial_t^h u(t),$$

where

$$\partial_t^h u(t) := \frac{u(t+h)-u(t)}{h}.$$

`Proof.` Let T be such that $\frac{T}{h}\in\mathbb{N}$, and assume that $\operatorname{spt} u\subset[-T,T]$. We choose $G:=F$ in the energy inequality (2.58), and we use (2.36), to obtain that

$$\int_\Omega \frac{F\left(s_i^h(y)\right)-F\left(s_{i-1}^h(y)\right)}{h}\,\mathrm{d}y \leq -\int_\Omega \left\langle \nabla\left[F'\left(s_i^h(y)\right)\right], \nabla c^\star\left[\nabla\left(F'(s_i^h(y))\right)\right]\right\rangle s_i^h(y)\,\mathrm{d}y,$$

for all $i\in\mathbb{N}$. Since $u\geq 0$, we deduce that

$$\sum_{i=1}^{T/h}\int_{t_{i-1}}^{t_i}\int_\Omega \frac{F\left(s_i^h(y)\right)-F\left(s_{i-1}^h(y)\right)}{h}u(t) \leq -\int_{\Omega_T} s^h\left\langle\nabla\left(F'(s^h)\right),\nabla c^\star\left[\nabla\left(F'(s^h)\right)\right]\right\rangle u(t). \tag{3.28}$$

By direct computations, the left hand side of the above inequality gives that

$$\sum_{i=1}^{T/h}\int_{t_{i-1}}^{t_i}\int_\Omega \frac{F\left(s_i^h(y)\right)-F\left(s_{i-1}^h(y)\right)}{h}u(t)$$

$$= \frac{1}{h} \int_{\Omega_T} F\left(s^h(t,y)\right) u(t) \; - \; \frac{1}{h} \int_{\Omega_h} F\left(s_0(y)\right) u(t)$$
$$- \frac{1}{h} \int_h^T \int_\Omega F\left(s^h(t-h)\right) u(t).$$

We use the substitution $\tau = t - h$ in the last integral, and the fact that $u(t+h) = 0$ for $t \in (T-h, T)$, to obtain that

$$\sum_{i=1}^{T/h} \int_{t_{i-1}}^{t_i} \int_\Omega \frac{F\left(s_i^h(y)\right) - F\left(s_{i-1}^h(y)\right)}{h} u(t) \tag{3.29}$$
$$= - \int_{\Omega_T} F\left(s^h(t,y)\right) \partial_t^h u(t) - \frac{1}{h} \int_{\Omega_h} F\left(s_0(y)\right) u(t).$$

We combine (3.28) and (3.29), and we let T go to ∞, to complete the proof of Lemma 3.3.2 □

Theorem 3.3.3 *Assume that* $c : \mathbb{R}^d \to [0, \infty)$ *is strictly convex, of class* C^1*, and satisfies* $c(0) = 0$ *and (HC4). Assume that* $F : [0, \infty) \to \mathbb{R}$ *is strictly convex, of class* $C^2((0,\infty))$*, and satisfies* $F(0) = 0,\ F \in C^2((0,\infty))$ *and (HF1) - (HF2). If* $s_0 \in \mathcal{P}_a(\Omega)$ *is such that* $s_0 + \frac{1}{s_0} \in L^\infty(\Omega)$*, and* $t \mapsto u(t)$ *is a non-negative function in* $C_c^\infty(\mathbb{R})$*, then*

$$\lim_{h \downarrow 0} \int_0^\infty \int_\Omega \left\langle s^h \sigma^h, \nabla\left(F'(s^h)\right) \right\rangle u(t) \; = \; \int_0^\infty \int_\Omega \left\langle s\sigma, \nabla\left(F'(s)\right) \right\rangle u(t), \tag{3.30}$$

where s *and* σ *are defined as in Lemma 3.2.1 and Lemma 3.3.1. Therefore, there is a subsequence of* $\left(\nabla c^\star\left[\nabla\left(F'(s^h)\right)\right]\right)_h$*, which converges weakly to* $\nabla c^\star\left[\nabla\left(F'(s)\right)\right]$ *in* $L^q((0,T) \times \Omega)$*, for* $0 < T < \infty$*.*

`Proof.` Let $T > 0$ be such that $\operatorname{spt} u \subset [-T, T]$, and assume that $s(t) = s_0$, for $t \le 0$. Denote by $(s^h)_h$ the subsequence of $(s^h)_h$, such that

(i) $(s^h)_{h \downarrow 0}$ converges to s a.e.,

(ii) $\{\nabla\left(F'(s^h)\right)\}_{h \downarrow 0}$ converges weakly to $\nabla\left(F'(s)\right)$, in $L^{q^\star}(\Omega_T)$,

and

(iii) $\{\sigma^h = \nabla c^\star\left[\nabla\left(F'(s^h)\right)\right]\}_{h \downarrow 0}$ converges weakly to σ, in $L^q(\Omega_T)$,

as in Proposition 3.2.5 and Lemma 3.3.1. We first show that

$$\lim_{h \downarrow 0} \int_{\Omega_T} \langle \sigma^h, s^h \nabla\left(F'(s)\right) \rangle u(t) = \int_{\Omega_T} \langle \sigma, s \nabla\left(F'(s)\right) \rangle u(t), \tag{3.31}$$

and

$$\lim_{h\downarrow 0}\int_{\Omega_T}\langle s^h\nabla c^\star\left[\nabla\left(F'(s)\right)\right],\nabla\left(F'(s^h)\right)-\nabla\left(F'(s)\right)\rangle\, u(t)=0. \tag{3.32}$$

By (ii), we have that $\nabla\left(F'(s)\right)\in L^{q^\star}(\Omega_T)$, and by (3.6), we see that $(s^h)_h$ is bounded in $L^\infty(\Omega_T)$. We use (i) and the dominated convergence theorem, to have that

$$\lim_{h\downarrow 0}\int_{\Omega_T}\left|(s^h-s)\nabla\left(F'(s)\right)\right|^{q^\star}=0,$$

i.e.

(iv) $\{s^h\nabla\left(F'(s)\right)\}_{h\downarrow 0}$ converges strongly to $s\nabla\left(F'(s)\right)$, in $L^{q^\star}(\Omega_T)$.

We combine (iii) - (iv), and the fact that $u\in C_c^\infty(\mathbb{R})$, to conclude (3.31). By Proposition 5.3.3, the convexity of c, and the fact that $c(z)\geq\beta\,|\,z\,|^q$, we have that

$$\begin{aligned}|\,\nabla c^\star(z)\,|^q\leq\frac{c\left(\nabla c^\star(z)\right)}{\beta} &= \frac{1}{\beta}\left(\langle z,\nabla c^\star(z)\rangle-c^\star(z)\right)\\ &\leq \frac{1}{\beta}\langle z,\nabla c^\star(z)\rangle\leq M(\beta,q)\,|\,z\,|^{q^\star}.\end{aligned}$$

We deduce that

$$\left|\nabla c^\star\left[\nabla\left(F'(s)\right)\right]\right|^q\leq M(\beta,q)\left|\nabla\left(F'(s)\right)\right|^{q^\star},$$

and then,

(v) $\nabla c^\star\left[\nabla\left(F'(s)\right)\right]\in L^q(\Omega_T)$.

We use (i), (v) and the dominated convergence theorem, to have that

$$\lim_{h\downarrow 0}\int_{\Omega_T}\left|(s^h-s)\nabla c^\star\left[\nabla\left(F'(s)\right)\right]\right|^q=0,$$

i.e.

(vi) $\{s^h\nabla c^\star\left[\nabla\left(F'(s)\right)\right]\}_{h\downarrow 0}$ converges strongly to $s\nabla c^\star\left[\nabla\left(F'(s)\right)\right]$, in $L^q(\Omega_T)$.

We combine (ii) and (vi), to conclude (3.32).

The proof of (3.30) follows directly from the following three claims.

Claim 1.

$$\int_{\Omega_T} \left\langle s\sigma, \nabla \left(F'(s)\right) \right\rangle u(t) \leq \liminf_{h\downarrow 0} \int_{\Omega_T} \left\langle s^h\sigma^h, \nabla \left(F'(s^h)\right) \right\rangle u(t).$$

`Proof.` Because $c^\star$ is convex, and u and s^h are non-negative, we have that

$$\int_{\Omega_T} s^h \left\langle \nabla c^\star \left[\nabla \left(F'(s^h)\right)\right] - \nabla c^\star \left[\nabla \left(F'(s)\right)\right], \nabla \left(F'(s^h)\right) - \nabla \left(F'(s)\right) \right\rangle u(t) \geq 0,$$

and then,

$$\begin{aligned} &\liminf_{h\downarrow 0} \int_{\Omega_T} \langle \sigma^h, s^h \nabla \left(F'(s)\right)\rangle u(t) \\ &\quad \leq \liminf_{h\downarrow 0} \int_{\Omega_T} \langle s^h\sigma^h, \nabla \left(F'(s^h)\right)\rangle u(t) \\ &\quad\quad + \limsup_{h\downarrow 0} \int_{\Omega_T} \langle s^h \nabla c^\star \left[\nabla \left(F'(s)\right)\right], \nabla \left(F'(s)\right) - \nabla \left(F'(s^h)\right)\rangle u(t). \end{aligned} \tag{3.33}$$

We combine (3.31) - (3.33), to conclude Claim 1.

Claim 2.

$$\begin{aligned} &\limsup_{h\downarrow 0} \int_{\Omega_T} \left\langle s^h\sigma^h, \nabla \left(F'(s^h)\right) \right\rangle u(t) \\ &\quad \leq \int_{\Omega} \left[s_0 F'(s_0) - F^\star \left(F'(s_0)\right) \right] u(0) \\ &\quad\quad + \int_{\Omega_T} \left[s(t,x) F'\left(s(t,x)\right) - F^\star \left(F'\left(s(t,x)\right)\right) \right] u'(t). \end{aligned}$$

`Proof.` We first observe that

$$\lim_{h\downarrow 0} \int_{\Omega_T} F(s^h) \partial_t^h u(t) = \int_{\Omega_T} F(s) u'(t). \tag{3.34}$$

Indeed, it is clear that

$$\begin{aligned} &\left| \int_{\Omega_T} F(s^h)\partial_t^h u(t) - F(s)u'(t) \right| \\ &\quad \leq \int_{\Omega_T} |\, F(s^h) - F(s)\,|\,|u'(t)\,| + \int_{\Omega_T} |\, F(s^h)\,|\,|\partial_t^h u(t) - u'(t)\,|. \end{aligned} \tag{3.35}$$

Because of the maximum/minimum principle of Proposition 2.2.1, and the continuity of F, we have that $\left(F(s^h)\right)_h$ is bounded in $L^\infty(\Omega_T)$. We let h go to 0 in (3.35), and we use (i), the fact that $u \in C_c^\infty(I\!R)$, and the Lebesgue

dominated convergence theorem, to conclude (3.34).
Recall that Lemma 3.3.2 gives that

$$\limsup_{h\downarrow 0}\int_{\Omega_T}\left\langle s^h\sigma^h,\nabla\left(F'(s^h)\right)\right\rangle u(t)$$
$$\leq \liminf_{h\downarrow 0}\frac{1}{h}\int_0^h\int_\Omega F(s_0)u(t)+\limsup_{h\downarrow 0}\int_{\Omega_T}F(s^h)\partial_t^h u(t),$$

and by (3.34) and the continuity of u, we obtain that

$$\limsup_{h\downarrow 0}\int_{\Omega_T}\left\langle s^h\sigma^h,\nabla\left(F'(s^h)\right)\right\rangle u(t) \quad (3.36)$$
$$\leq\int_\Omega F(s_0)u(0)+\int_{\Omega_T}F\left(s(t,x)\right)u'(t).$$

Since $F\in C^1\left(0,\infty\right))$ is strictly convex and satisfies (HF1), we have that

$$F(a)=aF'(a)-F^\star\left(F'(a)\right), \quad (3.37)$$

for $a>0$. We substitute (3.37) in (3.36) for $a=s(t,x)$ and $a=s_0(x)$, to conclude Claim 2.

Claim 3.

$$\int_\Omega\left[\,s_0F'(s_0)-F^\star\left(F'(s_0)\right)\right]\,u(0)$$
$$+\int_{\Omega_T}\left[\,s(t,x)F'\left(s(t,x)\right)-F^\star\left(F'\left(s(t,x)\right)\right)\right]\,u'(t)$$
$$\leq\int_{\Omega_T}\left\langle s\sigma,\nabla\left(F'(s)\right)\right\rangle u(t).$$

`Proof.` Set $\xi(t,x):=F'\left(s(t,x)\right)u(t)$, for $(t,x)\in\mathbb{R}\times\Omega$. Because of (i) - (ii), the maximum/minimum principle of Proposition 2.2.1, and the fact that $F\in C^2\left((0,\infty)\right)$, we have that $F'(s)\in L^\infty(\Omega_T)$ and $\nabla\left(F'(s)\right)\in L^{q^\star}(\Omega_T)$. Let $(\varphi_j)_j$ be a sequence in $C^\infty(\Omega_T)$, such that

$$\varphi_j\to F'(s)\quad\text{in }\ L^{q^\star}(\Omega_T),$$

and

$$\nabla\varphi_j\to\nabla\left(F'(s)\right)\quad\text{in }\ L^{q^\star}(\Omega_T),\ \text{ as }\ j\to\infty.$$

It is clear that $\xi_j:=\varphi_ju$ is admissible in (2.63). Therefore, we use Proposition 3.1.2 and the backward derivative $\partial_t^{-h}\,\xi_j(t,x):=\frac{\xi_j(t,x)-\xi_j(t-h,x)}{h}$ in (2.63), to have that

$$\int_{\Omega_T}(s_0-s^h)\partial_t^{-h}\,\xi_j+\int_{\Omega_T}\langle s^h\sigma^h,\nabla\xi_j\rangle=0(h^{\epsilon(q)}), \quad (3.38)$$

where $\epsilon(q) = \min(1, q-1)$. Letting j go to ∞ in (3.38), we obtain that

$$\int_{\Omega_T} (s_0 - s^h)\partial_t^{-h}\,\xi + \int_{\Omega_T} \langle \sigma^h, s^h \nabla \left(F'(s)\right)\rangle\, u(t) = 0(h^{\epsilon(q)}).$$

We let h go to 0 in the subsequent equality, and we use (3.31), to conclude that

$$\lim_{h\downarrow 0} \int_{\Omega_T} (s_0 - s^h)\,\partial_t^{-h}\,\xi + \int_{\Omega_T} \langle \sigma, s \nabla \left(F'(s)\right)\rangle\, u(t) = 0. \tag{3.39}$$

Since $\operatorname{spt} u \subset [-T, T]$, we have that

$$\int_{\Omega_T} s_0\,\partial_t^{-h}\xi = -\frac{1}{h}\int_{-h}^{0}\int_{\Omega} s_0(x)\xi(t,x),$$

and then,

$$\lim_{h\downarrow 0} \int_{\Omega_T} s_0\,\partial_t^{-h}\,\xi = -\int_{\Omega} s_0(x)\,\xi(0,x) = -\int_{\Omega} s_0 F'(s_0)u(0). \tag{3.40}$$

We combine (3.39), (3.40) and (i), to have that

$$\int_{\Omega_T} \langle \sigma, s \nabla \left(F'(s)\right)\rangle\, u(t) = \lim_{h\downarrow 0} \int_{\Omega_T} s(t,x)\partial_t^{-h}\xi(t,x) + \int_{\Omega} s_0 F'(s_0)u(0). \tag{3.41}$$

By direct computations, we obtain that

$$\begin{aligned} s(t,x)\partial_t^{-h}\xi(t,x) \;&=\; s(t,x)\,F'\left(s(t,x)\right)\partial_t^{-h}\,u(t) \\ &\quad + \frac{1}{h}\,s(t,x)\,u(t-h)\left[\,F'\left(s(t,x)\right) - F'\left(s(t-h,x)\right)\right]. \end{aligned}$$

Since $F^\star$ is convex, and $(F^\star)' = (F')^{-1}$, we have that

$$\left(F'(b) - F'(a)\right) b \geq F^\star\left(F'(b)\right) - F^\star\left(F'(a)\right),$$

for $a, b > 0$, and then, we deduce that

$$\begin{aligned} s(t,x)\,\partial_t^{-h}\xi(t,x) \;&\geq\; s(t,x)\,F'\left(s(t,x)\right)\,\partial_t^{-h}u(t) \\ &\quad + \frac{1}{h}\,u(t-h)\left[\,F^\star\left(F'\left(s(t,x)\right)\right) - F^\star\left(F'\left(s(t-h,x)\right)\right)\right]. \end{aligned}$$

We integrate both sides of the subsequent inequality over Ω_T, and we use that $u = 0$ on $(T-h, T)$ (for h small enough), and $s(t,x) = s_0(x)$ for $t \in (-h, 0)$, to obtain that

$$\begin{aligned} \int_{\Omega_T} s(t,x)\partial_t^{-h}\xi(t,x) \;&\geq\; \int_{\Omega_T} \left[\,s(t,x)F'\left(s(t,x)\right) - F^\star\left(F'\left(s(t,x)\right)\right)\right]\partial_t^{-h}u(t) \\ &\quad - \frac{1}{h}\int_0^h u(t-h)\int_{\Omega} F^\star\left(F'(s_0)\right). \end{aligned}$$

We let h go to 0 in the above inequality, to deduce that

$$\begin{aligned}\lim_{h\downarrow 0}\int_{\Omega_T} & s(t,x)\partial_t^{-h}\xi(t,x) \qquad (3.42)\\ \geq & \int_{\Omega_T}\big[\, s(t,x)F'\left(s(t,x)\right) - F^\star\left(F'\left(s(t,x)\right)\right)\big]\, u'(t)\\ & - \int_\Omega F^\star\left(F'(s_0)\right) u(0).\end{aligned}$$

We combine (3.41) and (3.42), to conclude Claim 3.

Finally, we show that

$$\sigma = \nabla c^\star\left[\,\nabla\left(F'(s)\right)\,\right], \tag{3.43}$$

which combined with Lemma 3.3.1 completes the proof of Theorem 3.3.3. Let $\epsilon > 0$, $\varphi \in C^\infty(\Omega)$, and set $\omega_\epsilon(t,x) := F'\left(s(t,x)\right) - \epsilon\psi(x)$, where ψ is such that $\nabla\psi = \varphi$. It is clear that $\nabla\omega_\epsilon \in L^{q^\star}(\Omega_T)$, and

$$\left|\,\nabla c^\star\left(\nabla\omega_\epsilon\right)\,\right|^q \leq M(\beta,q)\,|\,\nabla\omega_\epsilon\,|^{q^\star},$$

as in the proof of (v) in (3.32). We deduce that $\nabla c^\star\left(\nabla\omega_\epsilon\right) \in L^q(\Omega_T)$. We use that $c^\star$ is convex, and s^h and u are non-negative, to have that

$$\int_{\Omega_T} s^h\left\langle \nabla c^\star\left[\nabla\left(F'(s^h)\right)\right] - \nabla c^\star\left(\nabla w_\epsilon\right), \nabla\left(F'(s^h)\right) - \nabla w_\epsilon\right\rangle u(t) \,\geq\, 0.$$

We let h go to 0 in the above inequality, to obtain that

$$\begin{aligned}\limsup_{h\downarrow 0}\int_{\Omega_T}\langle s^h\sigma^h, \nabla\left(F'(s^h)\right)\rangle\, u(t) - \liminf_{h\downarrow 0}\int_{\Omega_T}\langle\sigma^h, s^h\nabla w_\epsilon\rangle\, u(t)&\\ - \liminf_{h\downarrow 0}\int_{\Omega_T}\langle s^h\nabla c^\star(\nabla w_\epsilon), \nabla\left(F'(s^h)\right) - \nabla w_\epsilon\rangle\, u(t) \geq 0.&\qquad (3.44)\end{aligned}$$

As in the proof of (3.31) and (3.32), we have that

$$\liminf_{h\downarrow 0}\int_{\Omega_T}\langle\sigma^h, s^h\nabla w_\epsilon\rangle\, u(t) = \int_{\Omega_T}\langle\sigma, s\nabla w_\epsilon\rangle\, u(t), \tag{3.45}$$

and

$$\begin{aligned}\liminf_{h\downarrow 0}\int_{\Omega_T} & \langle s^h\nabla c^\star(\nabla w_\epsilon), \nabla\left(F'(s^h)\right) - \nabla w_\epsilon\rangle\, u(t) \qquad (3.46)\\ & = \int_{\Omega_T}\langle s\nabla c^\star(\nabla w_\epsilon), \nabla\left(F'(s)\right) - \nabla w_\epsilon\rangle\, u(t).\end{aligned}$$

We combine (3.30) and (3.44) - (3.46), to have that

$$\int_{\Omega_T}\langle s\sigma - s\nabla c^\star(\nabla w_\epsilon), \nabla\left(F'(s)\right) - \nabla w_\epsilon\rangle\, u(t) \geq 0.$$

We divide the subsequent inequality by ϵ, and we let ϵ go to 0, to obtain that

$$\int_{\Omega_T} \langle s\sigma - s\nabla c^\star \left[\nabla \left(F'(s) \right) \right], \varphi(x)\, u(t) \rangle \geq 0.$$

Choosing $-\varphi$ in place of φ, we get that

$$\int_{\Omega_T} \langle s\sigma - s\,\nabla c^\star \left[\nabla \left(F'(s) \right) \right], \varphi(x)\, u(t) \rangle = 0.$$

And since φ and $u \geq 0$ are arbitrary test functions, we deduce that

$$s\sigma = s\,\nabla c^\star \left[\nabla \left(F'(s) \right) \right].$$

But, (i) and the maximum/minimum principle of Proposition 2.2.1 give that $s \neq 0$. Then, we conclude (3.43) □

3.4 Existence and uniqueness of solutions

In this section, we state and prove two theorems of existence and uniqueness of solutions to problem (1.5). In the first theorem, we assume that the potential $V = 0$, and in the second theorem, we consider arbitrary V.

Theorem 3.4.1 *(Case $V = 0$).*
Assume that $c : \mathbb{R}^d \to [0, \infty)$ is strictly convex, of class C^1, and satisfies $c(0) = 0$ and (HC4). Assume that $F : [0, \infty) \to \mathbb{R}$ is strictly convex, of class $C^2\left((0, \infty)\right)$, and satisfies $F(0) = 0$ and (HF1) - (HF2). If $s_0 \in \mathcal{P}_a(\Omega)$ is such that $s_0 + \frac{1}{s_0} \in L^\infty(\Omega)$, and $V = 0$, then, problem (1.5) has a unique weak solution $s : [0, \infty) \times \Omega \to [0, \infty)$, in the sense that

(i). $s + \frac{1}{s} \in L^\infty\left((0, \infty); L^\infty(\Omega)\right)$, $\nabla \left(F'(s)\right) \in L^{q^\star}\left((0,T) \times \Omega\right)$, *for* $0 < T < \infty$, *and*

(ii). *for* $\xi \in C_c^2(\mathbb{R} \times \mathbb{R}^d)$,

$$\int_0^\infty \int_\Omega \Big\{ -s\frac{\partial \xi}{\partial t} + \langle s\,\nabla c^\star \left[\nabla \left(F'(s) \right) \right], \nabla\xi \rangle \Big\} = \int_\Omega s_0(x)\, \xi(0, x)\, dx. \tag{3.47}$$

`Proof.` By Proposition 3.2.5, $(s^h)_h$ converges to s a.e., for a subsequence. And since $s^h \geq 0$ for all h, we deduce that $s \geq 0$. We combine Proposition 3.2.5 and the maximum/minimum principle of Proposition 2.2.1, to conclude that $s + \frac{1}{s} \in L^\infty\left((0,\infty); L^\infty(\Omega)\right)$. By Lemma 3.3.1, $\nabla\left(F'(s)\right) \in L^{q^\star}\left((0,T) \times \Omega\right)$, for $0 < T < \infty$. This proves (i).
Recall that Theorem 3.3.3 gives that $\nabla c^\star \left[\nabla \left(F'(s) \right) \right] \in L^q\left((0,T) \times \Omega\right)$

for $0 < T < \infty$, and Proposition 2.2.1 and Proposition 3.2.5 imply that $s \in L^\infty\left((0,T)\times\Omega\right)$. We deduce that $s\nabla c^\star\left[\nabla\left(F'(s)\right)\right] \in L^q\left((0,T)\times\Omega\right)$, for $0<T<\infty$. Now, let $\xi \in C^2_c(I\!\!R \times I\!\!R^d)$ be such that $\operatorname{spt}\xi(.,x) \subset [-T,T]$ for some $0<T<\infty$ and for all $x\in\Omega$, and set $\Omega_T := (0,T)\times\Omega$. Because of Proposition 2.5.1 and Proposition 3.1.2, we have that

$$\lim_{h\downarrow 0}\int_{\Omega_T}\Big\{(s_0 - s^h)\,\partial_t^h\xi + \langle\, s^h\nabla c^\star\Big[\nabla\Big(F'(s^h)\Big)\Big], \nabla\xi\,\rangle\Big\} = 0. \tag{3.48}$$

Lemma 3.2.1 gives that $(s^h)_h$ converges weakly to s in $L^1(\Omega_T)$, for a subsequence, and then, we have that

$$\begin{aligned} \lim_{h\downarrow 0}\int_{\Omega_T}(s_0 - s^h)\,\partial_t^h\xi &= \int_{\Omega_T}(s_0 - s)\,\frac{\partial\xi}{\partial t} \\ &= -\Big[\int_{\Omega_T} s\,\frac{\partial\xi}{\partial t} + \int_\Omega s_0(x)\xi(0,x)\Big]. \end{aligned} \tag{3.49}$$

By (3.6) and Proposition 3.2.5, $(s^h)_h$ is bounded in $L^\infty(\Omega_T)$ and converges a.e. to s, for a subsequence. We deduce that it converges strongly in $L^{q^\star}(\Omega_T)$. And since $\Big\{\nabla c^\star\left(\nabla\left(F'(s^h)\right)\right)\Big\}_h$ converges weakly to $\nabla c^\star\left(\nabla\left(F'(s)\right)\right)$ in $L^q(\Omega_T)$, for a subsequence (Theorem 3.3.3), we conclude that

(iii) $\Big\{s^h\nabla c^\star\left(\nabla\left(F'(s)\right)\right)\Big\}_h$ converges weakly to $s\nabla c^\star\left(\nabla\left(F'(s)\right)\right)$ in $L^1(\Omega_T)$, for a subsequence.

We combine (3.48) - (3.49) and (iii), and we use the fact that $\operatorname{spt}\xi(.,x)\subset[-T,T]$, to conclude (3.47).
Here, we prove uniqueness of the solution to (1.5) when $\frac{\partial s}{\partial t}\in L^1\left((0,T)\times\Omega\right)$, for $0<T<\infty$. By using the arguments in [16], we can extend the proof to the general case. In fact, assumption (1.2) would not be required here; one just need to notice that $\langle\nabla c^\star(z_1)-\nabla c^\star(z_2), z_1-z_2\rangle \geq 0$ by the convexity of $c^\star$.
Let $T>0$, and assume that s_1 and s_2 are solutions of (1.5) with the same initial data, such that $N\leq s_j\leq M$, and $\frac{\partial s_j}{\partial t}\in L^1\left((0,T)\times\Omega\right)$, $j=1,2$. Since $\nabla\left(F'(s_j)\right)\in L^{q^\star}\left((0,T)\times\Omega\right)$, and

$$\Big|\nabla c^\star\left[\nabla\left(F'(s_j)\right)\right]\Big|^q \leq M(\beta,q)\Big|\nabla\left(F'(s_j)\right)\Big|^{q^\star},$$

we have that $\nabla c^\star\left[\nabla\left(F'(s_j)\right)\right]\in L^q\left((0,T)\times\Omega\right)$. For $\delta>0$, we define

$$\Omega_T \ni (t,x)\mapsto \xi_\delta(t,x) := \varphi_\delta\left(F'\left(s_1(t,x)\right) - F'\left(s_2(t,x)\right)\right),$$

where

$$\varphi_\delta(\tau) := \begin{cases} 0 & \text{if} \quad \tau\leq 0 \\ \frac{\tau}{\delta} & \text{if} \quad 0\leq\tau\leq\delta \\ 1 & \text{if} \quad \tau\geq\delta \end{cases}$$

Using ξ_δ as a test function (or a smooth approximation of ξ_δ, if needed) in the equations satisfied by the solutions s_1 and s_2, we have that

$$\int_{\Omega_T} \xi_\delta \, \partial_t(s_1 - s_2) = -\int_{\Omega_T} \langle s_1 \nabla c^\star \left[\nabla \left(F'(s_1) \right) \right] - s_2 \nabla c^\star \left[\nabla \left(F'(s_2) \right) \right], \nabla \xi_\delta \rangle,$$

which reads as

$$\begin{aligned} &\int_{\Omega_T} \xi_\delta \, \partial_t(s_1 - s_2) \\ &\quad = -\frac{1}{\delta} \int_{\Omega_T^{(1,2)}} s_1 \langle \nabla c^\star \left[\nabla \left(F'(s_1) \right) \right] - \nabla c^\star \left[\nabla \left(F'(s_2) \right) \right], \nabla \left(F'(s_1) - F'(s_2) \right) \rangle \\ &\qquad -\frac{1}{\delta} \int_{\Omega_T^{(1,2)}} (s_1 - s_2) \langle \nabla c^\star \left[\nabla \left(F'(s_2) \right) \right], \nabla \left(F'(s_1) - F'(s_2) \right) \rangle, \end{aligned}$$

where $\Omega_T^{(1,2)} := \Omega_T \cap [\, 0 < F'(s_1) - F'(s_2) < \delta \,]$. Because $c^\star$ is convex, the first term on the right hand side of the above equality is non-positive. And since $F \in C^1\left((0,\infty)\right)$ is strictly convex and satisfies (HF1), and $N \le s_1,\, s_2 \le M$, we have on $\Omega_T^{(1,2)}$, that

$$|\, s_1 - s_2 \,| = \left| \left[(F^\star)' \circ F' \right](s_1) - \left[(F^\star)' \circ F'(s_2) \right] \right| \le \delta \sup_{\tau \in [\, F'(N), F'(M) \,]} (F^\star)''(\tau).$$

We deduce that

$$\begin{aligned} &\int_{\Omega_T} \xi_\delta \, \partial_t(s_1 - s_2) \\ &\quad \le \sup_{\tau \in [\, F'(N), F'(M) \,]} (F^\star)''(\tau) \int_{\Omega_T^{(1,2)}} \left| \langle \nabla c^\star \left[\nabla \left(F'(s_2) \right) \right], \nabla \left(F'(s_1) - F'(s_2) \right) \rangle \right|. \end{aligned}$$

We let δ go to 0 in the subsequent inequality, and we use that $\varphi_\delta \to \mathbb{I}_{[0,\infty)}$ and $[\, F'(s_1) - F'(s_2) \ge 0 \,] = [\, s_1 - s_2 \ge 0 \,]$, to have that

$$\int_{\Omega_T} \partial_t \left[\, (s_1 - s_2)^+ \, \right] \le 0,$$

which reads as

$$\int_\Omega [\, s_1(T) - s_2(T) \,]^+ \le \int_\Omega [\, s_1(0) - s_2(0) \,]^+ = 0,$$

for $0 < T < \infty$. Interchanging s_1 and s_2 in the above argument, we conclude that $s_1 = s_2$ □

Theorem 3.4.2 *(General case).*
Assume that $V : \bar{\Omega} \to [0,\infty)$ *is convex, of class* C^1, *and* $c : \mathbb{R}^d \to [0,\infty)$

is strictly convex, of class C^1, and satisfies $c(0) = 0$ and (HC4). Assume that $F : [0,\infty) \to I\!R$ is strictly convex, of class $C^2((0,\infty))$, and satisfies $F(0) = 0$ and (HF1) - (HF2). If $s_0 \in \mathcal{P}_a(\Omega)$ is such that $s_0 + \frac{1}{s_0} \in L^\infty(\Omega)$, then, problem (1.5) has a unique weak solution $s : [0,\infty) \times \Omega \to [0,\infty)$, in the sense that

(i). $s + \frac{1}{s} \in L^\infty((0,\infty); L^\infty(\Omega))$, $\nabla(F'(s)) \in L^{q^\star}((0,T)\times\Omega)$, *for* $0 < T < \infty$, *and*

(ii). *for all* $\xi \in C_c^2(I\!R \times I\!R^d)$,

$$\int_0^\infty \int_\Omega \Big\{ - s\frac{\partial \xi}{\partial t} + \langle s \nabla c^\star \left[\nabla \left(F'(s) + V\right) \right], \nabla \xi \rangle \Big\} = \int_\Omega s_0(x)\, \xi(0,x)\, dx. \tag{3.50}$$

`Proof.` The proof of the uniqueness of solution is similar to that of Theorem 3.4.1. We prove here the existence of solution. Let $\xi \in C_c^2(I\!R \times I\!R^d)$ be such that $\operatorname{spt}\xi \subset [-T,T]$ for some $0 < T < \infty$, and set $\Omega_T := (0,T)\times\Omega$. Because of Proposition 2.5.1 and Proposition 3.1.2, we have that

$$\lim_{h\downarrow 0} \int_{\Omega_T} \Big\{ (s_0 - s^h)\, \partial_t^h \xi + \langle\, s^h \nabla c^\star \Big[\nabla \Big(F'(s^h) + V \Big) \Big], \nabla \xi\, \rangle \Big\} = 0. \tag{3.51}$$

We show that the following claim suffices to conclude Theorem 3.4.2.

Claim: For $0 < T < \infty$, the following estimates hold:

$$\| s^h \|_{L^\infty(R;L^\infty(\Omega))} \le \| s_0 \|_{L^\infty(\Omega)}, \tag{3.52}$$

$$\int_{\Omega_T} s^h \Big| \nabla \Big(F'(s^h) \Big) \Big|^{q^\star} \le M(\Omega, T, F, s_0, V, q, \alpha), \tag{3.53}$$

and the energy inequality in time-space

$$\int_{\Omega_\infty} \langle s^h \nabla \Big(F'(s^h) + V \Big), \nabla c^\star \Big[\nabla \Big(F'(s^h) + V \Big) \Big] \rangle\, u(t) \le \frac{1}{h} \int_{\Omega_h} [\, F(s_0) + s_0 V\,]\; u(t) + \int_{\Omega_\infty} \Big[F(s^h) + s^h V \Big]\; \partial_t^h\, u(t), \tag{3.54}$$

where u is a non-negative function in $C_c^\infty(I\!R)$.

Indeed, because of (3.52), there exists $s : [0,\infty) \times \Omega \to [0,\infty)$, such that

(iii) $(s^h)_h$ converges to s, weakly in $L^1(\Omega_T)$, for a subsequence.

As a consequence,

$$\lim_{h\downarrow 0}\int_{\Omega_T}(s_0-s^h)\,\partial_t^h\,\xi=\int_{\Omega_T}(s_0-s)\,\frac{\partial\xi}{\partial t}. \tag{3.55}$$

Using (3.52) and (3.53), we deduce the space-compactness and the time-compactness of $(s^h)_h$ in $L^1(\Omega_T)$, as in the case where $V=0$. Consequently,

(iv) $(s^h)_h$ converges to s, strongly in $L^1(\Omega_T)$, for a subsequence.

Then, we use (iv), the energy inequality in space-time (3.54), and we follow the lines of the proof of Theorem 3.3.3, where we use $F'(s^h)+V$ in place of $F'(s^h)$, and $F(s^h)+s^hV$ in place of $F(s^h)$, to conclude that

(v) $\left\{\nabla c^\star\left[\nabla\left(F'(s^h)+V\right)\right]\right\}_h$ converges weakly to $\nabla c^\star\left[\nabla\left(F'(s)+V\right)\right]$, in $L^q(\Omega_T)$, for a subsequence.

Hence,

$$\lim_{h\downarrow 0}\int_{\Omega_T}\langle s^h\nabla c^\star\Big[\nabla\Big(F'(s^h)+V\Big)\Big],\nabla\xi\rangle$$
$$=\int_{\Omega_T}\langle s\nabla c^\star\left[\nabla\left(F'(s)+V\right)\right],\nabla\xi\rangle. \tag{3.56}$$

We combine (3.51) and (3.55) - (3.56), to conclude (3.50).
As in Theorem 3.4.1, (i) follows directly from (3.52), (3.53) and the minimum principle of Proposition 2.2.1.

`Proof of the Claim.` (3.52) is a direct consequence of the maximum principle of Proposition 2.2.1.
Because of Proposition 2.3.1 and the maximum/minimum principle of Proposition 2.2.1, we have that $P(s_i^h)\in W^{1,\infty}(\Omega)$, and $\nabla\left(F'(s_i^h)\right)\in L^\infty(\Omega)$. Then, choosing $G:=F$ in Theorem 2.4.2, the energy inequalities (2.58) and (2.59) read as

$$\int_\Omega F(s_{i-1}^h)-\int_\Omega F(s_i^h)\geq\int_\Omega\langle\nabla\Big(F'(s_i^h)\Big),S_i^h(y)-y\rangle\,s_i^h(y)\,\mathrm{d}y,$$

and

$$\int_\Omega s_{i-1}^h\,V-\int_\Omega s_i^h\,V\geq\int_\Omega\langle\,\nabla V\,,\,S_i^hy-y\,\rangle\,s_i^h(y)\,\mathrm{d}y,$$

where S_i^h is the c_h-optimal map that pushes s_i^h forward to s_{i-1}^h. We add both of the subsequent inequalities, and we use (2.36), to have that

$$E(s_{i-1}^h)-E(s_i^h) \tag{3.57}$$
$$\geq h\int_{\Omega_T}\langle\nabla\Big(F'(s_i^h)+V\Big),\nabla c^\star\Big[\nabla\Big(F'(s_i^h)+V\Big)\Big]\rangle\,s_i^h,$$

for $i \in \mathbb{N}$. We sum (3.57) over i, and we use that V and $s^h_{T/h}$ are non-negative, and Jensen's inequality, to have that

$$h \int_{\Omega_T} \langle \nabla \left(F'(s^h) + V \right), \nabla c^\star \left[\nabla \left(F'(s^h) + V \right) \right] \rangle \, s^h \leq E(s_0) - |\,\Omega\,| \, F\left(\frac{1}{|\,\Omega\,|} \right).$$

We conclude, as in the proof of (3.7), that

$$\int_{\Omega_T} s^h \left| \nabla \left(F'(s^h) + V \right) \right|^{q^\star} \leq \bar{M}(\Omega, T, F, s_0, q, \alpha). \tag{3.58}$$

On the other hand, because of (3.52) and the fact that $V \in C^1(\bar{\Omega})$, we have that

$$\left\| (s^h)^{1/q^\star} \nabla V \right\|_{L^{q^\star}(\Omega_T)} \leq \|\, s_0 \,\|^{1/q^\star}_{L^\infty(\Omega)} \|\, \nabla V \,\|_{L^\infty(\Omega)}. \tag{3.59}$$

We combine (3.58) - (3.59), and we use Minkowski's inequality, to conclude (3.53).

The proof of (3.54) follows the lines of the proof of Lemma 3.3.2 where we use the free energy inequality (3.57) in place of the internal energy inequality (2.58) □

Chapter 4

Open problems

In this chapter, we comment on two open problems related to (1.5). The first problem deals with the asymptotic behavior of solutions to (1.5): *what is the equilibrium solution of (1.5)? And how fast do solutions of (1.5) converge to equilibrium?* In the second problem, we address the following question: *given two solutions s_1 and s_2 of (1.5), does a contraction principle in the Wasserstein metric hold for some cost function $\tilde{c}$, that is,*

$$W_{\tilde{c}}\left(s_1(t), s_2(t)\right) \leq f(t)\, W_{\tilde{c}}\left(s_1(0), s_2(0)\right),$$

where f is a positive, non-increasing function on $[0,\infty)$, and $W_{\tilde{c}} := W_{\tilde{c}}^h$ when $h = 1$?

4.1 Asymptotic behavior of solutions

In this section, we comment on the first problem. We show that (1.5) has an equilibrium solution s_∞, and we argue on why one should expect solutions of (1.5) to decay exponentially fast to s_∞.

4.1.1 Equilibrium solution

Conjecture 4.1.1 *Assume that $F : [0,\infty) \to I\!R$ is strictly convex and satisfies (HF1), and $V : \Omega \to [0,\infty)$ is convex. If $s_0 \in \mathcal{P}_a(\Omega)$ is such that $E(s_0) < \infty$, then*

$$(\mathcal{I}): \quad \inf\left\{ E(s) := \int_\Omega \left[\, F(s) + sV \,\right] dx : \quad s \in \mathcal{P}_a(\Omega) \right\}$$

has a unique minimizer s_∞.
Furthermore, if $F \in C^2\left((0,\infty)\right)$ and $V \in C^1(\Omega)$, then s_∞ satisfies

$$s_\infty \nabla \left(F'(s_\infty) + V\right) = 0. \tag{4.1}$$

The function s_∞ defined by (4.1) is called *the equilibrium solution* of (1.5). Here and after, we assume that s_∞ is positive, and therefore, satisfies

$$\nabla\left(F'(s_\infty)+V\right)=0. \tag{4.2}$$

`Proof.` Existence and uniqueness of the minimizer for $\mathcal{I}$ follow the lines of the proof of Proposition 2.1.1. To show (4.1), we proceed as in Proposition 2.3.1. Indeed, for $\psi\in C_c^\infty(\Omega,\mathbb{R}^d)$, we define ϕ_ϵ by (2.38), and s_ϵ to be the density function of $\mu_\epsilon:=(\phi_\epsilon)_\# s_\infty$. As in Proposition 2.3.1, we have that

$$\lim_{\epsilon\to 0}\int_\Omega\left[\frac{F(s_\epsilon)-F(s_\infty)}{\epsilon}+\frac{s_\epsilon-s_\infty}{\epsilon}V\right]\mathrm{d}x$$
$$=-\int_\Omega\left[\,P(s_\infty)\,\mathrm{div}\psi+s_\infty\langle\nabla V,\psi\rangle\,\right]\mathrm{d}x.$$

We use an integration by parts, and the fact that $\nabla\left(P(s_\infty)\right)=s_\infty\nabla\left(F'(s_\infty)\right)$, to deduce (4.1) □

4.1.2 Trend to equilibrium

We argue that the free energy $E\left(s(t)\right)$ of a solution s of problem (1.5) is a decreasing quantity, unless s coincides with the equilibrium solution s_∞, in which case the time derivative of $E\left(s(t)\right)$ vanishes. It then makes sense to expect solutions of (1.5) to decay to the equilibrium solution s_∞, in relative entropy. Using the generalized transport inequality established in [7], that is,

$$W_c\left(s,s_\infty\right)\le\frac{1}{\lambda}\left[E(s)-E(s_\infty)\right],$$

we expect to obtain a decay in the Wasserstein metric. It might also be possible to estimate the rate at which solutions tend to equilibrium, as we shall see in Conjecture 4.1.4. Following Cordero-Gangbo-Houdré [7], we introduce the *generalized relative Fisher information of s with respect to s_∞, measured against the cost $c^\star$*

$$I_{c^\star}(s/s_\infty):=\int_\Omega\langle\nabla\left(F'(s)-F'(s_\infty)\right),\nabla c^\star\left[\nabla\left(F'(s)-F'(s_\infty)\right)\right]\rangle\, s\,\mathrm{d}x.$$

As noted by the authors of [7], $I_{c^\star}(s/s_\infty)$ is actually the Fisher information

$$I(s/s_\infty)=\int_{R^d}\left|\,\nabla\ln\left(\frac{s}{s_\infty}\right)\,\right|^2 s\,\mathrm{d}x,$$

when $c(z)=\frac{|z|^2}{2}$ and $F(x)=x\ln(x)$. In this case, $s_\infty=\frac{e^{-V}}{\int_\Omega e^{-V}}$ is the Gibbs function, and the following well-known entropy dissipation equation

$$\frac{d}{dt}H(s)=-I(s(/s_\infty), \tag{4.3}$$

holds for the Fokker-Planck equation; here

$$H(\rho) := \int_\Omega (\rho \ln(\rho) + \rho V)\, \mathrm{d}x$$

is the entropy functional. We expect the following generalization of (4.3):

Conjecture 4.1.2 *(Energy dissipation)*
Assume that c, F and V satisfy the assumptions in Theorem 3.4.2. If s is a solution of (1.5), then

$$\frac{d}{dt} E(s(t)) = -I_{c^\star}(s/s_\infty) \leq 0. \tag{4.4}$$

`Sketch of proof.` Assume that s is smooth enough so that one can differentiate $E(s(t))$. We have that

$$\begin{aligned}
\frac{d}{dt} E(s(t)) &= \int_\Omega \left[F'(s) + V \right] \frac{\partial s}{\partial t}\, \mathrm{d}x \\
&= \int_\Omega \left[F'(s) + V \right] \operatorname{div}\left\{ s \nabla c^\star \left[\nabla \left(F'(s) + V \right) \right] \right\} \mathrm{d}x \\
&= -\int_\Omega \langle \nabla \left(F'(s) + V \right), \nabla c^\star \left[\nabla \left(F'(s) + V \right) \right] \rangle\, s\, \mathrm{d}x \\
&= -I_{c^\star}(s/s_\infty),
\end{aligned}$$

where we use (4.2) in the last equality. And since $s \geq 0$, and $\langle z, \nabla c^\star(z) \rangle \geq 0$ for $z \in \mathbb{R}^d$ (Proposition 5.3.3), we conclude that $I_{c^\star}(s/s_\infty) \geq 0$ □

Before stating our convergence results, we recall the *generalized transport inequality* and *Logarithmic-Sobolev inequality* of Cordero-Gangbo-Houdré [7]. These inequalities will be the key ingredients in the trend to equilibrium of the solutions to (1.5). Here, we will not assume that $\lambda = 1$, contrarily to [7]. This will allow us to obtain a sharper rate in the convergence to equilibrium, when the cost function is $c(z) = \frac{|z|^q}{q}$, $q > 1$. Following Cordero-Gangbo-Houdré [7], we assume that the potential V is *c-uniform convex*, that is,

$$V(b) - V(a) \geq \langle \nabla V(a), b - a \rangle + \lambda\, c(a - b), \tag{4.5}$$

for $a, b \in \Omega$, and for some $\lambda > 0$. When $c(z) = \frac{|z|^2}{2}$, (4.5) is equivalent to the uniform convexity of V, that is, $\mathrm{Hess}\,(V) \geq \lambda\, \mathrm{Id}$ if $V \in C^2(\Omega)$. Note that, since $\lambda c \geq 0$, (4.5) implies that V is convex, and so, our existence and uniqueness results of Theorems 3.4.1 and 3.4.2 still hold.

Proposition 4.1.3 *(Cordero-Gangbo-Houdré)*
Assume that $c : \mathbb{R}^d \to [0, \infty)$ is even, strictly convex, of class C^1, and

satisfies $c(0) = 0$ *and (HC4). Assume that* $F : [0,\infty) \to I\!R$ *satisfies* $F \in C\left([0,\infty)\right) \cap C^2\left((0,\infty)\right),\ F(0)=0,\ \lim_{t\to\infty} \frac{F(t)}{t} = \infty,$ *and* $(0,\infty) \ni t \mapsto t^d F(t^{-d})$ *is convex and non-increasing. Assume that* $V : \bar{\Omega} \to [0,\infty)$ *is of class* C^1 *and satisfies (4.5). If* $s_\infty \in \mathcal{P}_a(\Omega)$ *satisfies (4.2), then, the generalized transport inequality*

$$W_c(s, s_\infty) \le \frac{1}{\lambda}\,[\,E(s) - E(s_\infty)\,] \tag{4.6}$$

holds, for $s \in \mathcal{P}_a(\Omega)$. *In addition, if* $s \in W^{1,\infty}(\Omega)$ *and* $s > 0$, *then,*

$$E(s) - E(s_\infty) + \lambda\, W_c(s_\infty, s) \le \int_\Omega \langle Id - S, \nabla\left[\,F'(s) - F'(s_\infty)\,\right]\rangle\, s. \tag{4.7}$$

Here $W_c := W_c^h$ *when* $h = 1$, *and* S *is the c-optimal map, such that* $S_\# s = s_\infty$.
Furthermore,

(i). if c is homogeneous of degree $q > 1$, *that is,* $c(z) = \frac{|\,z\,|^q}{q}$, *then,*

$$E(s) - E(s_\infty) \le \frac{1}{q^\star \lambda^{q^\star - 1}}\, I_{c^\star}(s/s_\infty). \tag{4.8}$$

(ii). if $\lambda \ge 1$, *and c is arbitrary, then,*

$$E(s) - E(s_\infty) \le I_{c^\star}(s/s_\infty). \tag{4.9}$$

(4.8) and (4.9) are called *generalized Logarithmic Sobolev inequalities.*

`Proof.` Because of the energy inequalities (2.58) and (2.59), where use (4.5) in place of the convexity of V, we have that

$$E(\rho_0) - E(\rho_1) \ge \int_\Omega \langle \nabla\left(F'(\rho_1) + V\right), S - \mathrm{Id}\rangle\, \rho_1 + \lambda\, W_c(\rho_0, \rho_1), \tag{4.10}$$

for $\rho_0,\ \rho_1 \in \mathcal{P}_a(\Omega)$, where S is the c-optimal map pushing ρ_1 forward to ρ_0. We set $\rho_0 = s$ and $\rho_1 = s_\infty$ in (4.10), and we use (4.2), to conclude (4.6). Next, we set $\rho_0 = s_\infty$ and $\rho_1 = s$ in (4.10), and we use (4.2), to conclude (4.7).
Now, assume that $c(z) = \frac{|\,z\,|^q}{q}$ and set $\tilde{c}(z) = \lambda\, c$. Clearly,

$$\tilde{c}^\star(z) = \lambda\, c^\star\left(\frac{z}{\lambda}\right) = \frac{|\,z\,|^{q^\star}}{q^\star \lambda^{q^\star - 1}}, \tag{4.11}$$

and then,

$$I_{c^\star}(s/s_\infty) = \int_\Omega \left|\,\nabla\left(F'(s) - F'(s_\infty)\right)\,\right|^{q^\star} s\,. \tag{4.12}$$

We combine (4.7) and Young's inequality

$$\langle y, z\rangle \leq \tilde{c}(y) + \tilde{c}^{\star}(z), \tag{4.13}$$

to obtain that

$$\begin{aligned} & E(s) - E(s_{\infty}) + \lambda\, W_c(s_{\infty}, s) \\ & \quad \leq \int_{\Omega} \tilde{c}\,(\mathrm{Id} - S)\, s\, \mathrm{d}y + \int_{\Omega} \tilde{c}^{\star}\left(\nabla\left(F'(s) - F'(s_{\infty})\right)\right) s\, \mathrm{d}y. \end{aligned}$$

We use (4.11), (4.12) and the fact that c is even in the subsequent inequality, to conclude (4.8).
Next, assume that $\lambda \geq 1$ and c is arbitrary. Using Young's inequality (4.13) on the cost c, and the fact that c is even, we deduce from (4.7), that

$$E(s) - E(s_{\infty}) \leq (1-\lambda)\, W_c(s_{\infty}, s) + \int_{\Omega} c^{\star}\left(\nabla\left(F'(s) - F'(s_{\infty})\right)\right) s\, \mathrm{d}y.$$

And since $1-\lambda \leq 0$ and $\langle z, \nabla c^{\star}(z)\rangle \geq c^{\star}(z)$ (see Proposition 5.3.3), we conclude (4.9) □

Conjecture 4.1.4 *(Trend to equilibrium)*
In addition to the hypotheses of Theorem 3.4.2, assume that c is even. Let s be a solution of (1.5).

(i). If $c(z) = \frac{|z|^q}{q}$, $q > 1$, then,

$$\begin{aligned} W\left(s(t), s_{\infty}\right) & \leq \frac{1}{\lambda}\left[E\left(s(t)\right) - E(s_{\infty})\right] \\ & \leq \frac{1}{\lambda}\, e^{-(q^{\star}\lambda^{q^{\star}-1})\, t}\left[E(s_0) - E(s_{\infty})\right], \end{aligned} \tag{4.14}$$

for $t \in [0, \infty)$.

(ii). If $\lambda \geq 1$, and c is arbitrary, then,

$$\begin{aligned} W\left(s(t), s_{\infty}\right) & \leq \frac{1}{\lambda}\left[E\left(s(t)\right) - E(s_{\infty})\right] \\ & \leq \frac{1}{\lambda}\, e^{-t}\left[E(s_0) - E(s_{\infty})\right], \end{aligned} \tag{4.15}$$

for $t \in [0, \infty)$.

`Sketch of proof.` Because of (4.4) and (4.8), we have that

$$\frac{d}{dt}\left[E\left(s(t)\right) - E(s_{\infty})\right] \leq -q^{\star}\lambda^{q^{\star}-1}\left[E\left(s(t)\right) - E(s_{\infty})\right],$$

which reads as

$$\frac{d}{dt}\left\{ e^{(q^\star \lambda^{q^\star-1})\,t}\,\left[\,E\left(s(t)\right)-E(s_\infty)\,\right]\right\}\leq 0,$$

We combine (4.6) and the subsequent inequality, to conclude (4.14). The proof of (4.15) is similar □

Remark 4.1.5 *Conjecture 4.1.4 extends a result of Carillo-Jüngel-Markowich-Toscani-Unterreiter [5], which states that:*

if $s_0 \in \mathcal{P}_a(\Omega)\cap L^\infty(\Omega)$, *and* $V:\mathbb{R}^d\to\mathbb{R}$ *and* $f:[0,\infty)\to\mathbb{R}$ *satisfy:*

(H1)*:* $\inf_\Omega V = 0,\ V = W/_\Omega,$ *where* $W\in C^2(\mathbb{R}^d,\mathbb{R})$, *and* $Hess(W)\geq \lambda\, Id$, *for some* $\lambda\geq 0$,

(H2)*:* f *is continuous, strictly increasing,* $f(0)=0$, *and* $f/_{(0,\infty)}\in C^3\left((0,\infty)\right)$,

(H3)*:* $f(u)\leq \frac{d}{d-1}\,uf'(u)$, *for* $u>0$,

and some technical assumptions on f, *then the solution* s *of*

$$\begin{cases} \frac{\partial s}{\partial t} = div\left(s\nabla V+\nabla\left(f(s)\right)\right) & on \quad \Omega\times(0,\infty)\\[2ex] s(t=0)=s_0 & on \quad \Omega\\[2ex] \left(s\nabla V+\nabla\left(f(s)\right)\right)\cdot\nu = 0 \end{cases} \tag{4.16}$$

satisfies

$$E\left(s(t)\right)-E(s_\infty)\leq e^{-2\lambda\,t}\,\left[\,E(s_0)-E(s_\infty)\,\right], \tag{4.17}$$

for $t\geq 0$.

Notice that (4.16) is equivalent to (1.5) when $c(z)=\frac{|z|^2}{2}$ and $F''(x)=\frac{f'(x)}{x}$, $x>0$, and the statement (4.17) is included in (4.14) for that uqadratic cost function c. Now, we show that, under assumptions **(H1) - (H3)**, Conjecture 4.1.4 still holds. Indeed, because of **(H2)**, and the fact that $F''(x)=\frac{f'(x)}{x}$, we have that $F\in C\left([0,\infty)\right)\cap C^4\left((0,\infty)\right)$ is strictly convex. Define the pressure

$$P(x):=\begin{cases} xF'(x)-F(x), & x>0\\ 0 & x=0,\end{cases}$$

and set

$$(0,\infty)\ni x\mapsto A(x):=x^d\,F(x^{-d}).$$

Because of **(H2)**, we have that $P = f$, and by direct computations, we show that

$$A'(x) = -dx^{d-1} f(x^{-d}),$$

and

$$A''(x) = d^2 x^{d-2} \left[x^{-d} f(x^{-d}) - \left(1 - \frac{1}{d}\right) f(x^{-d}) \right].$$

We use **(H2)** and **(H3)**, to obtain that $A'(x) < 0$ and $A''(x) > 0$, for $x > 0$. This shows that A is convex and non-increasing. Then, we can apply Conjecture 4.1.4, to deduce (4.17) □

4.2 Contraction in the Wasserstein metric

To avoid technicalities, we assme that the potential V is zero. Let s_1 and s_2 be two solutions of (1.5) in the sense of Theorem 3.4.1, and set $W_{\tilde{c}} = W_{\tilde{c}}^h$, where $h = 1$ and $\tilde{c} : \mathbb{R}^d \to \mathbb{R}$ is an arbitrary convex cost function. We address the following question, which ensures uniqueness of the solution to (1.5) : *does the contraction principle*

$$W_{\tilde{c}}\left(s_1(t), s_2(t)\right) \le W_{\tilde{c}}\left(s_1(0), s_2(0)\right) \tag{4.18}$$

hold for $t \in [0, \infty)$*?*

`Sketch of proof.` For $t \in (0, \infty)$ fixed, denote by γ_t the $\tilde{c}$-optimal measure in $\Gamma\left(s_1(t), s_2(t)\right)$, and let $\tau \in [t, \infty)$. Since s_j, $j = 1, 2$, satisfies

$$\begin{cases} \frac{\partial s_j}{\partial t} + \operatorname{div}\left(s_j\, U_{s_j}\right) = 0 \\ s_j(\tau = t) = s_j(t) \\ U_{s_j} := -\nabla c^\star \left[\, \nabla \left(F'(s_j)\right) \right], \end{cases} \tag{4.19}$$

we have, using the relation between Lagrangian and Eulerian descriptons, that

$$s_j(\tau) = \left(\Phi_j(\tau)\right)_{\#} s_j(t), \tag{4.20}$$

where the trajectories $\Phi_j(\tau)$ are defined by

$$\begin{cases} \frac{\partial \Phi_j(\tau)}{\partial \tau} = U_{s_j}\left(\tau, \Phi_j(\tau)\right) \\ \Phi_j(t) = \mathrm{id}_\Omega. \end{cases} \tag{4.21}$$

Because of (4.20), $\gamma_\tau := \left(\Phi_1(\tau) \times \Phi_2(\tau)\right)_{\#} \gamma_t \in \Gamma\left(s_1(\tau), s_2(\tau)\right)$, and then, we have that

$$\frac{d}{d\tau}\Big/_{\tau = t} W_{\tilde{c}}\left(s_1(\tau), s_2(\tau)\right)$$

$$= \lim_{\tau \downarrow t} \frac{W_{\tilde{c}}\left(s_1(\tau), s_2(\tau)\right) - W_{\tilde{c}}\left(s_1(t), s_2(t)\right)}{\tau - t}$$
$$\leq \lim_{\tau \downarrow t} \frac{1}{\tau - t} \int_{\Omega\times\Omega} \left[\, \tilde{c}\left(\Phi_1(\tau,x) - \Phi_2(\tau,y)\right) - \tilde{c}(x-y) \,\right] \mathrm{d}\gamma_t(x,y).$$

We use (4.21), and the fact that $\tilde{c}$ is convex, to have that

$$\frac{d}{d\,\tau}\Big/_{\tau=t} W_{\tilde{c}}\left(s_1(\tau), s_2(\tau)\right)$$
$$\leq \int_{\Omega\times\Omega} \langle \nabla\tilde{c}(x-y), U_{s_1}(t,x) - U_{s_2}(t,y)\rangle \,\mathrm{d}\gamma_t(x,y). \tag{4.22}$$

We combine (4.19) and (4.22), to conclude that

$$\frac{d}{d\,\tau}\Big/_{\tau=t} W_{\tilde{c}}\left(s_1(\tau), s_2(\tau)\right) \leq \tag{4.23}$$
$$- \int_{\Omega\times\Omega} \langle \nabla\tilde{c}(x-y), \nabla c^\star \left[\, \nabla\left(F'(s_1(t,x))\right) \right] - \nabla c^\star \left[\, \nabla\left(F'(s_2(t,x))\right) \right] \rangle \,\mathrm{d}\gamma_t(x,y).$$

Furhermore, because of the energy inequality (2.58), we have that

$$\begin{aligned} \int_\Omega F\left(s_1(\tau)\right) - \int_\Omega F\left(s_2(\tau)\right) \quad &\geq \quad \int_\Omega \langle \nabla\left(F'(s_2(\tau))\right), S - \mathrm{id}\rangle\, s_2(\tau) \qquad (4.24) \\ &= \quad \int_{\Omega\times\Omega} \langle \nabla\left(F'(s_2(\tau,y))\right), x - y\rangle \,\mathrm{d}\gamma_t(x,y), \end{aligned}$$

and similarly,

$$\int_\Omega F\left(s_2(\tau)\right) - \int_\Omega F\left(s_1(\tau)\right) \geq \int_{\Omega\times\Omega} \langle \nabla\left(F'(s_1(\tau,x))\right), y - x\rangle \,\mathrm{d}\gamma_t(x,y). \tag{4.25}$$

We add (4.24) and (4.25), to obtain that

$$\int_{\Omega\times\Omega} \langle x - y, \nabla\left(F'(s_2(\tau,y))\right) - \nabla\left(F'(s_1(\tau,x))\right)\rangle \,\mathrm{d}\gamma_t(x,y) \leq 0. \tag{4.26}$$

`Case 1:` $c(z) = \frac{|z|^2}{2}$

We set $\tilde{c}(z) = \frac{|z|^2}{2}$, combine (4.23) and (4.26), and use that $\nabla\tilde{c} = \nabla c^\star = \mathrm{id}_{R^d}$, to have that

$$\frac{d}{d\,\tau}\Big/_{\tau=t} W_2\left(s_1(\tau), s_2(\tau)\right) \leq 0.$$

This shows that

$$W_2\left(s_1(\tau), s_2(\tau)\right) \leq W_2\left(s_1(0), s_2(0)\right), \tag{4.27}$$

for $t \in [0, \infty)$. Hence, one expects the contraction principle (4.27) to hold for the class of PDE

$$\frac{\partial s}{\partial t} = \mathrm{div}\,\{s\nabla\left(F'(s)\right)\},$$

which includes the heat equation and the porous medium equation. A similar result may be obtained for the Fokker-Planck type equation

$$\frac{\partial s}{\partial t} = \mathrm{div}\,\{s\nabla\left(F'(s) + V\right)\}, \tag{4.28}$$

if one removes the assumption that $V = 0$. Indeed, assume that the potential V is uniformly convex with $\mathrm{Hess}(V) \geq \lambda\,\mathrm{id}$. Following the arguments presented above, and using the (potential) energy inequality

$$\int_\Omega s_1(\tau)V - \int_\Omega s_2(\tau)V \geq \int_\Omega \langle\nabla V(y), x - y\rangle\,\mathrm{d}\gamma_t(x,y) \;+\; \lambda\,W_2\left(s_1(\tau), s_2(\tau)\right),$$

we have that

$$\frac{d}{d\tau}\Big/_{\tau=t} W_2\left(s_1(\tau), s_2(\tau)\right) \leq 2\lambda\,W_2\left(s_1(\tau), s_2(\tau)\right).$$

We deduce the following contraction principle for (4.28)

$$W_2\left(s_1(\tau), s_2(\tau)\right) \leq e^{-2\lambda t}\,W_2\left(s_1(0), s_2(0)\right). \tag{4.29}$$

In particular, if s_2 coincides with the equilibrium solution s_∞ of (4.28), (4.29) reads as

$$W_2\left(s_1(t), s_\infty\right) \leq e^{-2\lambda t}\,W_2\left(s_1(0), s_\infty\right), \tag{4.30}$$

which shows that solutions of (4.28) decay exponentially fast to the equilibrium solution s_∞ with the rate 2λ. A similar result was obtained by Otto [18] when $F(x) = \frac{x^m}{m-1}$. We expect that (4.30) would extend Otto's result to more general energy density functions F.

`Case 2:` General cost functions

When the cost function c is not homogeneous of degree 2 (the case of the p-Laplacian equation, for example), we have not been able to establish (4.18). The difficulty here, is due to the fact that we could not compare the term on the right hand side of (4.23) and that on the left hand side of (4.26), which are actually identical when $c(z) = \frac{|z|^2}{2}$. We note that this difficulty does not come from the infinite-dimensional character of equation (1.5), but, it is due to the form of that equation. Indeed, consider a finite-dimensional ODE type of (1.5) when $V = 0$, that is,

$$\dot{x}(t) = -\nabla c^\star\left[\,\nabla f\left(x(t)\right)\right], \tag{4.31}$$

where $[0,\infty) \ni t \mapsto x(t) \in I\!\!R^d$, and $f : I\!\!R^d \to I\!\!R$ is convex. By a direct computation using (4.31), we have that

$$\frac{d}{dt}\left[\,\tilde{c}\,(x(t)-y(t))\,\right] = -\langle \nabla\tilde{c}(x-y), \nabla c^\star\,(\nabla f(x)) - \nabla c^\star\,(\nabla f(y))\rangle,$$

where $\tilde{c} : I\!\!R^d \to [0,\infty)$ is an arbitrary convex cost function. It clearly appears that, when c is not homogeneous of degree 2, the mere convexity of f is not sufficient to ensure that $\tilde{c}\,(x(t), y(t))$ is non-increasing in t, even if $\tilde{c} = c$ or $\tilde{c} = \frac{|z|^2}{2}$. Consequently, there is not much hope to expect (4.18) to hold when the cost function c is not homogeneous of degree 2. Still, one can ask the following questions: *are there possible physically acceptable conditions to impose on f, so that (4.18) holds for conveniently chosen cost functions $\tilde{c}$? Or, can one establish a contraction principle with another metric equivalent to the Wasserstein metric?*

Chapter 5

Appendix

In this chapter, we collect results of previous authors used in this work, and we establish intermediate results needed in the previous chapters. Throughout this chapter, $c : \mathbb{R}^d \to [0,\infty)$ is a convex function, and μ and ν are Borel probability measures on $\mathbb{R}^d$. We denote by $S := \mathrm{id} + \nabla c^\star(-\nabla v)$ the c-optimal map that pushes ν forward to μ. Here v is a c-concave function, that is,

$$v(y) := u^c(y) = \inf_{x \in R^d} \{ c(x-y) - u(x) \},$$

where $u : \mathbb{R}^d \to \mathbb{R}$ is a measurable function. We recall that $S^{-1} := T = \mathrm{id} + \nabla c^\star(\nabla u)$ is the c-optimal map that pushes μ forward to ν, and $u = v^c$. For $t \in [0,1]$, we define the interpolant map

$$S_t := (1-t)\mathrm{id} + tS,$$

the interpolant measure

$$\mu_{1-t} := S_{t\#}\nu,$$

and the time-dependent cost function

$$c_{(t)}(z) := t\, c\left(\frac{-z}{t}\right),$$

for $z \in \mathbb{R}^d$.

5.1 Optimality of the interpolant map

In this section, we recall a result due to Villani [23], that is, S_t is the $c_{(t)}$-optimal map that pushes ν forward to μ_{1-t}, for $t \in (0,1]$. Following Villani [23], we introduce the time-dependent Monge problem $(TDMP)_\tau$, and the time-independent Monge problem $(TIMP)_\tau$, where $\tau \in (0,1]$:

$$(TDMP)_\tau : \qquad \inf \Big\{ \int_{R^d} \int_0^\tau c\,(\dot{z}_t(y))\, \mathrm{d}t\, \mathrm{d}\nu(y) : \text{ for a.e. } y,\ [0,\tau] \ni t \mapsto z_t(y) \text{ is } C^0,\ \text{piecewise } C^1,\ z_0(y) = y \text{ and } z_{\tau\#}\nu = \mu_{1-\tau} \Big\},$$

and

$$(TIMP)_\tau : \quad \inf\left\{ \int_{R^d} c_{(\tau)}\,(y - A(y))\,\mathrm{d}\nu(y) : \quad A_{\#}\nu = \mu_{1-\tau} \right\}.$$

The following proposition is the first step toward showing the relation between $(TDMP)_\tau$ and $(TIMP)_\tau$.

Proposition 5.1.1 *Let $x, y \in \mathbb{R}^d$ and $\tau \in (0,1]$. Then*

(i).

$$(\mathcal{C})_\tau : \qquad \inf\Big\{ \int_0^\tau c\,(\dot z_t(y))\,dt : \text{ for a.e. } y,\ [0,\tau] \ni t \mapsto z_t(y) \text{ is } C^0,\ \text{piecewise } C^1,\ z_0(y) = y \text{ and } z_\tau(y) = x \Big\} = \tau\, c\left(\frac{x-y}{\tau}\right) = \tau\, c\left(\frac{z_\tau(y) - y}{\tau}\right).$$

Therefore, $(TDMP)_\tau$ and $(TIMP)_\tau$ have the same total cost.

(ii). If c is strictly convex, the infimum in $(\mathcal{C}_\tau)$ is uniquely attained at $z_t(y) := y + \frac{t}{\tau}(x-y)$.

`Proof.` Because of Jensen's inequality, we have that

$$\int_0^\tau c\,(\dot z_t(y))\,\mathrm{d}\,t \ge \tau\, c\left(\frac{1}{\tau}\int_0^\tau \dot z_t(y)\,\mathrm{d}t\right) = \tau\, c\left(\frac{x-y}{\tau}\right), \qquad (5.1)$$

for all paths $[0,\tau] \ni t \mapsto z_t(y)$. Moreover, the path $t \mapsto z_t(y) := y + \frac{t}{\tau}(x-y)$ is admissible in $(\mathcal{C}_\tau)$, and

$$\int_0^\tau c\,(\dot z_t(y))\,\mathrm{d}\,t = \tau\, c\left(\frac{x-y}{\tau}\right). \qquad (5.2)$$

We combine (5.1) and (5.2), to conclude (i).
If c is strictly convex, the infimum in $(\mathcal{C}_\tau)$ is unique. Because of (5.2), it is attained at $z_t(y) := y + \frac{t}{\tau}(x-y)$ □

Corollary 5.1.2 *Let $\tau \in (0,1]$.*

(i). If $[0,\tau] \ni t \mapsto z_t(y)$ is an infimum of $(TDMP)_\tau$, then for ν a.e., $t \mapsto z_t(y)$ is also an infimum of $(\mathcal{C}_\tau)$.

(ii). Assume that c is strictly convex. If $[0,\tau] \ni t \mapsto z_t(y)$ is the unique minimizer for $(TDMP)_\tau$, then z_τ is the unique minimizer for $(TIMP)_\tau$.

`Proof.` (i) Let z_t be an infimum of $(TDMP)_\tau$. If there is an admissible path Z_t in $(C)_\tau$, such that

$$\int_0^\tau c\left(\dot{Z}_t(y)\right)\,\mathrm{d}t < \int_0^\tau c\left(\dot{z}_t(y)\right)\,\mathrm{d}t,$$

we will have that

$$\int_{R^d}\int_0^\tau c\left(\dot{Z}_t(y)\right)\,\mathrm{d}t < \int_{R^d}\int_0^\tau c\left(\dot{z}_t(y)\right)\,\mathrm{d}t,$$

which yields a contradiction.
(ii) Now assume that c is strictly convex, and let $[0,\tau] \ni t \mapsto z_t$ be a miminizer for $(TDMP)_\tau$. Because of the following relaxed formulation of $(TDMP)_\tau$,

$$\inf\Big\{\int_{R^d\times R^d}\int_0^\tau c\left(\dot{z}_t(x,y)\right)\,\mathrm{d}\,t\mathrm{d}\gamma(x,y):\ \text{for a.e. } y,\ [0,\tau]\ni t\mapsto z_t(x,y)$$
$$\text{is } C^0,\ \text{piecewise } C^1,\ z_0(x,y)=x,\ z_\tau(x,y)=y \text{ and } \gamma\in\Gamma(\nu,\mu_{1-\tau})\Big\},$$

we have that z_t is the unique minimizer for $(TDMP)_\tau$. By Proposition 5.1.1 - (i), $(TDMP)_\tau$ and $(TIMP)_\tau$ have the same total cost, and by Proposition 5.1.1 and Corollary 5.1.2 - (i), the total cost associated with $(TDMP)_\tau$ is

$$\int_{R^d}\tau\, c\left(\frac{z_\tau(y)-y}{\tau}\right)\mathrm{d}\,\nu(y) = \int_{R^d} c_{(\tau)}\left(y-z_\tau(y)\right)\,\mathrm{d}\nu(y).$$

We deduce that

$$\inf\Big\{\int_{R^d} c_{(\tau)}\left(y-A(y)\right)\,\mathrm{d}\nu(y):\quad A_\#\nu=\mu_{1-\tau}\Big\}$$
$$=\int_{R^d} c_{(\tau)}\left(y-z_\tau(y)\right)\,\mathrm{d}\nu(y).$$

Hence, z_τ is the unique minimizer for $(TIMP)_\tau$ □

Theorem 5.1.3 *Assume that c is strictly convex. For all $\tau\in(0,1]$, the path $[0,\tau]\ni t\mapsto S_t(y):=y+t\nabla c^\star\left(-\nabla v(y)\right)$ is the unique minimizer for $(TDMP)_\tau$. Therefore, S_τ is the $c_{(\tau)}$-optimal map that pushes ν forward to $\mu_{1-\tau}$.*

`Proof.` We distinguish two cases.
Case 1: $\tau=1$
By Proposition 5.1.1 and Corollary 5.1.2, the minimizer $[0,1]\ni t\mapsto z_t$ for $(TDMP)_1$ satisfies

$$z_t(y)=y+t\left(z_1(y)-y\right). \tag{5.3}$$

Since $S(y) = y + \nabla c^\star(-\nabla v(y))$ is the unique minimizer for $(TIMP)_1$, Corollary 5.1.2 - (ii) gives that

$$z_1(y) = S(y) = y + \nabla c^\star(-\nabla v(y)). \tag{5.4}$$

We combine (5.3) and (5.4), to conclude that $z_t = S_t$, for $t \in [0,1]$.
<u>Case 2:</u> $0 < \tau < 1$
Assume by contradiction that there is a strictly better path $[0,\tau] \ni t \mapsto \tilde{z}_t$ than $[0,\tau] \ni t \mapsto S_t$, and define on $[0,1]$ the path

$$z_t := \begin{cases} \tilde{z}_t & \text{if} \quad t \in [0,\tau] \\ S_t & \text{if} \quad t \in [\tau,1] \end{cases}$$

Clearly, z_t is admissible in $(TDMP)_1$, and

$$\begin{aligned} \int_{R^d}\int_0^1 c(\dot{z}_t(y))\,\mathrm{d}t\,\mathrm{d}\nu(y) &= \int_{R^d}\int_0^\tau c\left(\dot{\tilde{z}}_t(y)\right)\mathrm{d}t\,\mathrm{d}\nu(y) \\ &\quad + \int_{R^d}\int_\tau^1 c\left(\dot{S}_t(y)\right)\mathrm{d}t\,\mathrm{d}\nu(y) \\ &< \int_{R^d}\int_0^1 c\left(\dot{S}_t(y)\right)\mathrm{d}t\,\mathrm{d}\nu(y), \end{aligned}$$

which contradicts the fact that $[0,1] \ni t \mapsto S_t$ is optimal in $(TDMP)_1$. Hence, $[0,\tau] \ni t \mapsto S_t$ is the minimizer for $(TIMP)_\tau$, for all $\tau \in (0,1]$.
Now, we use Corollary 5.1.2 - (ii), to deduce that S_τ is the unique minimizer for $(TIMP)_\tau$. □

5.2 Jacobian equations for optimal transport maps

In this section, we collect some results established by Cordero [6], which are needed in this work, and we sketch their proofs. In fact, we give a precise meaning to $\nabla S(y)$, and we establish few properties of this matrix. In addition, we show that the interpolant measure μ_{1-t} is absolutely continuous with respect to Lebesgue, and ∇S_t satisfies similar properties as ∇S. Following Cordero [6], we introduce the definitions of the terms we shall need in this section.

Definitions

- A multivalued mapping $T : I\!\!R^d \to I\!\!R^d$ is an application from $I\!\!R^d$ to the subsets of $I\!\!R^d$. We also denote it by

$$T : D(T) \to T(D(T)),$$

where

$$D(T) := \{\, x \in I\!\!R^d : \ T(x) \neq \emptyset \,\}$$

is called the domain of definition of T, and

$$T\left(D(T)\right) := \bigcup_{x \in D(T)} T(x).$$

- The uniqueness domain of the multivalued mapping T is defined by

$$\mathrm{dom}\,(T) := \{\, x \in D(T) : \ T(x) \text{ is a singleton} \,\}.$$

  Hence, $T : \mathrm{dom}\,(T) \to \mathbb{R}^d$ is a single-valued mapping.

- The inverse of a multivalued mapping $T : D(T) \to T\left(D(T)\right)$ is the multivalued mapping $T^{-1} : T\left(D(T)\right) \to D(T)$, defined by

$$T^{-1}(y) := \{\, x \in D(T) : \ y \in T(x) \,\}.$$

- A multivalued mapping $T : D(T) \to T\left(D(T)\right)$ is differentiable at $x \in D(T)$, if $x \in (\mathrm{dom}\,(T)) \cap (\mathrm{int}\,(D(T)))$, and there exists a linear map $d\,T_x : \mathbb{R}^d \to \mathbb{R}^d$, such that

$$\sup_{y \in T(x+u)} |\, y + Tx - d\,T_x(u) \,| = o(|\, u \,|).$$

  $d\,T_x$ is called the differential of T at x.

Theorem 5.2.1 *(Jacobian equation for optimal transport maps)* *Assume that c is strictly convex and $c, c^\star \in C^2(\mathbb{R}^d)$. Assume that μ and ν are compactly supported in $\bar{\Omega}$, and are absolutely continuous with respect to Lebesgue, with f and g as their respective density functions. Then, there is a subset K of Ω, of full measure for ν, such that the followings hold for $y \in K$*

(i). *$\nabla S(y)$ is well defined, and is diagonalizable with positive eigenvalues.*

(ii). *The pointwise Jacobian $det(\nabla S)$ satisfies*

$$0 \neq g(y) = f\left(S(y)\right)\, det\, \left(\nabla S(y)\right).$$

In addition, if $g > 0$ a.e., then

(iii). *the pointwise divergence $div(S)$ is integrable on Ω, and*

$$\int_\Omega div\,(Sy - y)\,\xi(y)\,dy \leq -\int_\Omega \langle\, Sy - y\,,\, \nabla\xi \,\rangle\, dy,$$

for $\xi \geq 0$ in $C_c^\infty(\mathbb{R}^d)$.

`Sketch of proof.` Recall that $S(y) = y + \nabla c^\star\left(-\nabla v(y)\right)$, where $v : \bar\Omega \to \bar\Omega$ is c-concave, that is,

$$v(y) := u^c(y) = \inf_{x \in \bar\Omega} \{c(x-y) - u(x)\}$$

for $y \in \bar\Omega$, and $u : \bar\Omega \to \bar\Omega$ is a measurable function. Because $c \in C^2(\mathbb{R}^d)$ is c-concave, and $\bar\Omega$ is compact, we have that

(iv) v is semi-concave on Ω.

Let ∂v denote the super-differential of v, that is, the multivalued mapping $\partial v : D(\partial v) = \Omega \to \mathbb{R}^d$, defined by

$$\partial v(y) := \{\, l \in \mathbb{R}^d : \; v(y+z) - v(y) \le \langle l, z\rangle + o(|\,z\,|)\,\}.$$

Because of (iv), Aleksandrov's theorem gives that

(v) there exists $K_1 \subset \Omega$, of full Lebesgue measure, such that, ∂v is differentiable at $y \in K_1$, and $D^2 v(y) := d(\partial v)_y$ is a symmetric matrix.

We use (v), and the fact that $c^\star \in C^2(\mathbb{R}^d)$, to conclude that

(vi) the multivalued mapping $\tilde S : \Omega \ni y \to y + \nabla c^\star\left(-\partial v(y)\right)$ is differentiable at $y \in K_1$, and

$$(d\,\tilde S)_y = \mathrm{id} - D^2 c^\star\left(-\partial v(y)\right) D^2 v(y).$$

Now, consider the c-super-differential $\partial^c v$ of v, that is, the multivalued mapping $\partial^c v : D(\partial^c v) = \bar\Omega \to \bar\Omega$, defined by

$$\begin{aligned} \partial^c v(y) &:= \{\, x \in \bar\Omega : \; v(y) + v^c(x) = c(x-y)\,\} \\ &= \{\, x \in \bar\Omega : \; c(x-y) - v(y) \le c(x-z) - v(z), \ \forall\, z \in \bar\Omega\,\}. \end{aligned}$$

Since c and $c^\star$ are continuously differentiable, and c is convex, we have that

$$\partial^c v(y) \subset y + \nabla c^\star\left(-\partial v(y)\right) = \tilde S(y). \tag{5.5}$$

Furthermore, because v is semi-concave, Rademacher's theorem gives that

(vii) the set $D^1 v \subset \Omega$ where v is differentiable, is a set of full Lebesgue measure.

Moreover, we have that

$$K_1 \subset D^1 v, \quad \text{and} \quad \partial^c v(y) = \{S(y)\}, \ \ \forall\, y \in D^1 v. \tag{5.6}$$

We combine (v) - (vii), (5.5) - 5.6, and we use the fact that ν is absolutely continuous with respect to Lebesgue, to obtain that $\nu(K_1) = 1$, and

$$\nabla S(y) := d(\tilde{S})_y = \mathrm{id} + D^2 c^\star \left(-\nabla v(y)\right) D^2 v(y),$$

for $y \in K_1$. We set

$$K := \{\, y \in K_1 : \ \partial u = \partial v^c \text{ is differentiable at } S(y),\ y \text{ is a Lebesgue point of } g,\ S(y) \text{ is a Lebesgue point of } f, \text{ and } g(y) \neq 0 \,\}.$$

Since $\nu(K_1) = 1$, $T := S^{-1} = \mathrm{id} + \nabla c^\star(\nabla u)$ is the c-optimal map, such that $T_{\#}\mu = \nu$, and the Lebesgue points of f and g are sets of full Lebesgue measure, we have that $\nu(K) = 1$. Moreover, for $y \in K$, $-D^2 v(y)$ and $D^2 c^\star \left(-\nabla v(y)\right)$ are respectively symmetric and symmetric positive definite matrices, and because of the c-concavity of v, we have that

$$D^2 c^\star \left(-\nabla v(y)\right) \geq D^2 v(y).$$

We deduce that $\nabla S(y)$ is diagonalizable with non-negative eigenvalues (see Lemma A.4, [15]). Interchanging S and T in the above argument, we conclude that $\nabla S(y)$ is invertible, and therefore, $\nabla S(y)$ has positive eigenvalues. This proves (i).

Now, let $y \in K$. Because μ is absolutely continuous with respect to Lebesgue, $S(y) = \partial^c v(y)$, and $\mu = S_{\#}\nu$, we have that

$$\begin{aligned} f\left(S(y)\right) &= \lim_{r \to 0} \frac{\mu\left[\, B_r\left(S(y)\right) \right]}{\mathrm{vol}\left[\, B_r\left(S(y)\right) \right]} \\ &= \lim_{r \to 0} \frac{\nu\left[\, S^{-1}\left(B_r(Sy)\right) \right]}{\mathrm{vol}\left[\, S^{-1}\left(B_r(Sy)\right) \right]} \lim_{r \to 0} \frac{\mathrm{vol}\left[\, (\partial^c v)^{-1}\left(B_r(Sy)\right) \right]}{\mathrm{vol}\left[\, B_r(Sy) \right]}. \end{aligned}$$

We use that the density function of ν is g, and that $(\partial^c v)^{-1} = \partial^c u$ is differentiable at $S(y)$ with differential $d\left((\partial^c v)^{-1}\right)_{S(y)} = \left[\, d(\partial^c v)_y \,\right]^{-1}$, to conclude that

$$f\left(S(y)\right) = \frac{g(y)}{\det\left[\, d(\partial^c v)_y \,\right]}.$$

Hence,

$$0 \neq g(y) = f\left(S(y)\right) \det\left[\, d(\partial^c v)_y \,\right] = f\left(S(y)\right) \det\left(\nabla S(y)\right).$$

The proof of (iii) follows the ideas of the proof of Proposition A.4 - (c) [15]. Indeed, let $\left(S^{(\epsilon)}\right)_{\epsilon \downarrow 0}$ denote the approximate sequence to S, constructed as in [15]. We have that $\left(S^{(\epsilon)} - \mathrm{id}\right) \to (S - \mathrm{id})$ in $L^1(\Omega)$, and $\mathrm{div}\left(S^{(\epsilon)}(y) - \mathrm{id}\right) \to \mathrm{div}\left(S(y) - \mathrm{id}\right)$ for $y \in K$, and K is a set of full measure for $\nu = g(y)\,\mathrm{d}y$. Since $g > 0$ a.e. and $g_{/K^c} = 0$, we deduce that K is of full measure Lebesgue measure. Hence, we follow the lines of the proof of

Proposition A.4 - (c) [15], to conclude (iii) □

The next theorem asserts that, the interpolant map S_t, viewed as a multivalued mapping, is injective on its uniqueness domain, and the interpolant measure μ_{1-t} is absolutely continuous with respect to Lebesgue. For the proof of this theorem, we refer to [6].

Theorem 5.2.2 *(Injectivity and density of the interpolant map)*
Assume that c is strictly convex. Then,

(i). for $t \in (0,1)$, the multivalued mapping $M_t := (1-t)\,id + t\,\partial^c v$ is injective on $dom(M_t) = dom(\partial^c v)$. Therefore, if $c \in C^2(\mathbb{R}^d)$, the single-valued mapping $S_t = (M_t)_{/D^1 v}$ is injective. Here, $D^1 v \subset \Omega$ is the set where v is differentiable, which is clearly a set of full Lebesgue measure.

(ii). Furthermore, if $c,\, c^\star \in C^2(\mathbb{R}^d)$, and μ and ν have compact support in $\bar{\Omega}$ and are absolutely continuous with respect to Lebesgue, then, the interpolant measure $\mu_{1-t} = S_{t\#}\nu$ is absolutely continuous with respect to Lebesgue.

As a corollary of Theorem 5.2.1 and Theorem 5.2.2, we have the following Jacobian equation for the interpolant map S_t.

Corollary 5.2.3 *(Jacobian equation for the interpolant map)*
Assume that c is strictly convex and $c,\, c^\star \in C^2(\mathbb{R}^d)$. Assume that μ and ν are compactly supported in $\bar{\Omega}$, and are absolutely continuous with respect to Lebesgue, with f and g as their respective density functions. If f_{1-t} denote the density of μ_{1-t} for $t \in (0,1)$, then, there exists $\tilde{K} \subset \Omega$ of full measure for ν, such that

$$0 \neq g(y) = f_{1-t}\left(S_t(y)\right)\, det\left[\,(1-t)\,id + t\nabla S(y)\,\right], \tag{5.7}$$

for $y \in \tilde{K}$.

`Proof.` Because of Theorem 5.1.3, there exists a $c_{(t)}$-concave function ψ_t, such that,

$$S_t(y) = y + \nabla c^\star_{(t)}\left(-\nabla\psi_t(y)\right)$$

is the $c_{(t)}$-optimal map that pushes ν forward to μ_{1-t}. Since c is convex, $c,\, c^\star \in C^2(\mathbb{R}^d)$, and $c^\star_{(t)}(z) = t\,c^\star(-z)$, we have that $c_{(t)}$ is convex, and $c_{(t)},\, c^\star_{(t)} \in C^2(\mathbb{R}^d)$. Furthermore, μ_{1-t} has compact support in $\bar{\Omega}$, and because of Theorem 5.2.2, it is absolutely continuous with respect to Lebesgue. We use Theorem 5.2.1 - (ii), to conclude (5.7) □

5.3 Intermediate results

For the convenience of the reader, we recall in the next two propositions, some properties of W_c^h and the energy functional $s \mapsto \int_\Omega F(s(x))\,\mathrm{d}x$. For the proof of the next proposition, we refer to Propositions A.1 and A.2 in [15], and Theorem 1.1-(iii) in [8].

Proposition 5.3.1 *Let $s_0 \in \mathcal{P}_a(\Omega)$, and $h > 0$. Assume that $F : [0, \infty) \to I\!R$ is convex, and $c : I\!R^d \to [0, \infty)$ is strictly convex. Then*

(i). $\mathcal{P}_a(\Omega) \ni s \mapsto W_c^h(s_0, s)$ and $\mathcal{P}_a(\Omega) \ni s \mapsto \int_\Omega F(s(x))\,dx$ are convex and weakly lower semi-continuous with respect to the L^1-norm. Moreover,

(ii). $\mathcal{P}_a(\Omega) \ni s \mapsto \int_\Omega F(s(x))\,dx$ is strictly convex, provided F is strictly convex.

The next proposition states that $\mathcal{P}_a(\Omega) \ni s \mapsto W_c^h(s_0, s)$ is continuous with respect to the L^1-norm.

Proposition 5.3.2 *Let s_0, $s_1 \in \mathcal{P}_a(\Omega)$, and $h > 0$. Assume that $c : I\!R^d \to [0, \infty)$ is strictly convex. If $\left(s_1^{(n)}\right)_n$ is a sequence in $\mathcal{P}_a(\Omega)$, converging to s_1 in $L^1(\Omega)$, then*

$$\lim_{n\to\infty} W_c^h\left(s_0, s_1^{(n)}\right) = W_c^h(s_0, s_1).$$

`Proof.` Let S such that $S_\# s_1 = s_0$ and $\gamma \in \Gamma(s_0, s_1)$ be optimal in $W_h^c(s_0, s_1)$, and S_n such that $(S_n)_\# s_1^{(n)} = s_0$ and $\gamma^{(n)} \in \Gamma\left(s_0, s_1^{(n)}\right)$ be optimal in $W_c^h\left(s_0, s_1^{(n)}\right)$. We first recall the proof of the following estimate, due to Otto [15] :

$$W_c^h\left(s_0, s_1^{(n)}\right) - W_c^h(s_0, s_1) \le \sup_{x,y\in\Omega} c\left(\frac{x-y}{h}\right) \int_\Omega \left(s_1^{(n)} - s_1\right)_+, \tag{5.8}$$

where,

$$\left(s_1^{(n)} - s_1\right)_+ := \max\left(s_1^{(n)} - s_1, 0\right).$$

Indeed, since s_1, $s_1^{(n)} \in \mathcal{P}_a(\Omega)$, we have that,

$$\int_\Omega \left(s_1^{(n)} - s_1\right)_+ = \int_\Omega \left(s_1 - s_1^{(n)}\right)_+ := \lambda. \tag{5.9}$$

Consider $\tilde{\gamma}$, defined by

$$\begin{aligned}\int_{R^d\times R^d} & \xi(x,y)\,\mathrm{d}\tilde{\gamma}(x,y) \\ &= \int_\Omega \xi\left(S(y), y\right)\,\min\left(s_1^{(n)}, s_1\right)(y)\,\mathrm{d}y \\ &+ \frac{1}{\lambda}\int_\Omega\int_\Omega \xi(x,\tilde{y})\left(s_1^{(n)} - s_1\right)_+(\tilde{y})\left(s_1 - s_1^{(n)}\right)_+(y)\,\mathrm{d}\tilde{y}\,\mathrm{d}y.\end{aligned} \tag{5.10}$$

Because $S_{\#}s_1 = s_0$, $\min\left(s_1^{(n)}, s_1\right) + \left(s_1^{(n)} - s_1\right)_+ = s_1^{(n)}$, and $\min\left(s_1^{(n)}, s_1\right) + \left(s_1 - s_1^{(n)}\right)_+ = s_1$, we have that $\tilde{\gamma} \in \Gamma(s_0, s_1)$. We use (5.9) - (5.10), and the fact that $c \geq 0$ is continuous, and $\min\left(s_1^{(n)}, s_1\right) \leq s_1$, to conclude that

$$\begin{aligned} W_c^h\left(s_0, s_1^{(n)}\right) \;&-\; W_h^c(s_0, s_1) \\ &\leq \int_{R^d \times R^d} c\left(\frac{x-y}{h}\right) \mathrm{d}\tilde{\gamma}(x,y) - \int_{R^d \times R^d} c\left(\frac{x-y}{h}\right) \mathrm{d}\gamma(x,y) \\ &\leq \int_\Omega c\left(\frac{S(y)-y}{h}\right)\left(s_1 - s_1^{(n)}\right)_+(y)\,\mathrm{d}y \cdot \frac{1}{\lambda}\int_\Omega \left(s_1^{(n)} - s_1\right)_+(\tilde{y})\,\mathrm{d}\tilde{y} \\ &\leq \sup_{x,y\in\Omega} c\left(\frac{x-y}{h}\right)\int_\Omega \left(s_1^{(n)} - s_1\right)_+(y)\,\mathrm{d}y. \end{aligned}$$

This proves (5.8).

We let n go to ∞ in (5.8), and we use the fact that $\left(s_1^{(n)}\right)_n$ converges to s_1 in $L^1(\Omega)$, to conclude that,

$$\lim_{n\to\infty} W_c^h\left(s_0, s_1^{(n)}\right) \leq W_c^h(s_0, s_1). \tag{5.11}$$

Now, since $\operatorname{spt}\gamma^{(n)} \subset \bar{\Omega} \times \bar{\Omega}$, we have that $\left(\gamma^{(n)}\right)_n$ is tight. We deduce that $\left(\gamma^{(n)}\right)_n$ converges weakly to some $\bar{\gamma}$. Because $\gamma^{(n)} \in \Gamma\left(s_0, s_1^{(n)}\right)$, and $\left(s_1^{(n)}\right)_n$ converges to s_1 in $L^1(\Omega)$, we obtain that $\bar{\gamma} \in \Gamma(s_0, s_1)$. As a consequence,

$$\begin{aligned} W_c^h(s_0, s_1) \;&\leq\; \int_{\Omega\times\Omega} c\left(\frac{x-y}{h}\right) \mathrm{d}\bar{\gamma}(x,y) && (5.12) \\ &=\; \lim_{n\to\infty}\int_{\Omega\times\Omega} c\left(\frac{x-y}{h}\right) \mathrm{d}\tilde{\gamma}^{(n)}(x,y) = \lim_{n\to\infty} W_c^h\left(s_0, s_1^{(n)}\right). \end{aligned}$$

We combine (5.11) and (5.12), to conclude Proposition 5.3.2 □

The following estimates will be needed in the previous chapters.

Proposition 5.3.3 *Assume that c is strictly convex, of class C^1, and satisfies $c(0) = 0$ and $\lim_{|x|\to\infty} \frac{c(x)}{|x|} = \infty$. Then*

$$\langle\, z, \nabla c^\star\,(z)\,\rangle \;\geq\; c^\star(z) \;\geq\; 0, \tag{5.13}$$

for $z \in I\!R^d$. In addition, if $c(z) \geq \beta\,|\,z\,|^q$, for some $\beta > 0$ and $q > 1$, then

$$\langle\, z, \nabla c^\star(z)\,\rangle \leq M(\beta, q)\,|\,z\,|^{q^\star}, \tag{5.14}$$

where $M(\beta, q)$ is a constant which only depends on β and q.

`Proof.` Since c is strictly convex, differentiable and satisfies $\lim_{|x|\to\infty} \frac{c(x)}{|x|} = \infty$, we have that $c^\star \in C^1(I\!R^d)$ is convex. Then,

$$\langle z, \nabla c^\star(z)\rangle = c^\star(z) + c\,(\nabla c^\star(z)) \geq c^\star(z). \tag{5.15}$$

Because $c(0) = 0$ and 0 minimizes c, we have that $c^\star(0) = 0$ and 0 minimizes $c^\star$. We conclude that $c^\star(z) \geq 0$.
Now, assume that $c(z) \geq \beta\, z\,|^q$. Since $c^\star \in C^1(I\!R^d)$ is convex and non-negative, we have that

$$\langle\, z,\, \nabla c^\star(z)\,\rangle \leq c^\star(2z) - c^\star(z) \leq c^\star(2z). \tag{5.16}$$

Moreover, because $c(z) \geq \beta\, z\,|^q$, we have that

$$c^\star(2z) \leq M(\beta, q)\,|\,z\,|^{q^\star}. \tag{5.17}$$

We combine (5.16) and (5.17), to conclude (5.14) □

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