Nonlinear Stability for Steady Channel Flow with Exponential Temperature Dependent Viscosity

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Abstract

We discuss the nonlinear stability of flow of a Navier-Stokes fluid with temperature-dependent viscosity through an open parallel-sided channel with constant wall temperatures. Using the *energy method* of Reynolds [10] and Orr [7], we obtain a critical value for the Reynolds number below which the flow tends to remain stable under *arbitrary* disturbances.

Keywords: Energy Stability, Convection, Navier-Stokes, Channel Flow.
AMS Subject Classification Code: 76E06,76E15,76E30.

1   Introduction

The present work deals with the stability for flow in a channel with viscosity of the fluid depending upon temperature. In particular, we propose to determine critical Reynolds and Peclét numbers, below which the flow in a channel tends to retain its steady, fully developed form under *arbitrary* disturbances of the fluid motion and thermal diffusion. This hydrodynamical stability problem has been addressed previously by Potter & Graber [8], Schafer & Herwig [13], Wall & Wilson [17] and more recently by Wall & Nagata [18]. For

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a thorough introduction to this problem, the reader is referred to [15], [17] and [18] and references therein. In [8], the authors consider the linear stability of channel flow using a particular temperature-dependent viscosity model relevant to water for which viscosity decreases with temperature. Neglecting any disturbances of the basic-state temperature distribution, they obtain a modified fourth-order Orr-Sommerfeld equation which is solved numerically. Their calculation suggests that the critical Reynolds number decreases monotonically with the temperature difference across the walls. Wall and Wilson [17] discuss the same linear stability problem for four different viscosity-temperature models but in the general case where disturbances to the basic state temperature distribution are permitted. Using particular disturbances to the basic state they derive a sixth order eigenvalue problem which generalizes the Orr-Sommerfeld equations obtained by Potter & Graber. Their numerical solution to the eigenvalue problem is in agreement with the previous results of [8],[13].

A prolific mathematical contribution to the same class of problems that we are studying, has been made by Straughan and his coworkers (see [5, 4, 3, 15, 16] and references therein) where they consider the nonlinear stability problem as opposed to the linear stability case dealt by the earlier mentioned authors. In fact, these authors are able to obtain sharp estimates for global stability by resorting to a numerical solution of the corresponding Euler-Lagrange equation for the relevant energy functional. Yet other works on the subject of nonlinear stability, of the rest state, for convection problems have been made by Rionero and co-workers [11, 12] and Capone & Gentile [1, 2] based on the Lyapunov method where the potential energy is split into a linear and a nonlinear part. Rigorous analysis of the appropriate potentials yields the required conditions for stability.

Our work essentially follows from these recent papers in an attempt to analyze the stability problem for the most general class of perturbations for the viscosity model $\mu = \mu_0 e^{-KT}$ [17]. Experimental results confirm that this model is most appropriate for many real fluids [19]. In fact, this is particularly relevant in geophysical applications, when describing the Earth's mantle [6]. The energy stability argument employed in this paper is based on the work of Reynolds [10] and Orr [7] and has been recently revived by Serrin [14]. This technique provides a very simple method for analyzing the most general problem of nonlinear stability. The particular advantage of our technique lies in its ability to deal with the nonlinearities
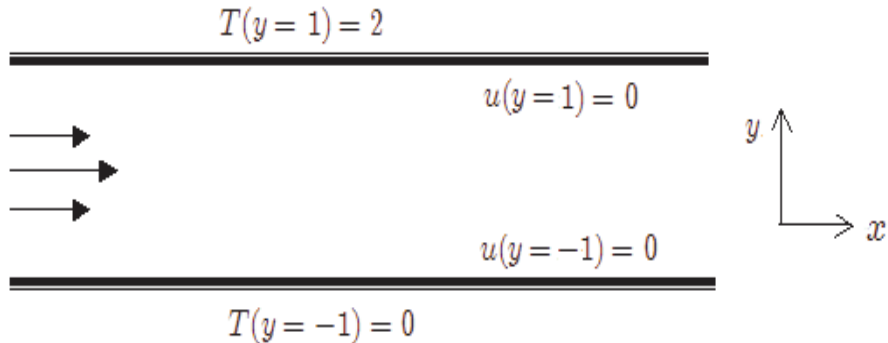
Figure 1: A schematic of the problem under consideration.

in a more effective and simple way than done previously. To this end, we employ the maximum principle for the convection-diffusion equation along with some fundamental inequalities, which permit us to decouple the temperature and velocity perturbations in a convenient manner. We are able to show that a certain steady solution of interest [17] remains nonlinearly stable below certain critical parameters, namely the Reynolds and Peclét numbers.

The paper employs standard mathematical notation. We employ standard indicial notation and the Einstein's repeated-index summation convention throughout this paper. We use the notation ∂_i to denote the partial derivative $\frac{\partial}{\partial x_i}$. By Ω, we refer to the domain, $\mathbf{R} \times (-1, 1) \times \mathbf{R}$, which represents the channel with its walls as the boundary, $\partial\Omega$. The function space of interest is $L^2(\Omega)$ with the usual norm

$$||u|| = \left(\int_\Omega |u|^2 \right)^{1/2}.$$

The outline of the paper is as follows. In section 2, we discuss the steady state solution of interest and obtain the perturbation equations and the corresponding energy equation. In section 3 we provide the proof of the stability theorem and section 4 concludes with an example where the physical significance of the main theorem is discussed.

2 Problem Formulation

2.1 Governing equations

The physical setting for our problem is as follows. We consider the motion of a viscous fluid in a channel of infinite length and height $2L$ (i.e. $-L \leq y \leq L$), with the origin placed at the centerline of the channel, which is along the x direction. In this subsection, we discuss the central governing equations for the problem and obtain the steady state solution of interest. The equations for the problem are given by the well known Navier-Stokes and heat equations, namely

$$\left.\begin{aligned} \rho(\partial_t u + u \cdot \operatorname{grad} u) &= \operatorname{div}(\tau) \\ u(-L) = u(L) &= 0 \end{aligned}\right\} \tag{1}$$

where ρ is the density and τ is the well known stress tensor for the Newtonian fluid under consideration, given by

$$\tau = -pI + 2\mu(T)D(u).$$

Here, p represents the pressure, I the identity tensor and $D(u)$ the symmetric part of the velocity gradient, i.e. $\frac{1}{2}\left(\operatorname{grad} u + \operatorname{grad}^T u\right)$. Also note that in our problem, the viscosity is dependent upon the temperature, T. In addition, we have the incompressibility equation,

$$\operatorname{div} u = 0 \tag{2}$$

and the heat equation,

$$\left.\begin{aligned} \partial_t T + u \cdot \operatorname{grad} T &= K\Delta T \\ T(-L) = T_l, \quad T(L) &= T_u \end{aligned}\right\} \tag{3}$$

where K is the thermal diffusivity coefficient, T_u, T_l are the temperatures at $y = L$ and $y = -L$ respectively. For sake of convenience, we non-dimensionalize our governing equations with suitable choice of the variables:

$$x^* = \frac{x}{L}, \ u^* = \frac{u}{V}, \ p^* = \frac{p}{\rho V^2}, \ t^* = \frac{t}{L/V}, \ \mu^* = \frac{\mu}{M}, T^* = 2\frac{T - T_l}{T_u - T_l}.$$

Substituting these into the equations (1)-(3), we get, upon simplification the non-dimensional version of the governing equations, namely

$$Re(\partial_t u + u \cdot \operatorname{grad} u) = -\operatorname{grad} p + 2\operatorname{div}(\mu(T)D(u)) \tag{4}$$

$$\mathrm{div}u = 0 \tag{5}$$

$$\partial_t T + u \cdot \mathrm{grad}\, T = \frac{1}{P_e}\Delta T \tag{6}$$

where we have finally discarded the superscript, $*$, for sake of convenience. Also, recall that we define the non-dimensional Reynolds number as $Re = LV\rho/M$, the Péclet number as $Pe = ReP_r$ and the Prandtl number as $P_r = M/\rho K$. The non-dimensional boundary conditions representing no-slip and fixed temperature at the channel walls are

$$u(y = \pm 1) = 0,\ T(y = -1) = 0,\ T(y = 1) = 2. \tag{7}$$

Hence, the governing equations for this problem expressed in non-dimensional form are the Navier-Stokes and heat equations, given by

$$Re(\partial_t u + u \cdot \mathrm{grad}\, u) = -\mathrm{grad}\, p + 2\mathrm{div}(\mu(T)D(u)) \tag{8}$$

$$\mathrm{div}u = 0 \tag{9}$$

$$\partial_t T + u \cdot \mathrm{grad}\, T = \frac{1}{P_e}\Delta T \tag{10}$$

where $\mu = \mu(T)$, $Re = LV\rho/M$ is the Reynolds number and $Pe = ReP_r$ is the Péclet number and $P_r = M/\rho\kappa$, the Prandtl number. The non-dimensional boundary conditions representing no-slip and fixed temperature at the channel walls are

$$u(x, y = \pm 1, z, t) = 0,\ T(x, y = -1, z, t) = 0,\ T(x, y = 1, z, t) = 2 \tag{11}$$

In this paper we will consider the stability of a basic state which is of the form

$$u(x, y, z, t) = (u_0(y), 0, 0),\ p(x, y, z, t) = p_0(x), T(x, y, z, t) = T_0(y) \tag{12}$$

Substituting these into the governing equations (8)-(10) and boundary conditions (11) and taking μ of the form $\mu = e^{-KT}$, suggests that (see [17]) u_0 and T_0 must be of the form

$$u_0(y) = -\frac{2}{K}(1 + \coth K + (y - \coth K)e^{K(1+y)})) \tag{13}$$

$$T_0(y) = 1 + y. \tag{14}$$

Since in most liquids, the viscosity drops with the temperature [1, 19], we will require that $K \geq 0$.

2.2 Perturbation equations

The perturbation equations can be easily obtained from the governing equations. We begin with considering perturbations of the forms

$$u(x, y, z, t) = u_0 + w(x, y, z, t)$$

$$T(x, y, z, t) = T_0(y) + \theta(x, y, z, t)$$

$$p(x, y, z, t) = p_0(x) + \pi(x, y, z, t)$$

where $u = (u_1, u_2, u_3)$, $u_0 = (u_0(y), 0, 0)$ and $w = (w_1, w_2, w_3)$ indicate three dimensional vector fields for the fluid flow. So, as a result $D(u) = D(u_0) + D(w)$, where, u_0, $D(u_0)$, T_0, and p_0 are the velocity, symmetric part of velocity gradient, temperature and pressure of the basic state, while w, $D(w)$, θ and π are perturbations to basic state of velocity, symmetric part of velocity gradient, temperature and pressure, respectively. Note that both basic state and perturbed state have to satisfy Navier-Stokes and heat equations (8)-(10). Substituting the perturbed velocity, temperature and pressure to the Navier-Stokes and heat equations gives us

$$\left.\begin{aligned} Re(\partial_t(u_0 + w) + (u_0 + w) \cdot \mathrm{grad}\ (u_0 + w)) = -\mathrm{grad}\ (p_0 + \pi) + 2\mathrm{div}(\mu(T_0 + \theta)D(u_0 + w)) \\ \mathrm{div}w = 0 \\ \partial_t(T_0 + \theta) + (u_0 + w) \cdot \mathrm{grad}\ (T_0 + \theta) = \frac{1}{P_e}\Delta(T_0 + \theta) \end{aligned}\right\} \tag{15}$$

with boundary conditions

$$\left.\begin{aligned} w(x, -1, z, t) &= w(x, 1, z, t) = 0 \\ \theta(x, -1, z, t) &= \theta(x, 1, z, t) = 0. \end{aligned}\right\} \tag{16}$$

3 The Main Theorem

In this section we shall attempt to employ a universal stability argument to find conditions for stability of the basic state. First, we state a Maximum/Minimum Principle Lemma for the equation 10 [9].

Lemma 1 *Let $\Theta(\mathbf{x}, t)$ satisfy equation (10). Then, if*

$$\Theta_1 \leq \Theta(\mathbf{x}, 0) \leq \Theta_0$$

for $\Theta_0 > \Theta_1$ for all $\mathbf{x} \in \Omega$, then

$$\Theta_1 \leq \Theta(\mathbf{x}, t) \leq \Theta_0$$

for all $\mathbf{x} \in \Omega$ and for all times, t.

Proof:

For proof of this theorem see [9]. $\square$

Remark:

An immediate consequence of this Lemma is that

$$0 \leq T_0(y) = T(x, y, z, 0) \quad \leq \quad 2 \tag{17}$$

$$\Rightarrow 0 \leq T(x, y, z, t) = T_0(y) + \theta(x, y, z, t) \quad \leq \quad 2. \tag{18}$$

In order to obtain the energy equation, we multiply both sides of equation $(15)_1$ by w and equation $(15)_3$ by θ. Then we integrate over the domain, Ω, integrate by parts and upon using the boundary conditions and simplification, we obtain the energy equation,

$$
\begin{aligned}
\frac{d\mathcal{E}(t)}{dt} \quad = \quad & -\int_\Omega w_i w_j \partial_j u_{0i} - \frac{2}{Re} \int_\Omega e^{-K(T_0+\theta)} D_{ij}(w) D_{ij}(w) - \frac{2}{Re} \int_\Omega e^{-K(T_0+\theta)} D_{ij}(u_0) D_{ij}(w) \\
& - \frac{\lambda}{Pe} \int_\Omega \partial_i \theta \partial_i \theta - \lambda \int_\Omega \theta w_i \partial_i T_0
\end{aligned}
\tag{19}
$$

where λ is an arbitrary positive constant and $\mathcal{E}(t) = \frac{1}{2}(||w||^2 + \lambda||\theta||^2)$ is the energy of the flow and thermal perturbations. In the theorem below we shall derive a condition for universal stability.

Theorem 1 *For arbitrary velocity and temperature perturbations w and θ respectively, satisfying equations $(15), (16)$, there exists a positive constant, ϵ, such that*

$$\frac{d}{dt}\mathcal{E}(t) \leq -\epsilon\mathcal{E}(t) \tag{20}$$

where $\mathcal{E}(t) = (||w||^2 + \lambda||\theta||^2)$. Furthermore, the system $(8) - (10)$ is nonlinearly stable provided the Reynolds and Peclét numbers respectively satisfy

$$Re < Re_c \equiv min\left(\frac{2\delta_1 e^{-2K}}{(1+\delta_1)c_p^2(\alpha_1 + 0.5\gamma\lambda)}, \quad \frac{\delta_2\alpha_1^2 K^2\gamma(1+\delta_1)e^{6K}}{\lambda}\right)$$

and

$$Pe < Pe_c \equiv \frac{2\lambda Re_c}{(1 + \delta_1)\alpha_1^2 c_p^2 K^2 e^{6K}}$$

where δ_1, δ_2, γ and λ are arbitrary positive constants and $\alpha_1 = max|grad\ u_0|$.

Proof:

The proof follows from finding appropriate estimates on each of the terms in equation (19). We shall denote the i-th term on the right hand side of this equation by (19_i). Therefore upon suitable application of Holder's and Young's inequalities and also using equations (14) and (18), we obtain

$$
\begin{align}
(19_1) \quad &\leq \quad max|grad\ u_0|||w||^2, \tag{21}\\
(19_2) \quad &\leq \quad -\frac{2e^{-2K}}{Re}||grad\ w||^2 \tag{22}\\
(19_4) \quad &= \quad -\frac{\lambda}{Pe}||grad\ \theta||^2 \tag{23}\\
(19_5) \quad &\leq \quad \lambda max|grad\ T_0|||\theta||||w|| \leq \lambda(\frac{1}{2\gamma}||\theta||^2 + \frac{\gamma}{2}||w||^2) \tag{24}
\end{align}
$$

where in the estimate for equation (19_5), we employ the Young's Inequality, with positive constant γ.

For the remaining term, we use Holder's inequality and equations (17) and (18) to obtain

$$
\begin{align}
(19_3) \quad &\leq \quad \frac{2}{Re}max|grad\ u_0|||e^{-K\theta}||||D(w)||\\
&\leq \quad \frac{2max|grad\ u_0|c_p K e^{2K}}{Re}||grad\ \theta||||D(w)||\\
&\leq \quad \frac{2e^{2K}max|grad\ u_0|}{Re}(\frac{K^2 c_p^2}{2\beta}||grad\ \theta||^2 + \frac{\beta}{2}||grad\ w||^2). \tag{25}
\end{align}
$$

In the final inequality, we use the Poincaré's inequality, with the corresponding constant c_p, followed by the Young's inequality with the constant $\beta > 0$. Hence, combining terms, we get

8

$$\frac{d\mathcal{E}(t)}{dt} \leq \alpha_1||w||^2 - \frac{2e^{-2K}}{Re}||\text{grad } w||^2 + \frac{\alpha_1\beta e^{2K}}{Re}||\text{grad } w||^2 + \frac{\gamma}{2}\lambda||w||^2 + \frac{\lambda}{2\gamma}||\theta||^2$$

$$- \frac{\lambda}{Pe}||\text{grad } \theta||^2 + \frac{\alpha_1 K^2 c_p^2 e^{2K}}{\beta Re}||\text{grad } \theta||^2$$

$$= (\alpha_1 + \frac{\gamma\lambda}{2})||w||^2 + (\frac{\beta\alpha_1 e^{2K}}{Re} - \frac{2e^{-2K}}{Re})||\text{grad } w||^2 + \frac{\lambda}{2\gamma}||\theta||^2$$

$$+ (\frac{\alpha_1 K^2 c_p^2 e^{2K}}{\beta Re} - \frac{\lambda}{Pe})||\text{grad } \theta||^2,$$

where $\alpha_1 = \max|\text{grad } u_0|$. Hence, if we choose (i) $\frac{2e^{-4K}}{\alpha_1} > \beta = \frac{2e^{-4K}}{(1+\delta_1)\alpha_1}$ and (ii) $\frac{Re\lambda\beta}{\alpha_1 K^2 c_p^2 e^{2K}} > Pe = \frac{Re\lambda\beta}{(1+\delta_2)\alpha_1 K^2 c_p^2}e^{-2K}$, with δ_1 and δ_2 both positive constants, then it follows that

$$\frac{d\mathcal{E}(t)}{dt} \leq [\alpha_1 + \frac{\gamma\lambda}{2} - \frac{2\delta_1 e^{-2K}}{(1+\delta_1)c_p^2 Re}]||w||^2 + [\frac{\lambda}{2\gamma} - \frac{\delta_2\alpha_1 K^2 e^{2K}}{\beta Re}]||\theta||^2$$

$$= [\alpha_1 + \frac{\gamma\lambda}{2} - \frac{2\delta_1 e^{-2K}}{(1+\delta_1)c_p^2 Re}]||w||^2 + \lambda[\frac{1}{\gamma} - \frac{\delta_2\alpha_1^2 K^2(1+\delta_1)e^{6K}}{2\lambda Re}]||\theta||^2$$

$$\leq \max(\epsilon_1, \epsilon_2)\mathcal{E}(t) \tag{26}$$

where

$$\epsilon_1 = \alpha_1 + \frac{\gamma\lambda}{2} - \frac{2\delta_1 e^{-2K}}{(1+\delta_1)c_p^2 Re}$$

and

$$\epsilon_2 = \frac{1}{2\gamma} - \frac{\delta_2\alpha_1^2 K^2(1+\delta_1)e^{6K}}{2\lambda Re}.$$

We define,

$$Re_1 \equiv \frac{2\delta_1 e^{-2K}}{(1+\delta_1)c_p^2(\alpha_1 + 0.5\gamma\lambda)}, Re_2 \equiv \frac{\delta_2\alpha_1^2 K^2\gamma(1+\delta_1)e^{6K}}{\lambda}$$

and

$$Re_c \equiv \min(Re_1, Re_2), Pe_c \equiv \frac{2\lambda Re_c}{(1+\delta_1)\alpha_1^2 K^2 c_p^2 e^{6K}}. \tag{27}$$

Then, $Re < Re_c$ and $Pe < Pe_c$ suggests that

$$\frac{d\mathcal{E}(t)}{dt} \leq -\epsilon\mathcal{E}(t), \quad -\epsilon = \max(\epsilon_1, \epsilon_2) \tag{28}$$

therefore implying nonlinear stability. □

4   Discussion

In this section, we shall attempt to numerically evaluate the critical Reynolds number, Re_c defined in equation (27), below which the steady velocity u_0 defined in Section 2.1 remains stable. We provide this critical value for $0 \leq K \leq 2$. The Poincaré constant for channel flow is $c_p = \pi$ [15]. Note that we are left free to choose several parameters, namely $\lambda, \gamma, \delta_1$ and δ_2. These parameters must be chosen appropriately so as to maximize the critical value, Re_c. Therefore, we first attempt to maximize both Re_1 and Re_2 before choosing the minimum.

It is not difficult to see that in order for these values to be a maximum, (i) $\lambda = 1 + \delta_1$ and without loss of generality, (ii) $\gamma = \frac{1}{(1+\delta_1)^2}$ and (iii) $\delta_2 = (1 + \delta_1)^3$. As a result we have

$$
\begin{aligned}
Pe_c &= \frac{2Re_c}{\alpha_1^2 c_p^2 K^2 e^{6K}} \\
Re_1 &= \frac{2\delta_1}{[\alpha_1(1+\delta_1)+0.5]c_p^2 e^{2K}} \\
Re_2 &= (1+\delta_1)K^2\alpha_1^2 e^{6K}.
\end{aligned}
\tag{29}
$$

Therefore, as $\delta_1 \to \infty$,

$$
Re_1 \to \frac{2}{\alpha_1 c_p^2 e^{2K}}, \quad Re_2 \to \infty
$$

and hence

$$
Re_c = Re_1 = \frac{2}{\alpha_1 c_p^2 e^{2K}}, \quad Pe_c = \frac{4}{\alpha_1^3 c_p^4 K^2 e^{8K}}.
$$

The critical values that we are able to obtain above are very small compared to the estimates made using a linearized analysis, however they possess the same exponential decay profile as those of Wall & Wilson [17]. We argue that the results of our theorem must be understood in the same sense as an existence argument, where smallness of initial data is often a necessary condition. We are able to establish the stability for the solution u_0, for nonzero Reynolds and Peclét numbers. The variations of Re_c and Pe_c versus K are indicated in figure 2. The area below the curves are the regions of stability. Therefore, for sufficiently slow flows and small thermal disturbances, the steady solution, u_0 in equation (13) is nonlinearly stable.
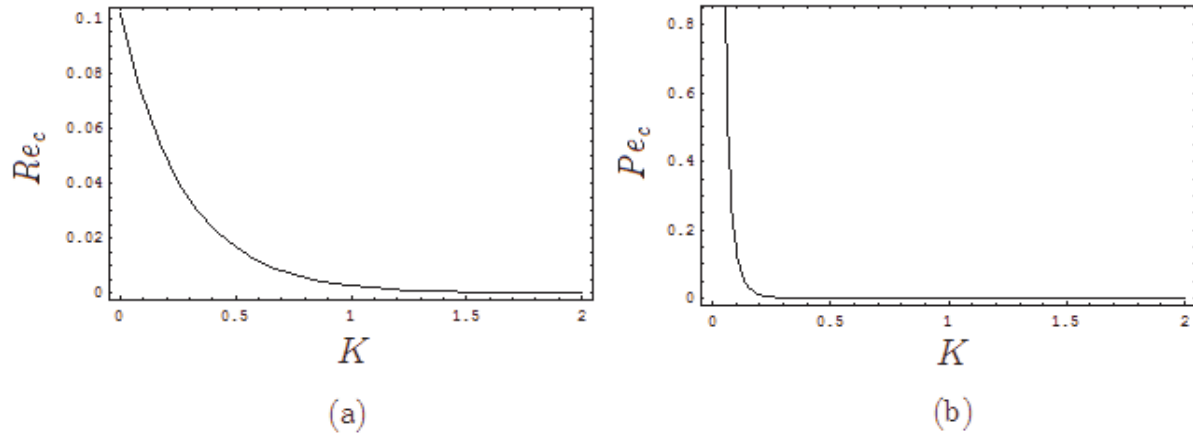
Figure 2: (a) Critical Reynolds number and (b) Critical Peclét number, versus K for $0 \leq K \leq 2$.

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