Existence of solutions to degenerate parabolic equations via the Monge-Kantorovich theory.

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Abstract

We obtain solutions of the nonlinear degenerate parabolic equation

$$\frac{\partial \rho}{\partial t} = \operatorname{div}\left\{\rho \nabla c^{\star} \left[\nabla \left(F'(\rho) + V\right)\right]\right\}$$

as a steepest descent of an energy with respect to a convex cost functional. The method used here is variational. It requires less uniform convexity assumption than that imposed by Alt and Luckhaus in their pioneering work [4]. In fact, their assumption may fail in our equation. This class of equations includes the Fokker-Planck equation, the porous-medium equation, the fast diffusion equation and the parabolic p-Laplacian equation.

Key Words: Monge-Kantorovich functional, doubly degenerate equation, steepest descent, gradient flow, energy inequality.

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1 Introduction

We consider a class of parabolic evolution equations, so-called doubly degenerate parabolic equations. These equations arise in many applications in physics and biology [12, 20, 21, 23]. They are used to model a variety of physical problems, e.g. the evolution of a fluid in a certain domain: porous-medium equation [19] and Fokker-Planck equation [13]. In this work, we focus on these parabolic equations of the form

$$\begin{cases} \frac{\partial b(u)}{\partial t} = \text{div } (a(b(u), \nabla u)) & \text{on } \quad (0,\infty)\times\Omega \\ u(t=0) = u_0 & \text{on } \quad \Omega \\ a(b(u),\nabla u)\cdot\nu = 0 & \text{on } \quad (0,\infty)\times\partial\Omega \end{cases} \tag{1}$$

where

$$a(b(u),\nabla u) := f(b(u))\,\nabla c^\star[\nabla(u+V)]$$

and $c^\star$ denotes the Legendre transform of a function $c : I\!\!R^d \to [0,\infty)$, that is,

$$c^\star(z) = \sup_{x\in I\!\!R^d}\{\langle x,z\rangle - c(x)\}$$

for $z \in I\!\!R^d$. Here Ω is a bounded domain of $I\!\!R^d$, ν is the outward unit normal to $\partial\Omega$, $b : I\!\!R \to I\!\!R$ is a monotone nondecreasing function, $V : \bar\Omega \to I\!\!R$ is a potential, $c : I\!\!R^d \to [0,\infty)$ is a convex function, f is a nonnegative real-valued function and $u_0 : \Omega \to I\!\!R$ is a measurable function. The unknown is $u : [0,\infty)\times\Omega \to I\!\!R$, $\quad u = u(t,x)$.

In a previous work, Alt and Luckhaus [4] proved existence of weak solutions to (1) when $V = 0$ under the following ellipticity condition on $a(t,z) := f(t)\nabla c^\star(z)$

$$\langle a(t,z_1) - a(t,z_2), z_1 - z_2\rangle \geq \lambda\,|\,z_1 - z_2\,|^p \tag{2}$$

for some $\lambda > 0$ and $p \geq 1$, and for all z_1, $z_2 \in I\!\!R^d$. This amounts to imposing that f is bounded below and the cost function c satisfies the ellipticity condition

$$\langle \nabla c^\star(z_1) - \nabla c^\star(z_2), z_1 - z_2\rangle \geq \lambda\,|\,z_1 - z_2\,|^p. \tag{3}$$

Note that when $c(z) = \frac{|\,z\,|^q}{q}$ or equivalently $c^\star(z) = \frac{|\,z\,|^p}{p}$ where $p > 1$ is the conjugate of $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, condition (3) reads as

$$\langle |\,z_1\,|^{p-2} z_1 - |\,z_2\,|^{p-2} z_2, z_1 - z_2\rangle \geq \lambda |\,z_1 - z_2\,|^p \tag{4}$$

which holds only if $p \geq 2$. In fact when $1 < p < 2$, the reverse inequality in (4) holds (see [9], pp. 13). In [4], the authors approximated (1) by a time discretization, and they used a Galerkin type argument to solve the resulting elliptic problems. In the same paper, they proved uniqueness of solutions to (1) when $V = 0$, assuming that (2) holds and that the distributional derivative $\frac{\partial b(u)}{\partial t}$ of a solution u of (1) is an integrable function. The last condition was removed by Otto in [17] where he used a technique called "doubling of variables" which was first introduced by Kružkov in [14]. This technique consists of doubling the time variable of two solutions of (1) and treating each solution as a constant with respect to the differential equation satisfied by the other solution.

In this work, we eliminate assumption (3), and we impose instead the following growth condition on the function c :

$$\beta\,|z|^q \leq c(z) \leq \alpha\,(|z|^q + 1) \tag{5}$$

for $z \in I\!R^d$ and for some α, $\beta > 0$ and $q > 1$. Notice that (5) is much weaker than the ellipticity condition (3) imposed by Alt and Luckhaus in [4]. Typical examples of functions which satisfy (5) but not (3) are $c(z) = \frac{|z|^q}{q}$ where $q > 2$. Indeed, for such functions c, we have that $c^\star(z) = \frac{|z|^p}{p}$ where $1 < p = \frac{q}{q-1} < 2$, and then, the reverse inequality in (3) and (4) hold.

We interpret (1) as a dissipative system and then, we introduce the internal energy density function $F : [0,\infty) \to I\!R$ satisfying $F' = b^{-1}$. Setting $\rho := b(u)$, $\rho_0 := b(u_0)$ and $f(x) = \max(x, 0)$, we rewrite (1) as

$$\begin{cases} \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\, U_\rho) = 0 & \text{on} \quad (0,\infty) \times \Omega \\ \rho(t=0) = \rho_0 & \text{on} \quad \Omega \\ \rho\, U_\rho \cdot \nu = 0 & \text{on} \quad (0,\infty) \times \partial\Omega. \end{cases} \tag{6}$$

Here

$$U_\rho := -\nabla c^\star \left[\nabla \left(F'(\rho) + V \right) \right]$$

denotes the vector field describing the average velocity of a fluid evolving with the continuity equation in (6), $\rho_0 : \Omega \to [0,\infty)$ is the initial mass density of the fluid, and the unknown $\rho : [0,\infty) \times \Omega \to [0,\infty)$, $\rho = \rho(t,x)$, is the mass density of the fluid at time t and position x of Ω.

The free energy associated with the fluid at time $t \in [0,\infty)$, is the sum of its internal energy and its potential energy,

$$E(\rho(t)) := \int_\Omega \left[F\left(\rho(t,x)\right) + \rho(t,x) V(x) \right] \mathrm{d}x.$$

Problem (6) includes the

- Linear Fokker-Planck equation:

$$\frac{\partial \rho}{\partial t} = \Delta\rho + \operatorname{div}(\rho\nabla V)$$

(where we use $c(z) = \frac{|z|^2}{2}$ and $F(x) = x \ln x$)

- Porous-medium and Fast diffusion equations:

$$\frac{\partial \rho}{\partial t} = \Delta\rho^m$$

($V = 0$, $c(z) = \frac{|z|^2}{2}$, and $F(x) = \frac{x^m}{m-1}$ with $1 \neq m \geq 1 - \frac{1}{d}$)

- Generalized heat equation:

$$\frac{\partial\rho}{\partial t} = \mathrm{div}\,\left(|\nabla\rho^{\frac{1}{p-1}}|^{p-2}\nabla\rho^{\frac{1}{p-1}}\right)$$

  ($V=0$, $c(z) := \frac{|z|^q}{q}$ with $\frac{1}{p}+\frac{1}{q}$, and $F(x) = \frac{1}{p-1}x\ln x$ with $p>1$)

- Parabolic p-Laplacian equation:

$$\frac{\partial\rho}{\partial t} = \mathrm{div}\,\left(|\nabla\rho|^{p-2}\nabla\rho\right)$$

  ($V=0$, $c(z) = \frac{|z|^q}{q}$ and $F(x) = \frac{x^m}{m(m-1)}$ with $m := \frac{2p-3}{p-1}$, $\frac{1}{p}+\frac{1}{q}=1$ and $p \geq \frac{2d+1}{d+1}$).

- Doubly degenerate diffusion equation (see [16]):

$$\frac{\partial\rho}{\partial t} = \mathrm{div}\,\left(|\nabla\rho^n|^{p-2}\nabla\rho^n\right) \tag{7}$$

  ($V=0$, $c(z) := \frac{|z|^q}{q}$ with $\frac{1}{p}+\frac{1}{q}=1$, and $F(x) = \frac{nx^m}{m(m-1)}$ with $m := n + \frac{p-2}{p-1}$ and $\frac{1}{p-1} \neq n \geq \frac{d-(p-1)}{d(p-1)}$).

The above restrictions on m, n and p are made so that F satisfies the assumptions (HF1) and (HF2) stated below.

We are interested in the following questions: under what conditions does (6) have a solution? Is the solution unique? What are the most relevant conditions on c, F and V which ensure that solutions converge asymptotically to an equilibrium?

In this work, we answered the first and the second questions. We proved existence and uniqueness of weak solutions to (6) when the initial mass density ρ_0 is bounded below and above, that is, $\rho_0 + \frac{1}{\rho_0} \in L^\infty(\Omega)$ (see Theorems 3.11 and 3.12). This restriction is made to simplify the proofs and not to bury fundamental facts into technical computations. We include in Remark 3.13 a method to extend our existence result to cases where $\frac{1}{\rho_0}$ fails to be bounded and where ρ_0 belong to a wider class of probability densities, e.g. $\rho_0 \in L^p(\Omega)$, $p \geq q$. In a coming paper [3], we establish large time asymptotics for solutions of (6).

Our approach in studying existence of solutions to (6) was inspired by the works of Jordan-Kinderlehrer-Otto [13] and Otto [16]. In [13], the authors observed that the Fokker-Planck equation can be interpreted as the gradient flow of the entropy functional

$$H(\rho) := \int_{R^d} (\rho\ln\rho + \rho V)\,\mathrm{d}x,$$

with respect to the Wasserstein metric d_2. Recall that d_2 is a metric on the set of probability measures on $\mathbb{R}^d$ with finite second moments defined by

$$d_2(\mu_0,\mu_1) := \left[\inf\left\{\int_{R^d\times R^d} \frac{|x-y|^2}{2}\,\mathrm{d}\gamma(x,y) : \quad \gamma\in\Gamma(\mu_0,\mu_1)\right\}\right]^{1/2}$$

where $\Gamma(\mu_0,\mu_1)$ denotes the set of probability measures on $\mathbb{R}^d\times\mathbb{R}^d$ having μ_0 and μ_1 as their marginals (see the definition below). This idea was extended by Otto in [16] for doubly degenerate diffusion equations of the form (7) when $p\geq 2$.

Now, we outline the proof of our existence theorem to (6). For the sake of illustration, we assume that $V = 0$. The proof consists of four main steps.

Step 1. We interpret (6) as a steepest descent of the internal energy functional

$$\mathcal{P}_a(\Omega) \ni \rho \mapsto E_i(\rho) := \int_\Omega F(\rho(x))\, \mathrm{d}x$$

against the Monge-Kantorovich work W_c^h, where $h > 0$ is a time-step size and $\mathcal{P}_a(\Omega)$ denotes the set of probability density functions $\rho : \Omega \to [0, \infty)$. In other words, given a mass density ρ_{k-1}^h of the fluid at time $t_{k-1} = (k-1)h$, we define the mass density ρ_k^h at time $t_k = kh$ as the unique minimizer of the variational problem

$$(P_k^h): \quad \inf_{\rho \in \mathcal{P}_a(\Omega)} \left\{ hW_c^h\left(\rho_{k-1}^h, \rho\right) + E_i(\rho) \right\} \tag{8}$$

(see Proposition 2.3). So at each time t, the system tends to decrease its internal energy $E_i(\rho)$ while trying to minimize the work to move from state $\rho(t)$ to state $\rho(t+h)$.

Step 2. We write the Euler-Lagrange equation to (P_k^h), and we show that

$$\frac{\rho_k^h - \rho_{k-1}^h}{h} = \mathrm{div}\left\{ \rho_k^h \nabla c^\star \left[\nabla\left(F'(\rho_k^h)\right)\right] \right\} + A_k(h) \tag{9}$$

weakly, for $k \in \mathbb{N}$ (see Proposition 2.6), where $A_k(h)$ tends to 0 as h goes to 0. Equation (9) explains why (8) is a discretization of (6).

Step 3. We define the approximate solution ρ^h to (6) as

$$\begin{cases} \rho^h(t,x) &= \rho_k^h(x) \text{ if } t \in ((k-1)h, kh],\ k \in \mathbb{N} \\ \rho^h(0,x) &= \rho_0(x) \end{cases}$$

and we deduce from (9) that ρ^h satisfies

$$\begin{cases} \frac{\partial \rho^h}{\partial t} = \mathrm{div}\left\{ \rho^h \nabla c^\star \left[\nabla\left(F'(\rho^h)\right)\right] \right\} + A(h) \quad \text{on} \quad (0,\infty) \times \Omega \\ \rho^h(t=0) = \rho_0 \quad \text{on} \quad \Omega \end{cases} \tag{10}$$

in a weak sense (see Proposition 2.9), where $A(h)$ is shown to be $0\left(h^{\epsilon(q)}\right)$ with $\epsilon(q)$ defined by $\epsilon(q) := \min(1, q-1)$ (see Proposition 3.2).

Step 4. We let h go to 0 in (10) and we show that $(\rho^h)_h$ converges to a function ρ which solves (6) in a weak sense. Here two convergence results are established: the weak convergence of $(\rho^h)_h$ to ρ in $L^1((0,T) \times \Omega)$ for $0 < T < \infty$ up to a subsequence, which proves that $\left(\frac{\partial \rho^h}{\partial t}\right)_h$ converges weakly to $\frac{\partial \rho}{\partial t}$ in the dual $\left[C_c^\infty(\mathbb{R} \times \mathbb{R}^d)\right]'$ of $C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$, and the weak convergence of the nonlinear term $\left(\mathrm{div}\{\rho^h \nabla c^\star \left[\nabla\left(F'(\rho^h)\right)\right]\}\right)_h$ to $\mathrm{div}\{\rho \nabla c^\star \left[\nabla\left(F'(\rho)\right)\right]\}$ in $\left[C_c^\infty(\mathbb{R} \times \mathbb{R}^d)\right]'$ for a subsequence.

The first convergence follows from the upper bound (26) in Proposition 2.3 which is a consequence of the Maximum Principle stated in Proposition 2.2 (see Lemma 3.3): *starting with a probability density function ρ_0 which is bounded above, that is, $\rho_0 \leq M$, the probability density function ρ_k^h - solution of (P_k^h) - is bounded above as well, that is,*

$\rho_k^h \leq M$ *for all* $k \in I\!N$. As a consequence, $(\rho^h)_h$ is bounded in $L^\infty((0,\infty)\times\Omega)$ and then, it converges to some ρ in $L^1((0,T)\times\infty)$ for $0 < T < \infty$ up to a subsequence. The second convergence is one of the most difficult tasks in the proof of the existence theorem. Its proof requires elaborated intermediate results. Here come some technical differences with the works in [4] and [16]. Indeed, due to the weaker condition (5) imposed on c compared to the stronger ellipticity condition (2) or (3) in [4], the methods used in [4] and [16] do not yield here strong convergence of the nonlinear term as in [4] and [16]. The best we could expect is the weak convergence of the nonlinear term, and to prove this convergence, we proceed as follows:

(i). First, we improve the previous convergence by showing that in fact, $(\rho^h)_h$ converges strongly to ρ for a subsequence in $L^1((0,T)\times\Omega)$ (see Proposition 3.7).

(ii). Then, we deduce that $\left(\mathrm{div}\{\rho^h\nabla c^\star\left[\nabla\left(F'(\rho^h)\right)\right]\}\right)_h$ converges weakly to $\mathrm{div}\{\rho\nabla c^\star\left[\nabla\left(F'(\rho)\right)\right]\}$ in $\left[C_c^\infty(I\!R\times I\!R^d)\right]'$ for a subsequence (see Theorem 3.10).

To prove (i), one needs to have a good control on the spatial derivative of ρ^h, for example, to show that $\left\{\nabla\left(F'(\rho^h)\right)\right\}_h$ is bounded in $L^{q^\star}((0,T)\times\Omega)$, $0 < T < \infty$. The main ingredient used to establish this result is the following *Mass Transportation type Energy Inequality*:

$$E_i(\tilde\rho_0) - E_i(\tilde\rho_1) \geq \int_\Omega \langle\nabla\left(F'(\tilde\rho_1)\right), \tilde S(y) - y\rangle\tilde\rho_1(y)\,\mathrm{d}y, \tag{11}$$

for $\tilde\rho_0$, $\tilde\rho_1 \in \mathcal{P}_a(\Omega)$. Here, $\tilde S$ denotes the c-optimal map that pushes $\tilde\rho_1$ forward to $\tilde\rho_0$ (see the definition in Proposition 1.1). A more general statement of the energy inequality is given in Theorem 2.8. Inequality (11) can be seen as a consequence of the *displacement convexity* of the internal energy functional $\mathcal{P}_a(\Omega) \ni \rho \mapsto E_i(\rho)$, that is, the convexity of

$$[0,1] \ni t \mapsto E_i(\tilde\rho_{1-t}),$$

where,

$$\tilde\rho_{1-t} := \left((1-t)\,\mathrm{id} + t\tilde S\right)_{\#}\tilde\rho_1 \tag{12}$$

is the shortest path (w.r.t. the Monge-Kantorovich functional W_c defined below) joining $\tilde\rho_1$ and $\tilde\rho_0$ in $\mathcal{P}_a(\Omega)$. When $c(z) = \frac{|z|^2}{2}$ in which case $\tilde S$ is the gradient of a convex function, the interpolation in (12) was first introduced by McCann in [15].

Indeed, setting $\tilde\rho_0 := \rho_{k-1}^h$ and $\tilde\rho_1 := \rho_k^h$ in (11) and using the Euler-Lagrange equation of (P_k^h), that is,

$$\frac{S_k^h - \mathrm{id}}{h} = \nabla c^\star\left[\nabla\left(F'(\rho_k^h)\right)\right], \tag{13}$$

where S_k^h is the $c\left(\frac{\cdot}{h}\right)$-optimal map that pushes ρ_k^h forward to ρ_{k-1}^h, we obtain that

$$h\int_\Omega\langle\nabla\left(F'(\rho_k^h)\right), \nabla c^\star\left[\nabla\left(F'(\rho_k^h)\right)\right]\rangle\,\rho_k^h \leq E_i(\rho_{k-1}^h) - E_i(\rho_k^h). \tag{14}$$

We integrate (14) over $t \in [0,T]$ and we use Jensen's inequality to deduce that

$$\int_0^T\int_\Omega\langle\nabla\left(F'(\rho^h)\right), \nabla c^\star\left[\nabla\left(F'(\rho^h)\right)\right]\rangle\rho^h \leq E_i(\rho_0) - |\,\Omega\,|F\left(\frac{1}{|\,\Omega\,|}\right). \tag{15}$$

Using condition (5) combined with (15) and the fact that $(\rho^h)_h$ is bounded in $L^\infty((0,\infty)\times\Omega)$, we obtain that

$$\int_0^T\int_\Omega \rho^h \left| \nabla\left(F'(\rho^h)\right)\right|^{q^\star} \le \text{cst} \quad \text{(see Lemma 3.3)}.$$

Then, we use that $\left(\frac{1}{\rho^h}\right)_h$ is bounded in $L^\infty((0,\infty)\times\Omega)$ (see (26)) which is a consequence of the Minimum Principle of Proposition 2.2, to conclude that $\left\{\nabla\left(F'(\rho^h)\right)\right\}_h$ is bounded in $L^{q^\star}((0,T)\times\Omega)$, $0<T<\infty$. This yields (i).
To prove (ii), we first use (13) and condition (5) to have that $\left\{\nabla c^\star\left[\nabla\left(F'(\rho^h)\right)\right]\right\}_h$ is bounded in $L^q(\Omega\times(0,\infty))$ (see Lemma 3.8), from which we deduce that $\left\{\nabla c^\star\left[\nabla\left(F'(\rho^h)\right)\right]\right\}_h$ converges weakly to some σ in $L^q(\Omega\times(0,T))$ for a subsequence and for all $0<T<\infty$. Next, we use (i) and the boundedness of $\left\{\nabla\left(F'(\rho^h)\right)\right\}_h$ in $L^{q^\star}(\Omega\times(0,\infty))$ to obtain that $\left\{\nabla\left(F'(\rho^h)\right)\right\}_h$ converges weakly to $\nabla(F'(\rho))$ in $L^{q^\star}(\Omega\times(0,T))$ for a subsequence (see Lemma 3.8). In the end, we extend the energy inequality (11) in time-space (see Lemma 3.9), and we combine the new inequality with the strong convergence of $(\rho^h)_h$ to ρ, the weak convergence of $\left\{\nabla\left(F'(\rho^h)\right)\right\}_h$ to $\nabla(F'(\rho))$ and the weak convergence of $\left\{\nabla c^\star\left[\nabla\left(F'(\rho^h)\right)\right]\right\}_h$ to σ, to establish that $\left(\text{div}\{\rho^h\nabla c^\star\left[\nabla\left(F'(\rho^h)\right)\right]\}\right)_h$ converges weakly to $\text{div}(\rho\sigma)$ for a subsequence, and that $\text{div}(\rho\sigma)=\text{div}\{\rho\nabla c^\star[\nabla(F'(\rho))]\}$ (see Theorem 3.10). The convexity of $c^\star$ plays an important role in this proof.

Notations

- Ω is an open, bounded, convex and smooth domain of $\mathbb{R}^d$, $d\ge 1$, and $\Omega_T := (0,T)\times\Omega$ for $0<T\le\infty$.
- $B_R(x)\subset\mathbb{R}^d$ denotes the open ball of radius R centered at x, $B_R(x)^c$ is the complement of $B_R(x)$ in $\mathbb{R}^d$, and $p^\star$ denotes the conjugate index of $p>1$, that is, $\frac{1}{p}+\frac{1}{p^\star}=1$.
- $\mathcal{P}_a(\Omega) := \left\{\rho:\Omega\to[0,\infty) \text{ measurable}, \int_\Omega \rho(x)\,\mathrm{d}x = 1\right\}$ and $\mathcal{P}_a^{(R)}(\Omega) := \left\{\rho\in\mathcal{P}_a(\Omega):\ \rho\le R \text{ a.e.}\right\}$ for $0<R<\infty$.
- If $\varphi:\Omega\to\mathbb{R}$, then $\|\varphi\|_{L^q(\Omega)}$ denotes the L^q-norm of φ, and $\text{spt}(\varphi)$ denotes the support of φ, that is the closure of $\{x\in\Omega:\ \varphi(x)\ne 0\}$.
- If $x=(x_1,\cdots,x_d)$ and $y=(y_1,\cdots,y_d)$ are vectors in $\mathbb{R}^d$, then $\langle x,y\rangle := \sum_{i=1}^d x_i y_i$, and $|x| := \sqrt{\langle x,x\rangle}$.
- If A is a convex subset of $\mathbb{R}^D$, $D\ge 1$, and if $G:A\to\mathbb{R}$ is convex function on A, then $G^\star:\mathbb{R}^D\to\mathbb{R}$ denotes the Legendre transform of G, that is,

$$G^\star(y) := \sup_{x\in R^d}\{\langle x,y\rangle - \bar{G}(x)\}, \text{ where } \bar{G}(x) := \begin{cases} G(x) & \text{if } x\in A \\ +\infty & \text{otherwise.}\end{cases}$$

By abuse of notations, we will identify G and $\bar{G}$.

- If A is a Borel subset of $I\!R^d$, then $|A|$ denotes the Lebesgue measure of A, and $I\!I_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$ denotes the characteristic function of A.
- If μ_0 and μ_1 are two nonnegative measures, by $\mu_0 << \mu_1$ we mean that μ_0 is absolutely continuous with respect to μ_1.

Throughout this paper, M and N are positive constants, a.e. (almost everywhere) refers to the d-dimensional Lebesgue measure, and

$$c_h(z) := c\left(\frac{z}{h}\right) \quad \text{and} \quad E_i(\rho) := \int_\Omega F\left(\rho(x)\right) \mathrm{d}x$$

for all $z \in I\!R^d$ and $\rho \in \mathcal{P}_a(\Omega)$.

Definitions

Probability measures with marginals. Let μ_0 and μ_1 be probability measures on $I\!R^d$. A Borel probability measure γ on the product space $I\!R^d \times I\!R^d$ is said to have μ_0 and μ_1 as its marginals if one of the following equivalent conditions holds:

(i). for Borel $A \subset I\!R^d$,

$$\gamma[A \times I\!R^d] = \mu_0[A] \text{ and } \gamma[I\!R^d \times A] = \mu_1[A].$$

(ii). For $(\varphi, \psi) \in L^1_{\mu_0}(I\!R^d) \times L^1_{\mu_1}(I\!R^d)$, where $L^1_{\mu_i}(I\!R^d)$ denotes the space of μ_i-integrable functions on $I\!R^d$ $(i = 1, 2)$,

$$\int_{R^d \times R^d} [\varphi(x) + \psi(y)] \, \mathrm{d}\gamma(x, y) = \int_{R^d} \varphi(x) \, \mathrm{d}\mu_0(x) + \int_{R^d} \psi(y) \, \mathrm{d}\mu_1(y).$$

We denote by $\Gamma(\mu_0, \mu_1)$ the set of all probability measures satisfying (i) or (ii). If μ_0 and μ_1 are absolutely continuous with respect to Lebesgue with ρ_0 and ρ_1 denoting their respective density functions, we simply write $\Gamma(\rho_0, \rho_1)$.

Push-forward mapping. Let μ_0 and μ_1 be probability measures on $I\!R^d$. A Borel map $T: I\!R^d \to I\!R^d$ is said to push μ_0 forward to μ_1, if

(i). $\mu_1[A] = \mu_0[T^{-1}(A)]$ for Borel $A \subset I\!R^d$, or equivalently

(ii). $\int_{R^d} \varphi(y) \, \mathrm{d}\mu_1(y) = \int_{R^d} \varphi\left(T(x)\right) \mathrm{d}\mu_0(x)$ for $\varphi \in L^1_{\mu_1}(I\!R^d)$.

Whenever (i) or (ii) holds, we write that $\mu_1 = T_\# \mu_0$, and we say that T pushes μ_0 forward to μ_1.
Similarly, if $\tau: I\!R^d \to I\!R^d \times I\!R^d$ is a Borel map, the push forward $\gamma = \tau_\# \mu_0$ of μ_0 by τ is the probability measure on $I\!R^d \times I\!R^d$ defined by $\gamma(A \times B) = \mu_0\left(\tau^{-1}(A \times B)\right)$ for Borel subsets A and B in $I\!R^d$, or equivalently $\int_{R^d \times R^d} \varphi(y, z) \, \mathrm{d}\gamma(y, z) = \int_{R^d} \varphi\left(\tau(x)\right) \mathrm{d}\mu_0(x)$ for

$\varphi \in L^1_\gamma(\mathbb{R}^d \times \mathbb{R}^d)$.

The next proposition is due to Caffarelli [5] and Gangbo-McCann [11]. It asserts the existence and uniqueness of the minimizer for the Monge-Kantorovich problem.

Proposition 1.1 *(Existence of optimal maps).*
Let $c : \mathbb{R}^d \to [0,\infty)$ be strictly convex, and $\rho_0, \rho_1 \in \mathcal{P}_a(\Omega)$. Then

(i). there is a function $v : \overline{\Omega} \to \mathbb{R}$ such that $T : \bar\Omega \to \bar\Omega$, $T := id - (\nabla c^\star) \circ \nabla u$ pushes ρ_0 forward to ρ_1, where $u(x) = \inf_{y \in \overline{\Omega}} \left\{ c(x-y) - v(y) \right\}$ for $x \in \overline{\Omega}$.

(ii). T is the unique minimizer (a.e. with respect to ρ_0) of the Monge problem

$$(\mathcal{M}) : \quad \inf \left\{ \int_\Omega c\,(x - T(x))\; \rho_0(x)\, dx, \quad T_\# \rho_0 = \rho_1 \right\}.$$

(iii). The joint measure $\gamma := (id \times T)_\# \rho_0$ uniquely solves the Kantorovich problem

$$(\mathcal{K}) : \quad \inf \left\{ \int_{R^d \times R^d} c\,(x-y)\; d\gamma(x,y), \quad \gamma \in \Gamma(\rho_0, \rho_1) \right\}.$$

(iv). T is one-to-one that is, there exists a map $S : \bar\Omega \to \bar\Omega$ pushing ρ_1 forward to ρ_0, such that $T\,(S(y)) = y$ a.e. with respect to ρ_1 while $S\,(T(x)) = x$ a.e. with respect to ρ_0.
Moreover, $S = id + \nabla c^\star(-\nabla v)$, where $v(y) = \inf_{x \in \overline{\Omega}} \left\{ c(x-y) - u(x) \right\}$ for $y \in \overline{\Omega}$. v is called the c-transform of u, and it is denoted by $v := u^c$.

We will refer to T (respectively S) as the c-optimal map that pushes ρ_0 (respectively ρ_1) forward to ρ_1 (respectively ρ_0), and γ will be called the c-optimal measure in $\Gamma(\rho_0, \rho_1)$.

Monge-Kantorovich functional. Let $c : \mathbb{R}^d \to [0,\infty)$ be strictly convex, $h > 0$, and $\rho_0,\ \rho_1 \in \mathcal{P}_a(\Omega)$. We define the Monge-Kantorovich functional for the cost $c\left(\frac{\cdot}{h}\right)$ by

$$W_c^h\,(\rho_0, \rho_1) \;:=\; \inf \left\{ \int_{R^d \times R^d} c\left(\frac{x-y}{h}\right)\, \mathrm{d}\gamma(x,y) : \quad \gamma \in \Gamma(\rho_0, \rho_1) \right\}.$$

If $c(z) = \frac{|z|^q}{q}$, we denote W_c^h by W_q^h. When $c(z) = \frac{|z|^2}{2}$ and $h = 1$, $d_2 := \sqrt{W_2^h}$ is called the Wasserstein metric.

We deduce from Proposition 1.1 that there exist a unique probability measure $\gamma \in \Gamma(\rho_0, \rho_1)$ and a unique map T that pushes ρ_0 forward to ρ_1 and whose inverse S pushes ρ_1 forward to ρ_0, such that

$$\begin{aligned} W_c^h(\rho_0, \rho_1) = \int_{R^d \times R^d} c\left(\frac{x-y}{h}\right) \mathrm{d}\gamma(x,y) \quad &= \quad \int_\Omega c\left(\frac{x - T(x)}{h}\right) \rho_0(x)\, \mathrm{d}x \\ &= \quad \int_\Omega c\left(\frac{S(y) - y}{h}\right) \rho_1(y)\, \mathrm{d}y. \end{aligned}$$

Assumptions

(HC1) : $c : \mathbb{R}^d \to [0, \infty)$ is such that $0 = c(0) < c(z)$ for all $z \neq 0$.

(HC2) : $\lim_{|x| \to \infty} \frac{c(x)}{|x|} = \infty$, i.e. c is coercive.

(HC3) : $\beta \, | \, z \, |^q \leq c(z) \leq \alpha \, (| \, z \, |^q + 1)$ for all $z \in \mathbb{R}^d$, where α, $\beta > 0$ and $q > 1$.

(HF1) : Either $\lim_{x \to +\infty} \frac{F(x)}{x} = +\infty$ (i.e. $F : [0, \infty) \to \mathbb{R}$ has a super-linear growth at $+\infty$), or $\lim_{x \to +\infty} \frac{F(x)}{x} = 0$ and $F'(x) < 0$ for all $x > 0$.

(HF2) : $(0, \infty) \ni x \mapsto x^d F(x^{-d})$ is convex.

We impose assumption (HF1) to ensure that the Legendre transform $F^\star$ of F is finite on $(-\infty, 0)$. In fact, the assumption (HF1) combined with the strict convexity of F and $F(0) = 0$ imply that $F^\star(x)$ is finite for all $x < 0$. Therefore, $F^\star\left(F'(x)\right)$ is finite for $x > 0$. Let us also point out here that assumption (HC3) implies (HC2), and (HC3) combined with $c(0) = 0$ imply both (HC1) and (HC2). So, we will omit (HC1) and (HC2) wherever (HC3) and $c(0) = 0$ are assumed. We warn the reader that assumptions (HC1) and (HC2) are introduced for the sole purpose to achieve maximum generality in our results. If preferred, the reader could replace these assumptions by (HC3) and $c(0) = 0$ in all our results.

Examples of cost and energy density functions which satisfy the above assumptions are:

- $c(z) = | \, z \, |^q$ where $q > 1$, or in general $c(z) = \sum_{i=1}^n A_i \, | \, z \, |^{q_i}$ where $n \in \mathbb{N}$, $q_i > 1$, and $A_i > 0$ (take $q = \max_{\{i=1, \cdots, n\}}(q_i) = q_{i_0}$ for some $i_0 \in \{1 \cdots, n\}$, $\beta = A_{i_0}$, and $\alpha = \sum_{i=1}^n A_i$ for assumptions (HC1)-(HC3) to hold).
- $F(x) = x \ln x$, $F(x) = \frac{x^m}{m-1}$, $m > 1$ or $1 - \frac{1}{d} \leq m < 1$, and $F(x) = \sum_{i=1}^n A_i \, F_i(x)$, where $n \in \mathbb{N}$, $A_i > 0$ and the F_i are like the examples given above.

2 Calculus of Variations on $\mathcal{P}_a(\Omega)$

We discretize (6) and we prove in section 2.1 that the problem

$$(P) : \quad \inf \left\{ I(\rho) := h W_c^h(\rho_0, \rho) + E_i(\rho), \quad \rho \in \mathcal{P}_a(\Omega) \right\} \tag{16}$$

admits a unique minimizer ρ_1. The reason why we minimize such a functional $I(\rho)$ will be clear in section 2.2 where we find the Euler-Lagrange equation of (P). In fact, we shall see that the Euler-Lagrange equation is nothing but the discretization of (6). In section 2.3, we show that

$$E_i(\rho_0) - E_i(\rho_1) \geq \left. \frac{d \, E(\rho_{1-t})}{dt} \right|_{t=0} \tag{17}$$

where ρ_{1-t} (defined by (12)) denotes the probability density obtained by interpolating ρ_0 and ρ_1 along the geodesic joining them in $\mathcal{P}_a(\Omega)$ equipped with W_c. We refer to (17) as the - internal - energy inequality. We shall see later that (17) is an essential ingredient in the proof of the convergence of the approximate sequence $(\rho^h)_h$ (see the definition in section 2.4) to a solution of (6).

Throughout this section, we assume that $F : [0,\infty) \to \mathbb{R}$ is strictly convex and twice continuously differentiable on $(0,\infty)$.

2.1 Existence of solutions to a minimization problem (P)

In this section, $h > 0$ and $\rho_0 \in \mathcal{P}_a(\Omega)$ is such that $\rho_0 \leq M$ a.e. We show that the problem

$$(P_R): \quad \inf\left\{ I(\rho) := hW_c^h(\rho_0,\rho) + E_i(\rho) : \quad \rho \in \mathcal{P}_a^{(R)}(\Omega) \right\} \tag{18}$$

admits a unique minimizer ρ_{1R} for $R \geq M$ (Proposition 2.1) and that $\rho_{1R} \in \mathcal{P}_a^{(M)}(\Omega)$ for $R > 2M$ that is, $0 \leq \rho_{1R} \leq M$ a.e. (Proposition 2.2). We deduce that (P) (defined by (16)) has a unique minimizer ρ_1 which satisfies $0 \leq \rho_1 \leq M$ a.e. (Proposition 2.3).

Proposition 2.1 *Let $R \geq M$, and assume that $c : \mathbb{R}^d \to [0,\infty)$ is strictly convex and satisfies (HC1). Then (P_R) has a unique minimizer ρ_{1R} which satisfies*

$$|\Omega| F\left(\frac{1}{|\Omega|}\right) \leq E_i(\rho_{1R}) \leq E_i(\rho_0). \tag{19}$$

`Proof:` Let I_{inf} denote the infimum of $I(\rho)$ over $\rho \in \mathcal{P}_a^{(R)}(\Omega)$. Since $\rho_0 \in \mathcal{P}_a^{(R)}(\Omega)$, $E_i(\rho_0) < \infty$ and $c(0) = 0$, we have that $I_{inf} \leq \frac{1}{h} E_i(\rho_0)$. Moreover, because of Jensen's inequality and the fact that $c \geq 0$ and $\rho \in \mathcal{P}_a(\Omega)$, we have that $I_{inf} \geq \frac{|\Omega|}{h} F\left(\frac{1}{|\Omega|}\right)$. We deduce that I_{inf} is finite. Now, let $\left(\rho^{(n)}\right)_n$ be a minimizing sequence for (P_R). We have that $\left(\rho^{(n)}\right)_n$ is bounded in $L^\infty(\Omega)$. As a consequence, $\left(\rho^{(n)}\right)_n$ converges weakly-$\star$ to a function ρ_{1R} in $L^\infty(\Omega)$, and then weakly in $L^1(\Omega)$ for a subsequence since Ω is bounded. Clearly, $\rho_{1R} \in \mathcal{P}_a^{(R)}(\Omega)$. Furthermore, because of Proposition 5.3.1 [2], we have that $\mathcal{P}_a(\Omega) \ni \rho \mapsto I(\rho)$ is weakly lower semi-continuous on $L^1(\Omega)$ as the sum of weakly lower semi-continuous functions. Therefore,

$$I(\rho_{1R}) \leq \liminf_{n\to\infty} I(\rho^{(n)}) = I_{inf} \leq I(\rho_{1R}),$$

which shows that ρ_{1R} is a minimizer of (P_R). The uniqueness of ρ_{1R} follows from the convexity of $\mathcal{P}_a(\Omega) \ni \rho \mapsto W_c^h(\rho_0,\rho)$ and the strict-convexity of $\mathcal{P}_a(\Omega) \ni \rho \mapsto \int_\Omega F(\rho)\,\mathrm{d}x$ (see Proposition 5.3.1 [2]).
Next, we observe that $I(\rho_{1R}) \leq I(\rho_0)$, and since $W_c^h(\rho_0,\rho_0) = 0$ and $W_c^h(\rho_0,\rho_{1R}) \geq 0$ (because of (HC1)), we deduce that $E_i(\rho_{1R}) \leq E_i(\rho_0)$. We use Jensen's inequality and the fact that $\rho_{1R} \in \mathcal{P}_a^{(R)}(\Omega)$ to conclude that $|\Omega| F\left(\frac{1}{|\Omega|}\right) \leq E_i(\rho_{1R}) \leq E_i(\rho_0)$. □

Proposition 2.2 *(Maximum/Minimum principle)*
Let $R > 2M$ and ρ_0 be such that $N \leq \rho_0 \leq M$ a.e. Assume that $c : \mathbb{R}^d \to [0,\infty)$ is strictly convex and satisfies (HC1). Then the minimizer ρ_{1R} of the problem (P_R) (defined by (18)) satisfies $N \leq \rho_{1R} \leq M$ a.e. Therefore, ρ_{1R} does not depend on R.

Proof: The proof we present here is similar to that in [18] where $c(z) = \frac{|z|^2}{2}$ and $F(x) = x\ln(x)$. Since the proof of "$\rho_{1R} \geq N$ a.e." is analogue to that of "$\rho_{1R} \leq M$ a.e.", we only prove that $\rho_{1R} \leq M$ a.e. Suppose by contradiction that $E := \{y \in \Omega : \rho_{1R}(y) > M\}$ has a positive Lebesgue measure. The idea is to come up with $\rho_{1R}^{(\epsilon)} \in \mathcal{P}_a^{(R)}(\Omega)$ such that $I(\rho_{1R}) > I(\rho_{1R}^{(\epsilon)})$. This contradicts the fact that ρ_{1R} is the minimizer of I over $\mathcal{P}_a^{(R)}(\Omega)$.
Let γ_R be the c_h-optimal measure in $\Gamma(\rho_0, \rho_{1R})$. We have that

$$\gamma_R(E^c \times E) > 0, \tag{20}$$

where $E^c := \mathbb{R}^d \setminus E$; otherwise

$$\begin{aligned} M|E| < \int_E \rho_{1R}(y)\,\mathrm{d}y &= \gamma_R(\mathbb{R}^d \times E) = \gamma_R(E \times E) \leq \gamma_R(E \times \mathbb{R}^d) \\ &= \int_E \rho_0(x)\,\mathrm{d}x \leq M\,|\,E\,|, \end{aligned}$$

which yields a contradiction. Consider the measure $\nu := \gamma_R \, \mathbb{I}_{E^c \times E}$ defined by

$$\int_{R^d \times R^d} \xi(x,y)\,\mathrm{d}\nu(x,y) = \int_{E^c \times E} \xi(x,y)\,\mathrm{d}\gamma_R(x,y),$$

for $\xi \in C_0(\mathbb{R}^d \times \mathbb{R}^d)$, or equivalently

$$\nu(F) = \gamma_R\,[\,F \cap (E^c \times E)\,]\,,$$

for Borel sets $F \subset \mathbb{R}^d \times \mathbb{R}^d$. Denote by ν_0 and ν_1 its marginals that is,

$$\int_{R^d \times R^d} [\,\varphi(x) + \psi(y)\,]\,\mathrm{d}\nu(x,y) = \int_{R^d} \varphi(x)\,\mathrm{d}\nu_0(x) + \int_{R^d} \psi(y)\,\mathrm{d}\nu_1(y),$$

for $\varphi, \psi \in C_0(\mathbb{R}^d)$. Since $\nu << \gamma_R$ and $\gamma_R \in \Gamma(\rho_0, \rho_{1R})$, we have that $\nu_0 << \rho_0(x)\,\mathrm{d}x$ and $\nu_1 << \rho_{1R}(y)\,\mathrm{d}y$. As a consequence, ν_0 and ν_1 are absolutely continuous with respect to Lebesgue. Denote by v_0 and v_1 their respective density functions. We have that

(i). $0 \leq v_0 \leq M$ a.e. and $0 \leq v_1 \leq R$ a.e., and

(ii). $v_0 = 0$ a.e. on E, and $v_1 = 0$ a.e. on E^c.

For $\epsilon \in (0,1)$, we define $\rho_{1R}^{(\epsilon)} := \rho_{1R} + \epsilon(v_0 - v_1)$ and the probability measure $\gamma_R^{(\epsilon)}$ by

$$\begin{aligned} \int_{R^d \times R^d} \xi(x,y)\,\mathrm{d}\gamma_R^{(\epsilon)}(x,y) &:= \int_{R^d \times R^d} \xi(x,y)\,\mathrm{d}\gamma_R(x,y) \\ &\quad + \epsilon \int_{E^c \times E} [\xi(x,x) - \xi(x,y)]\,\mathrm{d}\gamma_R(x,y), \end{aligned}$$

for $\xi \in C_0(\mathbb{R}^d \times \mathbb{R}^d)$. Because of (i), (ii) and the fact that $2M < R$, we have for small enough ϵ, that $0 \leq \rho_{1R}^{(\epsilon)} \leq R$ and

$$\int_\Omega \rho_{1R}^{(\epsilon)}(y)\,\mathrm{d}y = 1 + \epsilon\,[\gamma_R(E^c \times E) - \gamma_R(E^c \times E)] = 1.$$

Hence, $\rho_{1R}^{(\epsilon)} \in \mathcal{P}_a^{(R)}(\Omega)$. Moreover, since $\gamma_R \in \Gamma(\rho_0, \rho_{1R})$ and ν has marginals $\nu_0 = v_0(x)\,\mathrm{d}x$ and $\nu_1 = v_1(y)\mathrm{d}y$, we have that $\gamma_R^{(\epsilon)} \in \Gamma(\rho_0, \rho_{1R}^{(\epsilon)})$. Now, we show that $I(\rho_{1R}^{(\epsilon)}) < I(\rho_{1R})$, for ϵ small enough. Indeed,

$$I(\rho_{1R}^{(\epsilon)}) - I(\rho_{1R}) = h\left[W_c^h(\rho_0, \rho_{1R}^{(\epsilon)}) - W_c^h(\rho_0, \rho_{1R})\right] + \int_\Omega \left[F(\rho_{1R}^{(\epsilon)}) - F(\rho_{1R})\right]. \quad (21)$$

Because $\gamma_R^{(\epsilon)} \in \Gamma(\rho_0, \rho_{1R}^{(\epsilon)})$ and $c(0) = 0$, we have that

$$\begin{aligned} W_c^h(\rho_0, \rho_{1R}^{(\epsilon)}) - W_c^h(\rho_0, \rho_{1R}) &\leq \int_{R^d \times R^d} c\left(\frac{x-y}{h}\right) \mathrm{d}\gamma_R^{(\epsilon)}(x,y) \\ &\quad - \int_{R^d \times R^d} c\left(\frac{x-y}{h}\right) \mathrm{d}\gamma_R(x,y) \\ &= -\epsilon \int_{E^c \times E} c\left(\frac{x-y}{h}\right) \mathrm{d}\gamma_R(x,y). \end{aligned} \quad (22)$$

On the other hand, according to (i) and (ii), we have, for ϵ small enough, that

$$\rho_{1R}^{(\epsilon)} = \rho_{1R} - \epsilon v_1 \geq M - \epsilon v_1 > 0 \quad \text{on} \quad E, \quad (23)$$

and

$$\rho_{1R}^{(\epsilon)} = \rho_{1R} + \epsilon v_0 \geq \epsilon v_0 > 0 \quad \text{on} \quad E^c \cap [\, v_0 > 0\,]. \quad (24)$$

We combine (i), (ii), (23), (24), and the fact that $F \in C^2(0,\infty)$ is convex, and $\nu = \gamma_R \mathbb{I}_{E^c \times E}$ has marginals $\nu_0 = v_0(x)\,\mathrm{d}x$ and $\nu_1 = v_1(y)\,\mathrm{d}y$, to obtain that

$$\begin{aligned} \int_\Omega \left[F(\rho_{1R}^{(\epsilon)}) - F(\rho_{1R})\right] &= \int_{E^c} [\, F(\rho_{1R} + \epsilon v_0) - F(\rho_{1R})\,] \\ &\quad + \int_E [\, F(\rho_{1R} - \epsilon v_1) - F(\rho_{1R})\,] \\ &\leq \epsilon \left[\int_{E^c \cap [\, v_0 > 0\,]} F'(\rho_{1R} + \epsilon v_0) v_0 - \int_E F'(\rho_{1R} - \epsilon v_1) v_1\right] \\ &\leq \epsilon \left[\int_{E^c} F'(M + \epsilon v_0) v_0 - \int_E F'(M - \epsilon v_1) v_1\right] \\ &= \epsilon \left[\int_{E^c \times E} \big(F'(M + \epsilon v_0(x)) - F'(M - \epsilon v_1(y))\big)\, \mathrm{d}\gamma_R(x,y)\right]. \end{aligned}$$

Therefore,

$$\int_\Omega \left[\, F(\rho_{1R}^{(\epsilon)} - F(\rho_{1R})\,\right] = 0(\epsilon^2). \quad (25)$$

Combining (21), (22) and (25), we conclude for small enough ϵ that

$$I(\rho_{1R}^{(\epsilon)}) - I(\rho_{1R}) \leq \frac{-\epsilon h}{2} \int_{E^c \times E} c\left(\frac{x-y}{h}\right) \mathrm{d}\gamma_R(x,y) < 0$$

where the last inequality holds because of (HC1) and (20). □

Proposition 2.3 *(Existence and uniqueness of solutions for (P))*
Assume that $N \le \rho_0 \le M$ a.e. and $c : I\!R^d \to [0,\infty)$ is strictly convex and satisfies (HC1). Then $\rho_1 := \rho_{1R}$ (defined in Proposition 2.2) is the unique minimizer for the problem (P) defined by (16). Therefore,

$$N \le \rho_1 \le M \text{ a.e.} \tag{26}$$

and

$$|\,\Omega\,|\, F\left(\frac{1}{|\,\Omega\,|}\right) \le E_i(\rho_1) \le E_i(\rho_0). \tag{27}$$

`Proof:` Let $\rho \in \mathcal{P}_a(\Omega)$. Because of Proposition 1.4.1 [2], there exists a sequence $\left(\rho^{(R)}\right)_{R>2M}$ in $\mathcal{P}_a^{(R)}(\Omega)$ converging to ρ such that

$$\int_\Omega F\left(\rho^{(R)}\right) \le \int_\Omega F(\rho). \tag{28}$$

Since ρ_1 is the minimizer for (P_R) (see Proposition 2.2), we have using (28) that

$$hW_c^h(\rho_0,\rho_1) + \int_\Omega F(\rho_1) \le hW_c^h(\rho_0,\rho^{(R)}) + \int_\Omega F(\rho). \tag{29}$$

And since $\left(\rho^{(R)}\right)_R$ converges to ρ in $L^1(\Omega)$, Proposition 5.3.2 [2] gives that

$$\lim_{R\uparrow\infty} W_c^h\left(\rho_0,\rho^{(R)}\right) = W_c^h(\rho_0,\rho). \tag{30}$$

We let R go to ∞ in (29) and we use (30) to conclude that ρ_1 is a minimizer for (P). The uniqueness of the minimizer follows from the strict convexity of $\mathcal{P}_a(\Omega) \ni \rho \mapsto I(\rho)$ as in Proposition 2.1, and the statements (26) and (27) are direct consequences of Proposition 2.2 and (19). □

In the remaining of this section, we state two propositions needed to establish the convergence of the approximate solution ρ^h (defined by (65) below) of equation (6) as h goes to 0. The proposition stated below shows that the interpolant densities ρ_{1-t}, $t \in [0,1]$ (defined by (12)) between two probability densities ρ_0 and ρ_1 which are bounded above, are also bounded above.

Proposition 2.4 *Let ρ_0, $\rho_1 \in \mathcal{P}_a(\Omega)$ be such that $\rho_0, \rho_1 \le M$ a.e., and assume that $c : I\!R^d \to [0,\infty)$ is strictly convex, of class C^1 and satisfies $c(0) = 0$ and (HC3). Denote by S the c-optimal map that pushes ρ_1 forward to ρ_0, and define the interpolant map*

$$S_t := (1-t)\, id + tS,$$

for $t \in [\,0,1\,]$. Then, for nonnegative functions ξ in $C_c(I\!R^d)$, we have that

$$\int_\Omega \xi\,\left(S_t(y)\right)\, \rho_1(y)\, dy \;\le\; M \int_{R^d} \xi\,(x)\, dx. \tag{31}$$

`Proof:` We split the proof into two steps. In step 1, we prove (31) for sufficiently regular cost functions $c \in C^2(\mathbb{R}^d)$ whose Legendre transform $c^\star$ are also C^2. Under this assumption, the matrix ∇S is diagonalizable with positive eigenvalues, and $A \mapsto (\det A)^{1/d}$ is concave on the set of $d \times d$ diagonalizable matrices with positive eigenvalues. In step 2, we approximate a general cost function c by regular cost functions c_k of the type of Step 1, and we obtain (31) in the limit as k goes to ∞.

Step 1. Assume that c is strictly convex, and c, $c^\star \in C^2(\mathbb{R}^d)$.
Because of Proposition 4.1 in the Appendix, $\mu_{1-t} := (S_t)_\# \rho_1$ is absolutely continuous with respect to Lebesgue for all $t \in [0,1]$. Let ρ_{1-t} denote the density function of μ_{1-t}. Then (31) reads as

$$\int_\Omega \xi(x)\, \rho_{1-t}(x)\, \mathrm{d}x \;\leq\; M \int_{\mathbb{R}^d} \xi(x)\, \mathrm{d}x.$$

Thus, it suffices to show that $\rho_{1-t} \leq M$.
From Proposition 4.1, there exists a set $K \subset \Omega$ of full measure for $\mu_1 := \rho_1(y)\,\mathrm{d}y$ such that S_t is injective on K, and for $y \in K$ and $t \in [0,1]$, $\nabla S(y)$ is diagonalizable with positive eigenvalues, and

$$0 \neq \rho_1(y) = \rho_{1-t}\left(S_t(y)\right) \det\left[\nabla S_t(y)\right] \tag{32}$$

where $\nabla S_t(y) = (1-t)\mathrm{id} + t\nabla S(y)$. Since $\rho_0, \rho_1 \leq M$ a.e. and $S_\# \rho_1 = \rho_0$, we can choose K such that $\rho_1(y)$, $\rho_0\left(S(y)\right) \leq M$ for $y \in K$. We set $t = 1$ in (32) and we use that $\rho_0\left(S(y)\right) \leq M$ to deduce that

$$\det\left[\nabla S(y)\right] \geq \frac{\rho_1(y)}{M}. \tag{33}$$

Since $A \mapsto (\det A)^{1/d}$ is concave on the set of $d \times d$ diagonalizable matrices with positive eigenvalues, we have that

$$\left[\det \nabla S_t(y)\right]^{1/d} \geq (1-t) + t\left(\det\left[\nabla S(y)\right]\right)^{1/d}. \tag{34}$$

We use (33), (34) and the fact that $\rho_1(y) \leq M$, to obtain that

$$\det\left[\nabla S_t(y)\right] \geq \frac{\rho_1(y)}{M}. \tag{35}$$

We combine (32) and (35), and we use that S_t is injective on K to deduce that $\rho_{1-t} \leq M$ on $S_t(K)$. But, since $\mu_1(K^c) = 0$ and $\mu_{1-t} = (S_t)_\# \mu_1$, we have that $\mu_{1-t}\left[\left(S_t(K)\right)^c\right] = 0$ and then $\rho_{1-t} = 0$ on $\left[S_t(K)\right]^c$. We conclude that $\rho_{1-t} \leq M$.

Step 2. Assume that $c(0) = 0$ and c satisfies (HC3).
Since $c(0) = 0$ and c satisfies (HC3), there exists a sequence $(c_k)_k$ of strictly convex cost functions such that

$$\begin{cases} c_k,\, c_k^\star \in C^2(\mathbb{R}^d), \\ c_k \to c \;\text{ locally in }\; C^1(\mathbb{R}^d), \;\text{as } k \to \infty, \\ 0 = c_k(0) < c_k(z) \;\text{ for }\; z \neq 0 \end{cases} \tag{36}$$

(see [2], Proposition 1.3.1). Denote by S_k the c_k-optimal map that pushes ρ_1 forward to ρ_0, and set

$$S_k^{(t)} := (1-t)\,\mathrm{id} + tS_k$$

for $t \in [0,1]$. Lemma 2.2.2 [2] gives that $\left(S_k^{(t)}\right)_k$ converges to S_t a.e. on $[\rho_1 \neq 0]$ for a subsequence, and from Step 1, we have that

$$\int_\Omega \xi\left(S_k^{(t)}(y)\right)\rho_1(y)\,\mathrm{d}y \leq M \int_{\mathbb{R}^d} \xi(x)\,\mathrm{d}x. \tag{37}$$

We let k go to ∞ in (37), and we use that $0 \leq \xi \in C_c(\mathbb{R}^d)$ and Fatou's lemma to conclude (31). □

Next, we state a proposition needed in the next section to prove the strong convergence of the approximate solutions $(\rho^h)_h$ of (6) in $L^1\left((0,T)\times\Omega\right)$, for $0 < T < \infty$.

Proposition 2.5 *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be strictly convex of class $C^1(\mathbb{R})$, such that $\lim_{t\to\infty} \frac{g(t)}{t} = \infty$. Given M, $\delta > 0$, and p, $q > 1$, define*
$A_{M,\delta} := \Big\{(u_1,u_2) \in L^q(\Omega)^2 :\ \| u_j \|_{L^q(\Omega)} \leq M,\ \| g'(u_j) \|_{W^{1,p}(\Omega)} \leq M$ *and*
$\int_\Omega [\, f'(u_2) - f'(u_1)\,][\, u_2 - u_1\,] \leq \delta,\ \ (j=1,2)\Big\}$,
and set

$$\Lambda_M(\delta) := \sup_{(u_1,u_2)\in A_{M,\delta}} \| u_2 - u_1 \|_{L^1(\Omega)}.$$

Then

$$\lim_{\delta\downarrow 0} \Lambda_M(\delta) = 0.$$

`Proof:` Suppose by contradiction that there exist $\kappa > 0$ and $(u_j^\delta)_{\delta\downarrow 0}$, $(j=1,2)$ such that $(u_1^\delta, u_2^\delta) \in A_{M,\delta}$ and

$$\| u_2^\delta - u_1^\delta \|_{L^1(\Omega)} > \kappa. \tag{38}$$

By the Sobolev embedding theorem, $\left(g'(u_j^\delta)\right)_\delta$ converges strongly in $L^p(\Omega)$, and then a.e. for a (non-relabeled) subsequence. Since $g \in C^1(\mathbb{R})$ is strictly convex and has a super-linear growth at ∞, we have that $(g')^{-1}$ is continuous. We deduce that

(i). $(u_j^\delta)_\delta$ converges to some function u_j a.e., for $j = 1,2$.

We use (i), $\| u_j^\delta \|_{L^q(\Omega)} \leq M$ and the fact that $q > 1$ to conclude that $(u_j^\delta)_\delta$ converges strongly to u_j in $L^1(\Omega)$. And since $\| u_1^\delta - u_2^\delta \|_{L^1(\Omega)} > \kappa$, we obtain that

$$\| u_2 - u_1 \|_{L^1(\Omega)} > \kappa. \tag{39}$$

Next, we use (i), the convexity of f and the fact that $\int_\Omega \left[\, f'(u_2^\delta) - f'(u_1^\delta)\,\right]\left[\, u_2^\delta - u_1^\delta\,\right] \leq \delta$, to have that

$$0 \leq \int_\Omega \left[f'(u_2) - f'(u_1)\right][u_2 - u_1] \leq \liminf_{\delta\downarrow 0} \int_\Omega \left[f'(u_2^\delta) - f'(u_1^\delta)\right]\left[u_2^\delta - u_1^\delta\right] \leq 0.$$

This implies that

$$\left[f'\left(u_2(x)\right) - f'\left(u_1(x)\right)\right]\left[u_2(x) - u_1(x)\right] = 0 \ \text{ for a.e. } x \in \Omega. \tag{40}$$

Since $f \in C^1(\mathbb{R})$ is strictly convex, we have that f' is one-to-one and then (40) implies that $u_1(x) = u_2(x)$ for a.e. $x \in \Omega$. This yields a contradiction to (39). □

2.2 Properties of the minimizer for (P)

We establish the Euler-Lagrange equation for the problem (P) defined by (16), and we derive some properties of the minimizer for this problem. The next proposition is the first step toward showing that (P) is a discretization of equation (6), or in other words, (6) is the steepest descent of the internal energy functional E_i with respect to the Monge-Kantorovich functional W_c^h.

Proposition 2.6 *Let $\rho_0 \in \mathcal{P}_a(\Omega)$ be such that $N \leq \rho_0 \leq M$ a.e. Assume that $c : \mathbb{R}^d \to [0, \infty)$ is strictly convex, of class C^1 and satisfies (HC1) - (HC2). If ρ_1 denotes the minimizer for (P), then the following hold:*

$$\int_{\Omega\times\Omega} \langle \nabla c\left(\frac{x-y}{h}\right), \psi(y) \rangle \, d\gamma(x,y) + \int_\Omega P(\rho_1(y)) \, div \, \psi(y) \, dy = 0 \tag{41}$$

for $\psi \in C_c^\infty(\Omega, \mathbb{R}^d)$; here $P(x) := P_F(x) := x\, F'(x) - F(x)$ for $x \in (0, \infty)$, and γ is the c_h-optimal measure in $\Gamma(\rho_0, \rho_1)$. Moreover,

(i). $P(\rho_1) \in W^{1,\infty}(\Omega)$.

(ii). If S is the c_h-optimal map that pushes ρ_1 forward to ρ_0, then

$$\frac{S(y) - y}{h} = \nabla c^\star \left[\nabla(F'(\rho_1(y)))\right], \tag{42}$$

for a.e. $y \in \Omega$, and

$$\begin{aligned} \Big| \int_\Omega \frac{\rho_1(y) - \rho_0(y)}{h} \varphi(y) \, dy \quad &+ \quad \int_\Omega \rho_1(y) \langle \nabla c^\star \left[\nabla \left(F'(\rho_1(y))\right) \right], \nabla\varphi(y) \rangle \, dy \Big| \\ &\leq \quad \frac{1}{2h} \sup_{x\in\bar\Omega} \Big| D^2\varphi(x) \Big| \int_{\Omega\times\Omega} |\, x - y\,|^2 \, d\gamma(x,y), \end{aligned} \tag{43}$$

for $\varphi \in C^2(\bar\Omega)$.

`Proof:` Since $c \in C^1(\mathbb{R}^d)$ is strictly convex and satisfies (HC2), we have that $c^\star \in C^1(\mathbb{R}^d)$ and $(\nabla c)^{-1} = \nabla c^\star$. Following [13], let $\psi \in C_c^\infty(\Omega, \mathbb{R}^d)$ and consider the flow map $(\phi_\epsilon)_{\epsilon \in R}$ in $C^\infty(\Omega, \Omega)$ defined by

$$\begin{cases} \frac{\partial \phi_\epsilon}{\partial \epsilon} = \psi \circ \phi_\epsilon \\ \phi_0 = \text{id}. \end{cases} \tag{44}$$

We have that $det\,(\nabla\phi_\epsilon) \neq 0$, and

$$\frac{\partial(det\,\nabla\phi_\epsilon)}{\partial\epsilon}\Big|_{\epsilon=0} = \text{div}\,\psi. \tag{45}$$

Define the probability measure $\mu_\epsilon := (\phi_\epsilon)_\# \rho_1$ on Ω. Since ϕ_ϵ is a C^1-diffeomorphism, then μ_ϵ is absolutely continuous with respect to Lebesgue. Let ρ_ϵ denote its density function. Clearly, $\rho_\epsilon \in \mathcal{P}_a(\Omega)$ and

$$(\rho_\epsilon \circ \phi_\epsilon)\, det\,(\nabla\phi_\epsilon) = \rho_1 \quad \text{a.e.} \tag{46}$$

Next, define the probability measure $\gamma_\epsilon := (id \times \phi_\epsilon)_{\#}\gamma$ on $\Omega \times \Omega$, that is,

$$\int_{\Omega\times\Omega} \xi(x,y)\,\mathrm{d}\gamma_\epsilon(x,y) = \int_{\Omega\times\Omega} \xi\,(x,\phi_\epsilon(y))\,\mathrm{d}\gamma(x,y), \qquad \forall \xi \in C(\Omega\times\Omega).$$

We have that $\gamma_\epsilon \in \Gamma(\rho_0,\rho_\epsilon)$, and then, the mean-value theorem gives that

$$\begin{aligned}
&\frac{W_c^h(\rho_0,\rho_\epsilon) - W_c^h(\rho_0,\rho_1)}{\epsilon} \\
&\quad \le \int \frac{1}{\epsilon}\left[\, c_h(x-\phi_\epsilon(y)) - c_h(x-y)\,\right] \mathrm{d}\gamma(x,y) \\
&\quad = -\int \left\langle\, \nabla c_h\left[\, x-y+\theta(y-\phi_\epsilon(y))\,\right], \frac{\phi_\epsilon - \phi_0}{\epsilon}(y)\,\right\rangle \mathrm{d}\gamma(x,y),
\end{aligned}$$

where $\theta \in [0,1]$. Because of (44), we have that $|\,\frac{\phi_\epsilon-\phi_0}{\epsilon}\,| \le \|\,\psi\,\|_{L^\infty}$ for $\epsilon > 0$. Then, we use that $c \in C^1(I\!R^d)$, the Lebesgue dominated convergence theorem and (44) to obtain that

$$\limsup_{\epsilon\downarrow 0} \frac{W_c^h(\rho_0,\rho_\epsilon) - W_c^h(\rho_0,\rho_1)}{\epsilon} \le -\int \langle\, \nabla c_h(x-y), \psi(y)\,\rangle\,\mathrm{d}\gamma(x,y). \tag{47}$$

On the other hand, because of (46), we have that

$$\int_\Omega F\left(\rho_\epsilon(x)\right)\mathrm{d}x = \int_\Omega F(\rho_\epsilon\circ\phi_\epsilon(y))\det\nabla\phi_\epsilon(y)\,\mathrm{d}y = \int_\Omega F\left(\frac{\rho_1(y)}{\det\nabla\phi_\epsilon(y)}\right)\det\nabla\phi_\epsilon(y)\,\mathrm{d}y.$$

And since $F \in C^1\left((0,\infty)\right)$, we deduce by the mean-value theorem that

$$\begin{aligned}
&\int_\Omega \frac{F(\rho_\epsilon(x)) - F(\rho_1(x))}{\epsilon}\,\mathrm{d}x \\
&\quad = \frac{1}{\epsilon}\int_\Omega \left[\left(F\left(\frac{\rho_1}{det\,\nabla\phi_\epsilon}\right) - F(\rho_1)\right) det\,\nabla\phi_\epsilon + F(\rho_1)(det\,\nabla\phi_\epsilon - 1)\right] \\
&\quad = \int_\Omega \left[-F'\left(\rho_1 + \theta\left(\frac{\rho_1}{det\,\nabla\phi_\epsilon} - \rho_1\right)\right)\rho_1 \frac{det\,\nabla\phi_\epsilon - 1}{\epsilon}\right] \\
&\qquad + \int_\Omega \left[F(\rho_1)\frac{det\,\nabla\phi_\epsilon - 1}{\epsilon}\right],
\end{aligned} \tag{48}$$

where $\theta \in [0,1]$. We combine (44), (45) and (48) to have that

$$\lim_{\epsilon\downarrow 0}\int_\Omega \frac{F(\rho_\epsilon(y)) - F(\rho_1(y))}{\epsilon}\,\mathrm{d}y = -\int_\Omega P(\rho_1(y))\,\mathrm{d}iv\,\psi(y)\,\mathrm{d}y. \tag{49}$$

We use (47) and (49), to conclude that

$$\int_{\Omega\times\Omega} \langle\, \nabla c_h(x-y), \psi(y)\,\rangle\,\mathrm{d}\gamma(x,y) + \frac{1}{h}\int_\Omega P(\rho_1(y))\,\mathrm{div}\,\psi(y)\,\mathrm{d}y \le 0. \tag{50}$$

Substituting $\nabla c_h(z) = \frac{1}{h}\nabla c\left(\frac{z}{h}\right)$ in (50) and using that ψ is arbitrarily chosen in $C_c^\infty(\Omega, I\!R^d)$, we conclude (41).

(i). By (26), $N \le \rho_1 \le M$ a.e., and since $F \in C^1((0,\infty))$, we have that $P(\rho_1) \in L^\infty(\Omega)$. Now, let $\varphi \in C_c^\infty(\Omega)$ and for an arbitrary $i \in \mathbb{N}$, define $\psi = (\psi_j)_{j=1,\cdots,d} \in C_c^\infty(\Omega, \mathbb{R}^d)$ by $\psi_j := \delta_{ij}\,\varphi$, where δ_{ij} denotes the Kronecker symbol. Because of (41), we have that

$$\begin{aligned} \Big| \int_\Omega P(\rho_1(y)) \frac{\partial \varphi}{\partial z_i}(y) \Big| &= \Big| \int_{\Omega\times\Omega} \frac{\partial c}{\partial z_i}\left(\frac{x-y}{h}\right) \varphi(y)\, \mathrm{d}\gamma(x,y) \Big| \\ &\le \sup_{x,y\in\Omega} \Big| \frac{\partial c}{\partial z_i}\left(\frac{x-y}{h}\right) \Big| \int_\Omega |\,\varphi(y)\,| \rho_1(y)\, \mathrm{d}y \\ &\le M \,\|\, \varphi \,\|_{L^1(\Omega)} \sup_{x,y\in\Omega} \Big| \frac{\partial c}{\partial z_i}\left(\frac{x-y}{h}\right) \Big|. \end{aligned}$$

And since $c \in C^1(\mathbb{R}^d)$, we deduce (i).

(ii). Because $P(\rho_1) \in W^{1,\infty}(\Omega)$, we can integrate by parts in (41). We use that $\gamma \in \Gamma(\rho_0,\rho_1)$ and $S_\# \rho_1 = \rho_0$ to obtain that

$$\begin{aligned} \int_\Omega \langle \nabla c\left(\frac{S(y)-y}{h}\right), \psi(y) \,\rangle\, \rho_1(y)\, \mathrm{d}y &= \int_\Omega \langle\, \nabla\, [\, P(\rho_1(y))\,]\,, \psi(y) \,\rangle\, \mathrm{d}y \\ &= \int_\Omega \rho_1(y) \,\langle\, \nabla \left[F'(\rho_1(y))\right], \psi(y) \,\rangle\, \mathrm{d}y, \end{aligned}$$

for $\psi \in C_c^\infty(\Omega, \mathbb{R}^d)$. And since ψ is arbitrarily chosen, we deduce that

$$\nabla c\left(\frac{S(y)-y}{h}\right) \rho_1(y) = \nabla\left[\, F'(\rho_1(y))\,\right] \rho_1(y), \tag{51}$$

for a.e. $y \in \Omega$. We combine (51) and the fact that $(\nabla c)^{-1} = \nabla c^\star$ and $\rho_1 \neq 0$ a.e., to conclude (42).

Next, consider $\varphi \in C^2(\bar{\Omega})$. Taking the scalar product of both sides of (42) with $\rho_1(y)\nabla\varphi(y)$, and using that $\gamma = (\mathrm{id} \times S)_\#\, \rho_1$, we have that

$$\begin{aligned} &\frac{1}{h} \int_{\Omega\times\Omega} \langle\, y - x, \nabla\varphi(y) \,\rangle\, d\gamma(x,y) \\ &\quad = -\int_\Omega \langle\, \nabla c^\star \left[\, \nabla \left(F'\left(\rho_1(y)\right) + V(y)\right)\,\right], \nabla\, \varphi(y) \,\rangle\, \rho_1(y)\, \mathrm{d}y. \end{aligned} \tag{52}$$

Now, we express $\frac{1}{h}\int_{\Omega\times\Omega} \langle\, y-x, \nabla\varphi(y) \,\rangle\, d\gamma(x,y)$ in terms of $\int_\Omega \frac{\rho_1(y)-\rho_0(y)}{h}\, \varphi(y)\, \mathrm{d}y$. Since $\gamma \in \Gamma(\rho_0,\rho_1)$, we have that

$$\int_\Omega \frac{\rho_1(y)-\rho_0(y)}{h} \varphi(y)\, \mathrm{d}y = \frac{1}{h}\int_{\Omega\times\Omega} [\,\varphi(y) - \varphi(x)\,]\, \mathrm{d}\gamma(x,y).$$

Combining the above equality with the first order Taylor expansion of φ around y, we obtain that

$$\begin{aligned} &\Big| \frac{1}{h}\int_{\Omega\times\Omega} \langle\, y-x, \nabla\varphi(y) \,\rangle d\gamma(x,y) - \frac{1}{h}\int_\Omega \left(\rho_1(y) - \rho_0(y)\right) \varphi(y) \,\Big| \\ &\quad \le \frac{1}{2h} \sup_{x\in\bar{\Omega}} |\, D^2\varphi(x) \,| \int_{\Omega\times\Omega} |\, x-y \,|^2\, \mathrm{d}\gamma(x,y). \end{aligned} \tag{53}$$

We substitute (52) into (53) to conclude (43). □

2.3 Energy inequality

We establish an inequality relating the internal energy $E_i(\rho_0)$ and $E_i(\rho_1)$ of two probability density functions ρ_0 and ρ_1. This inequality will be called *Energy Inequality* and will be used later to improve compactness properties of the approximate sequence ρ^h (see the definition in section 2.4), to solutions of equation (6). First, we prove this inequality for smooth cost functions c whose Legendre transform $c^\star$ are C^2. Rather than using the density function F, we consider a more general function G which satisfies some assumptions to be specified below. The (internal) energy inequality reads as

$$\int_\Omega G(\rho_0(y))\,\mathrm{d}y - \int_\Omega G(\rho_1(y))\,\mathrm{d}y \geq -\int_\Omega P_G(\rho_1(y))\,\mathrm{div}\,(S(y)-y)\,\mathrm{d}y, \tag{54}$$

where S is the c-optimal map that pushes ρ_1 forward to ρ_0, and $P_G(x) := xG'(x) - G(x)$ is the thermodynamical pressure associated with G. For smooth cost functions c, this inequality is simply a consequence of the displacement convexity of $\mathcal{P}_a(\Omega) \ni \rho \mapsto \int_\Omega G(\rho(x))\,\mathrm{d}x$, that is, the convexity of $[0,1] \ni t \mapsto \int_\Omega G(\rho_{1-t}(x))\,\mathrm{d}x$, where ρ_{1-t} is the probability density obtained by interpolating ρ_0 and ρ_1 along the geodesic joining them in $(\mathcal{P}_a(\Omega), W_c)$ (see Proposition 4.1 in the Appendix). To prove (54), we rather follow a more direct approach using the following regularity property of the c-optimal map (Proposition 4.1): *if ρ_0, $\rho_1 \in \mathcal{P}_a(\Omega)$, c, $c^\star \in C^2(\mathbb{R}^d)$ and S is the c-optimal map that pushes ρ_1 forward to ρ_0, then $\nabla S(y)$ is diagonalizable with positive eigenvalues for $\mu_1 := \rho_1(y)\,dy$ - a.e. $y \in \Omega$. Moreover, the pointwise Jacobian $det\nabla S$ satisfies*

$$0 \neq \rho_1(y) = det\nabla S(y)\,\rho_0(S(y)) \tag{55}$$

for μ_1 - a.e. $y \in \Omega$.

Proposition 2.7 *(Energy inequality for regular cost functions)*
Let ρ_0, $\rho_1 \in \mathcal{P}_a(\Omega)$ be density functions of two Borel probability measures μ_0 and μ_1 on $\mathbb{R}^d$, respectively. Let $\bar{c} : \mathbb{R}^d \to [0,\infty)$ be strictly convex, such that $\bar{c}$, $\bar{c}^\star \in C^2(\mathbb{R}^d)$. Let $G : [0,\infty) \to \mathbb{R}$ be differentiable on $(0,\infty)$, such that $G(0) = 0$ and $(0,\infty) \ni x \mapsto x^d G(x^{-d})$ be convex and nonincreasing. Then, the internal energy inequality (54) holds. In addition, if $P_G(\rho_1) \in W^{1,\infty}(\Omega)$ and $\rho_1 > 0$ a.e., then

$$\int_\Omega G(\rho_0(y))\,dy - \int_\Omega G(\rho_1(y))\,dy \geq \int_\Omega \langle \nabla[\,G'\,(\rho_1(y))\,], S(y)-y\rangle \rho_1(y)\,dy. \tag{56}$$

`Proof:` Set

$$A(x) := x^d G(x^{-d}),\ x \in (0,\infty).$$

We have that

$$A'(x) = -dx^{d-1} P_G(x^{-d}). \tag{57}$$

Since A is nonincreasing, we have that $P_G \geq 0$, and then

(i). $(0,\infty) \ni x \mapsto \frac{G(x)}{x}$ is nondecreasing.

Proposition 4.1 gives that $\nabla S(y)$ is diagonalizable with positive eigenvalues and that (55) holds for μ_1 - a.e. $y \in \Omega$. So, $\rho_0(S(y)) \neq 0$ for μ_1 - a.e. $y \in \Omega$. We use that $G(0) = 0$, $S_{\#}\rho_1 = \rho_0$ and (55) to deduce that

$$\begin{aligned}\int_\Omega G(\rho_0(x))\,\mathrm{d}x &= \int_{[\rho_0 \neq 0]} \frac{G(\rho_0(x))}{\rho_0(x)}\rho_0(x)\,\mathrm{d}x = \int_\Omega \frac{G(\rho_0(S(y)))}{\rho_0(S(y))}\rho_1(y)\,\mathrm{d}y \\ &= \int_\Omega G\left(\frac{\rho_1(y)}{det\,\nabla S(y)}\right) \det\nabla S(y)\,\mathrm{d}y. \end{aligned} \tag{58}$$

Comparing the geometric mean $(\det \nabla S(y))^{1/d}$ to the arithmetic mean $\frac{\mathrm{tr}\,\nabla S(y)}{d}$, we have that

$$\frac{\rho_1(y)}{\det \nabla S(y)} \geq \rho_1(y)\left(\frac{d}{\mathrm{tr}\,\nabla S(y)}\right)^d.$$

Then, we deduce from (i) and the above inequality, that

$$G\left(\frac{\rho_1(y)}{det\,\nabla S(y)}\right) \det\nabla S(y) \geq \Lambda^d\, G\left(\frac{\rho_1(y)}{\Lambda^d}\right) = \rho_1(y) A\left(\frac{\Lambda}{\rho_1(y)^{1/d}}\right), \tag{59}$$

where,

$$\Lambda := \frac{tr\,\nabla S(y)}{d}.$$

Now, we use (57) and the convexity of A, to obtain that

$$\begin{aligned}\rho_1(y)A\left(\frac{\Lambda}{\rho_1(y)^{1/d}}\right) &\geq \rho_1(y)\left[A\left(\frac{1}{\rho_1(y)^{1/d}}\right) + A'\left(\frac{1}{\rho_1(y)^{1/d}}\right)\left(\frac{\Lambda - 1}{\rho_1(y)^{1/d}}\right)\right] \\ &= \rho_1(y)\left[\frac{G_1(\rho_1(y))}{\rho_1(y)} - d(\Lambda - 1)\frac{P_G(\rho_1(y))}{\rho_1(y)}\right] \\ &= G_1(\rho_1(y)) - P_G(\rho_1(y))\; tr\,(\nabla S(y) - \mathrm{id}). \end{aligned} \tag{60}$$

Combining (58) - (60), we conclude that

$$\begin{aligned}\int_\Omega G(\rho_0(y))\,\mathrm{d}y - \int_\Omega G(\rho_1(y))\,\mathrm{d}y &\geq -\int_\Omega P_G(\rho_1(y))\; tr\,(\nabla S(y) - \mathrm{id})\,\mathrm{d}y \\ &= -\int_\Omega P_G(\rho_1(y))\,\mathrm{div}\,(S(y) - y)\,\mathrm{d}y. \end{aligned}$$

Next, assume that $P_G(\rho_1) \in W^{1,\infty}(\Omega)$ and $\rho_1 > 0$ a.e. Since $P_G \geq 0$, we can approximate $P_G(\rho_1)$ by nonnegative functions in $C_c^\infty(\mathbb{R}^d)$. We use Proposition 4.1 - (iv) to obtain that

$$\begin{aligned}-\int_\Omega P_G(\rho_1(y))\,\mathrm{d}iv\,(S(y) - y)\,\mathrm{d}y &\geq \int_\Omega \langle \nabla[P_G(\rho_1(y))], S(y) - y\rangle\,\mathrm{d}y \qquad (61) \\ &= \int_\Omega \langle \nabla[G'(\rho_1(y))], S(y) - y\rangle\,\rho_1(y)\,\mathrm{d}y. \end{aligned}$$

We combine (54) and (61) to conclude (56). □

The next theorem extends the energy inequality (56) to general cost functions c.

Theorem 2.8 *(Energy inequality for general cost functions).*
Let $\rho_0,\, \rho_1 \in \mathcal{P}_a(\Omega)$ be such that $\rho_1 > 0$ a.e., and let $c : I\!\!R^d \to [0,\infty)$ be strictly convex, of class C^1 and satisfy $c(0) = 0$ and (HC3). Let $G : [0,\infty) \to I\!\!R$ be differentiable on $(0,\infty)$, such that $G(0) = 0$, $(0,\infty) \ni x \mapsto x^d G(x^{-d})$ be convex and nonincreasing, $\nabla\left(G'(\rho_1)\right) \in L^\infty(\Omega)$, and $P_G(\rho_1) \in W^{1,\infty}(\Omega)$. Denote by S the c-optimal map that pushes ρ_1 forward to ρ_0. Then,

$$\int_\Omega G(\rho_0(y))\, dy - \int_\Omega G(\rho_1(y))\, dy \geq \int_\Omega \langle \nabla[\, G'\left(\rho_1(y)\right)], S(y) - y\rangle \rho_1(y)\, dy. \tag{62}$$

`Proof:` Let $(c_k)_k$ be a sequence of regular cost functions satisfying (36), which exists because of Proposition 1.3.1 [2]. By Proposition 2.7, we have that

$$\int_\Omega G(\rho_0(y))\, \mathrm{d}y - \int_\Omega G(\rho_1(y))\, \mathrm{d}y \geq \int_\Omega \langle\, \nabla\left(G'\left(\rho_1(y)\right)\right), S_k(y) - y\,\rangle \rho_1(y)\, \mathrm{d}y, \tag{63}$$

for all $k \in I\!\!N$, where S_k denotes the c_k-optimal map that pushes ρ_1 forward to ρ_0. We let k go to ∞ in (63), and we use that $\nabla\left(G'(\rho_1)\right) \in L^\infty(\Omega)$ and $(S_k)_k$ converges to S in $L^2_{\rho_1}(\Omega, I\!\!R^d)$ (see Lemma 2.2.2 [2]), to conclude (62); here $L^2_{\rho_1}(\Omega, I\!\!R^d)$ denotes the set of measurable functions $\varphi : \Omega \to I\!\!R^d$ whose square are summable with respect to the measure $\mu_1 := \rho_1(y)\, \mathrm{d}y$, that is, $\int_\Omega |\, \varphi(y)\,|^2 \rho_1(y)\, \mathrm{d}y < \infty$. □

2.4 Approximate solutions to the parabolic equation

Throughout this section, we assume that $\rho_0 + \frac{1}{\rho_0} \in L^\infty(\Omega)$. For fixed $h > 0$ and $i \in I\!\!N$, we denote by ρ_i^h the minimizer of

$$(P_i^h): \quad \inf \Big\{ hW_c^h(\rho_{i-1}^h, \rho) + E_i(\rho): \quad \rho \in \mathcal{P}_a(\Omega) \Big\}, \tag{64}$$

where $\rho_0^h := \rho_0$ (see Proposition 2.3). We define the approximate solution ρ^h to (6) by

$$\rho^h(t,x) := \begin{cases} \rho_0(x) & \text{if} \quad t = 0 \\ \\ \rho_i^h(x) & \text{if} \quad t \in (t_{i-1}, t_i], \end{cases} \tag{65}$$

where $t_i = ih,\ i \in I\!\!N$. The next proposition shows that

$$\frac{\partial \rho^h}{\partial t} = \operatorname{div}\left\{ \rho^h\, \nabla c^\star \left[\, \nabla \left(F'(\rho^h) \right) \right] \right\} + \Lambda(h),$$

in a weak sense. We will show in the next section that

$$\|\, \Lambda(h)\, \|_{(W^{2,\infty}(\Omega))^\star} = 0\left(h^{\epsilon(q)}\right),$$

where $\epsilon(q) := \min(1, q-1)$.

Proposition 2.9 *Assume that $c : \mathbb{R}^d \to [0,\infty)$ is strictly convex, of class C^1 and satisfies (HC1) - (HC2). Then*

$$\Big| \int_0^T \int_\Omega (\rho_0 - \rho^h)\, \partial_t^h \xi \, dx\, dt \;+\; \int_0^T \int_\Omega \langle\, \rho^h \nabla c^\star \Big[\nabla \Big(F'(\rho^h) \Big) \Big] , \nabla \xi \,\rangle \, dx\, dt \Big| \tag{66}$$

$$\leq \; \frac{1}{2} \sup_{[0,T]\times\bar\Omega} \Big| D^2 \xi(t,x) \Big| \sum_{i=1}^{T/h} \int_{\Omega\times\Omega} |x-y|^2 \, d\gamma_i^h(x,y),$$

for all functions $\xi : \mathbb{R} \times \Omega \to \mathbb{R}$ such that $\xi(t,.) \in C^2(\bar\Omega)$ for $t \in \mathbb{R}$, and $spt\xi(.,x) \subset [-T,T]$ for $x \in \Omega$ and for some $T > 0$. Here,

$$\partial_t^h \xi(t,x) := \frac{\xi(t+h,x) - \xi(t,x)}{h},$$

and γ_i^h is the c_h-optimal measure in $\Gamma(\rho_{i-1}^h, \rho_i^h)$.

`Proof:` Without loss of generality, we assume that $\frac{T}{h} \in \mathbb{N}$. Because of (43), we have that

$$\Big| \int_\Omega A_i^h(t,x)\, \mathrm{d}x \Big| \leq B_i^h,$$

for $t \in (0,T)$, where

$$A_i^h(t,x) := \frac{\rho_i^h(x) - \rho_{i-1}^h(x)}{h}\, \xi(t,x) + \Big\langle \rho_i^h(x) \nabla c^\star \Big[\nabla \Big(F'\Big(\rho_i^h(x) \Big) \Big) \Big] , \nabla \xi(t,x) \Big\rangle,$$

and

$$B_i^h \; := \; \frac{1}{2h} \sup_{[0,T]\times\bar\Omega} \Big| D^2 \xi(t,x) \Big| \int_{\Omega\times\Omega} |\, x-y\,|^2 \, d\gamma_i^h(x,y).$$

We integrate the above inequality over $t \in (0,T)$ to obtain that

$$\Big| \sum_{i=1}^{T/h} \int_{t_{i-1}}^{t_i} \mathrm{d}t \int_\Omega A_i^h(t,x)\, \mathrm{d}x \Big| \; \leq \; h \sum_{i=1}^{T/h} B_i^h. \tag{67}$$

The right hand side of (67) gives that

$$h \sum_{i=1}^{T/h} B_i^h \; = \; \frac{1}{2} \sup_{[0,T]\times\bar\Omega} \Big| D^2 \xi(t,x) \Big| \sum_{i=1}^{T/h} \int_{\Omega\times\Omega} |\, x-y\,|^2 d\gamma_i^h(x,y), \tag{68}$$

while, on the left hand side, we have that

$$\sum_{i=1}^{T/h} \int_{t_{i-1}}^{t_i} \int_\Omega A_i^h(t,x)\, \mathrm{d}x\, \mathrm{d}t \;\; = \;\; \sum_{i=1}^{T/h} \int_{t_{i-1}}^{t_i} \int_\Omega \frac{\rho_i^h(x) - \rho_{i-1}^h(x)}{h}\, \xi(t,x)\, \mathrm{d}x\, \mathrm{d}t \tag{69}$$

$$+ \int_0^T \int_\Omega \Big\langle \rho^h \nabla c^\star \Big[\nabla \Big(F'(\rho^h) \Big) \Big] , \nabla \xi \Big\rangle \, \mathrm{d}x\, \mathrm{d}t.$$

By a direct computation, the first term on the right hand side of (69) gives that

$$\begin{aligned}\sum_{i=1}^{T/h}\int_{t_{i-1}}^{t_i}\int_\Omega \frac{\rho_i^h(x)-\rho_{i-1}^h(x)}{h}\,\xi(t,x)\,\mathrm{d}x\,\mathrm{d}t \quad &= \quad \frac{1}{h}\int_0^T\int_\Omega \rho^h(t,x)\,\xi(t,x)\,\mathrm{d}x\,dt \\ &\quad -\frac{1}{h}\sum_{i=2}^{T/h}\int_{t_{i-1}}^{t_i}\int_\Omega \rho^h(\tau-h,x)\,\xi(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau \\ &\quad -\frac{1}{h}\int_0^h\int_\Omega \rho_0(x)\,\xi(t,x)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

We use the substitution $\tau = t+h$ in the above expression, to obtain that

$$\begin{aligned}&\sum_{i=1}^{T/h}\int_{t_{i-1}}^{t_i}\int_\Omega \frac{\rho_i^h(x)-\rho_{i-1}^h(x)}{h}\,\xi(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \frac{1}{h}\int_0^T\int_\Omega \rho^h(t,x)\,\xi(t,x)\,\mathrm{d}x\,\mathrm{d}t \;-\; \frac{1}{h}\int_0^{T-h} \rho^h(t,x)\,\xi(t+h,x)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad\quad -\frac{1}{h}\int_0^h\int_\Omega \rho_0(x)\,\xi(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = -\int_0^T\int_\Omega \rho^h(t,x)\,\partial_t^h\,\xi(t,x)\,\mathrm{d}x\,\mathrm{d}t \;+\; \frac{1}{h}\int_{T-h}^T \rho^h(t,x)\,\xi(t+h,x) \\ &\quad\quad -\frac{1}{h}\int_0^h\int_\Omega \rho_0(x)\,\xi(t,x)\,dt\,\mathrm{d}x.\end{aligned}$$

Noting that

$$-\frac{1}{h}\int_0^h\int_\Omega \rho_0(x)\,\xi(t,x)\,\mathrm{d}t\,\mathrm{d}x = \int_0^T\int_\Omega \rho_0(x)\,\partial_t^h\xi(t,x)\,\mathrm{d}x\,\mathrm{d}t,$$

and $\xi(t+h)=0$ for $t\in(T-h,T)$, we deduce that

$$\begin{aligned}&\sum_{i=1}^{T/h}\int_{t_{i-1}}^{t_i}\int_\Omega \frac{\rho_i(x)-\rho_{i-1}^h(x)}{h}\,\xi(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_0^T\int_\Omega \Big(\rho_0(x)-\rho^h(t,x)\Big)\;\partial_t^h\xi(t,x)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{70}$$

We combine (67) - (70) to conclude (66). □

3 Existence and uniqueness of solutions

Below, we study the limit of inequality (66) as h goes to 0. The first three sections below deal with the limits of the three terms of (66), and the last section proves the existence theorem to equation (6) when ρ_0 is bounded below and above.

Here and after, ρ^h is defined as in (65), and $F:[0,\infty)\to I\!R$ is strictly convex, twice continuously differentiable on $(0,\infty)$ and satisfies $F(0)=0$ and (HF1)-(HF2).

3.1 Second moments of the optimal measures

We show that

$$\sum_{i=1}^{T/h} \int_{\Omega\times\Omega} |x-y|^2 \,\mathrm{d}\gamma_i^h(x,y) = 0(h^{\epsilon(q)}), \tag{71}$$

where $\epsilon(q) := \min(1, q-1)$, γ_i^h denotes the c_h-optimal measure in $\Gamma(\rho_{i-1}^h, \rho_i^h)$ and ρ_i^h is the unique minimizer of (64). The first step toward proving (71) is the next lemma which states that $\sum_{i=1}^{\infty} W_c^h(\rho_{i-1}^h, \rho_i^h)$ is bounded, uniformly in h.

Lemma 3.1 *Assume that $c : \mathbb{R}^d \to [0,\infty)$ is strictly convex and satisfies (HC1). Then*

$$\sum_{i=1}^{\infty} h\, W_c^h(\rho_{i-1}^h, \rho_i^h) \le E_i(\rho_0) - |\,\Omega\,| F\left(\frac{1}{|\,\Omega\,|}\right). \tag{72}$$

`Proof:` Let $T > 0$ be such that $\frac{T}{h} \in \mathbb{N}$. Since $c(0) = 0$, Proposition 2.3 gives that

$$h\, W_c^h(\rho_{i-1}^h, \rho_i^h) \le E_i(\rho_{i-1}^h) - E_i(\rho_i^h),$$

for all $i \in \mathbb{N}$. We sum both sides of the above inequality over i, to obtain that

$$\sum_{i=1}^{T/h} h\, W_c^h(\rho_{i-1}^h, \rho_i^h) \le E_i(\rho_0) - \int_\Omega F(\rho_{T/h}^h(x))\,\mathrm{d}x.$$

We apply Jensen's inequality on the integral term above, and we let T go to ∞ to conclude (72). □

Proposition 3.2 *Assume that $c : \mathbb{R}^d \to [0,\infty)$ is strictly convex and satisfies $c(0) = 0$ and (HC3). Then for $T > 0$ and $h \in (0,1)$ such that $\frac{T}{h} \in \mathbb{N}$, we have that*

$$\sum_{i=1}^{T/h} \int_{\Omega\times\Omega} |\,x-y\,|^2 \,d\gamma_i^h(x,y) \le M(\Omega, T, F, \rho_0, q, \beta)\, h^{\epsilon(q)}, \tag{73}$$

where $\epsilon(q) := \min(1, q-1)$.

`Proof.` Because of (HC3), we have that $c(z) \ge \beta\,|z|^q$, and then

$$\int_{\Omega\times\Omega} |\,x-y\,|^q \,\mathrm{d}\gamma_i^h(x,y) \;\le\; \frac{h^q}{\beta}\, W_c^h(\rho_{i-1}^h, \rho_i^h), \tag{74}$$

for $i \in \mathbb{N}$. We distinguish two cases, based on the values of q.

<u>*Case 1:*</u> $1 < q \le 2$.

Because of (74), we have for $i \in \mathbb{N}$, that

$$\begin{aligned} \int_{\Omega\times\Omega} |\,x-y\,|^2 \,\mathrm{d}\gamma_i^h(x,y) &\le \sup_{x,y\in\bar\Omega} |\,x-y\,|^{(2-q)} \int_{\Omega\times\Omega} |\,x-y\,|^q \,\mathrm{d}\gamma_i^h(x,y) \\ &\le \frac{(\operatorname{diam}\Omega)^{(2-q)}}{\beta}\, h^q\, W_c^h(\rho_{i-1}^h, \rho_i^h), \end{aligned}$$

where $\operatorname{diam}\Omega$ denotes the diameter of Ω. We sum both sides of the above inequality over i, and we use (72) to conclude that

$$\sum_{i=1}^{T/h} \int_{\Omega\times\Omega} |\, x-y\,|^2 \, \mathrm{d}\gamma_i^h(x,y) \; \leq M(\Omega, F, \rho_0, q, \beta)\, h^{q-1}.$$

<u>*Case 2:*</u> $q > 2$.
Because of Jensen's inequality and (74), we have that

$$\int_{\Omega\times\Omega} |\, x-y\,|^2 \, \mathrm{d}\gamma_i^h(x,y) \leq \left(\int_{\Omega\times\Omega} |\, x-y\,|^q \, \mathrm{d}\gamma_i^h(x,y) \right)^{2/q} \leq \frac{h^2}{\beta^{2/q}} \left[W_c^h(\rho_{i-1}^h, \rho_i^h) \right]^{2/q}.$$

We sum both sides of the above inequality over i, and we use Hölder's inequality on the right hand side term, to obtain that

$$\begin{aligned} \sum_{i=1}^{T/h} \int_{\Omega\times\Omega} |\, x-y\,|^2 \, \mathrm{d}\gamma_i^h(x,y) \quad &\leq \quad \frac{h^2}{\beta^{2/q}} \left(\frac{T}{h} \right)^{1-\frac{2}{q}} \left[\sum_{i=1}^{T/h} W_c^h(\rho_{i-1}^h, \rho_i^h) \right]^{2/q} \\ &= T^{1-\frac{2}{q}} \frac{h}{\beta^{2/q}} \left[\sum_{i=1}^{T/h} h\, W_c^h(\rho_{i-1}^h, \rho_i^h) \right]^{2/q}. \qquad (75) \end{aligned}$$

We combine (72) and (75) to conclude that

$$\sum_{i=1}^{T/h} \int_{\Omega\times\Omega} |\, x-y\,|^2 \, \mathrm{d}\gamma_i^h(x,y) \leq M(\Omega, T, F, \rho_0, q, \beta)\, h. \qquad \square$$

3.2 Strong convergence of the approximate solutions

We prove that $(\rho^h)_h$ is compact in $L^1(\Omega_T)$, for $0 < T < \infty$. The main ingredient in the proof is the energy inequality (62). It allows us to obtain a uniform bound in h of the $L^{q^\star}$- norm of $\nabla \left(F'(\rho^h) \right)$, which leads to the compactness of $(\rho^h)_h$ in $L^1(\Omega_T)$. We first show that $(\rho^h)_h$ converges weakly in $L^1(\Omega_T)$ for a subsequence. We introduce the following constant needed in the next lemma:

$$\overline{M}(\Omega, T, F, \rho_0, q, \alpha) := M(\alpha, q) \left(E_i(\rho_0) - |\,\Omega\,| F\left(\frac{1}{|\,\Omega\,|} \right) + \alpha T |\,\Omega\,|\, \|\, \rho_0 \,\|_{L^\infty(\Omega)} \right),$$

where $M(\alpha, q)$ is a constant which depends only on α and q.

Lemma 3.3 *Assume that $c : \mathbb{R}^d \to [0, \infty)$ is strictly convex, of class C^1 and satisfies (HC1). If $\rho_0 \in \mathcal{P}_a(\Omega) \cap L^\infty(\Omega)$, then*

$$\|\, \rho^h \,\|_{L^\infty((0,\infty); L^\infty(\Omega))} \leq \|\, \rho_0 \,\|_{L^\infty(\Omega)}. \qquad (76)$$

Therefore, there exist $\rho : [0, \infty) \times \Omega \to \mathbb{R}$ and a subsequence of $(\rho^h)_{h\downarrow 0}$ which converges to ρ weakly in $L^1(\Omega_T)$ for $0 < T < \infty$.
In addition, if $\frac{1}{\rho_0} \in L^\infty(\Omega)$ and c satisfies (HC3), then

$$\int_{\Omega_T} \rho^h \Big|\, \nabla \left(F'(\rho^h) \right)\, \Big|^{q^\star} \leq \overline{M}(\Omega, T, F, \rho_0, q, \alpha). \qquad (77)$$

Proof: Because of the upper bound in (26), we have that $\rho_i^h \leq \| \rho_0 \|_{L^\infty(\Omega)}$ for all $i \in \mathbb{N}$. Then $\| \rho^h(t) \|_{L^\infty(\Omega)} \leq \| \rho_0 \|_{L^\infty(\Omega)}$ for $t \in [0,\infty)$. We take the supremum of the previous inequality over $t \in (0,\infty)$ to deduce (76).
Due to (76), we have that $(\rho^h)_h$ is weakly precompact in $L^1(\Omega_T)$, for $0 < T < \infty$. We use the standard diagonal argument to conclude that $(\rho^h)_{h\downarrow 0}$ converges weakly to some function $\rho : [0,\infty) \times \Omega \to \mathbb{R}$ in $L^1(\Omega_T)$ for a subsequence.
Because of Proposition 2.6, the estimate (26) and the fact that $\nabla \left(P(\rho_i^h)\right) = \rho_i^h \nabla \left(F'(\rho_i^h)\right)$, we have that $P(\rho_i^h) \in W^{1,\infty}(\Omega)$ and $\nabla \left(F'(\rho_i^h)\right) \in L^\infty(\Omega)$ for $i \in \mathbb{N}$. We choose $G := F$ in the energy inequality (62), and we use (42) to obtain that

$$h \int_\Omega \left\langle \nabla \left(F'(\rho_i^h)\right), \nabla c^\star \left[\nabla \left(F'(\rho_i^h)\right) \right] \right\rangle \rho_i^h \leq \int_\Omega F(\rho_{i-1}^h) - \int_\Omega F(\rho_i^h).$$

We sum both sides of the subsequent inequality over i, and we use Jensen's inequality to deduce that

$$\int_{\Omega_T} \left\langle \nabla \left(F'(\rho^h)\right), \nabla c^\star \left[\nabla \left(F'(\rho^h)\right) \right] \right\rangle \rho^h \leq \int_\Omega F(\rho_0) - |\,\Omega\,| F\left(\frac{1}{|\,\Omega\,|}\right). \tag{78}$$

Using (139) of Proposition 4.2 in the Appendix, and the fact that $c(z) \leq \alpha\,(|\,z\,|^q + 1)$, we have that

$$\langle\, z, \nabla c^\star(z) \,\rangle \geq c^\star(z) \geq M(\alpha,q)\, |z\,|^{q^\star} - \alpha.$$

Then (78) implies that

$$M(\alpha,q) \int_{\Omega_T} \rho^h \left| \nabla \left(F'(\rho^h)\right) \right|^{q^\star} \leq \int_\Omega F(\rho_0) - |\,\Omega\,| F\left(\frac{1}{|\,\Omega\,|}\right) + \alpha \int_{\Omega_T} \rho^h. \tag{79}$$

We combine (76) and (79) to obtain that

$$M(\alpha,q) \int_{\Omega_T} \rho^h \left| \nabla \left(F'(\rho^h)\right) \right|^{q^\star} \leq \int_\Omega F(\rho_0) - |\,\Omega\,| F\left(\frac{1}{|\,\Omega\,|}\right) + \alpha T |\,\Omega\,|\, \| \rho_0 \|_{L^\infty(\Omega)}.$$

We divide both sides of the above inequality by $M(\alpha,q)$ to conclude (77). □

Lemma 3.4 *(Space-compactness)*
Assume that $c : \mathbb{R}^d \to [0,\infty)$ is strictly convex, of class C^1 and satisfies $c(0) = 0$ and (HC3). If $\rho_0 \in \mathcal{P}_a(\Omega)$ is such that $\rho_0 + \frac{1}{\rho_0} \in L^\infty(\Omega)$, then for $\eta \neq 0$ and $0 < T < \infty$,

$$\int_{\Omega_T^{(\eta)}} \left| \rho^h(t, x + \eta e) - \rho^h(t,x) \right| \leq M(\Omega, T, F, \rho_0, \alpha, q)\, |\eta|, \tag{80}$$

where e is a unit vector of $\mathbb{R}^d$, $\Omega^{(\eta)} := \{ x \in \Omega : \ dist\,(x, \partial\Omega) > |\,\eta\,| \}$, and $\Omega_T^{(\eta)} := (0,T) \times \Omega^{(\eta)}$.

Proof: Since $\rho_0 + \frac{1}{\rho_0} \in L^\infty(\Omega)$, (26) implies that $(\rho^h)_h$ is bounded below and above. Then, we use that $F \in C^2(0,\infty)$ to obtain that

$$\begin{aligned} \left\| \nabla \rho^h \right\|_{L^{q^\star}(\Omega_T)}^{q^\star} &= \int_{\Omega_T} \frac{1}{\rho^h \,[F''(\rho^h)\,]^{q^\star}}\, \rho^h \left| \nabla \left(F'(\rho^h)\right) \right|^{q^\star} \\ &\leq M(\Omega, \rho_0, F) \int_{\Omega_T} \rho^h \left| \nabla \left(F'(\rho^h)\right) \right|^{q^\star}. \end{aligned} \tag{81}$$

We combine (77) and (81) to conclude that $\left(\nabla\rho^h\right)_h$ is bounded in $L^{q^\star}(\Omega_T)$. As a consequence, $(\rho^h)_h$ is bounded in $W^{1,q^\star}(\Omega_T)$. Approximating ρ^h by $C^\infty(\Omega_T)$-functions and using the mean-value theorem and the fact that $(\nabla\rho^h)_h$ is bounded in $L^{q^\star}(\Omega_T)$, we have that

$$\int_{\Omega_T^{(\eta)}} |\, \rho^h(t, x+\eta e) - \rho^h(t,x)\,|^{q^\star} \leq M(\Omega, T, F, \rho_0, \alpha, q)\,|\,\eta\,|^{q^\star}. \tag{82}$$

We combine (82) and Hölder's inequality to conclude that

$$\begin{aligned}\int_{\Omega_T^{(\eta)}} \left| \rho^h(t, x+\eta e) - \rho^h(t,x) \right| &\leq |\,\Omega_T\,|^{1/q} \left(\int_{\Omega_T^{(\eta)}} \left| \rho^h(t, x+\eta e) - \rho^h(t,x) \right|^{q^\star} \right)^{1/q^\star} \\ &\leq M(\Omega, T, F, \rho_0, \alpha, q)\,|\,\eta\,|. \qquad \square\end{aligned}$$

Next, we focus on the time-compactness of $(\rho^h)_h$ in Ω_T for $0 < T < \infty$. The following constant will be needed in the next lemma:

$$\begin{aligned}&\overline{\overline{M}}(\Omega, T, F, \rho_0, q, \alpha, \beta) \\ &:= \frac{\|\,\rho_0\|_{L^\infty(\Omega)}^{1/q^\star}}{\left\| \frac{1}{\rho_0} \right\|_{L^\infty(\Omega)}^{1/q^\star}} M(q,\alpha,\beta) \left(E_i(\rho_0) - |\,\Omega\,| F\left(\frac{1}{|\,\Omega\,|}\right) + \alpha T |\,\Omega\,|\,\|\,\rho_0\,\|_{L^\infty(\Omega)} \right),\end{aligned}$$

where $M(q,\alpha,\beta)$ is a constant which only depends on q, α and β.

Lemma 3.5 *Assume that $c : \mathbb{R}^d \to [0,\infty)$ is strictly convex, of class C^1 and satisfies $c(0) = 0$ and (HC3). If $\rho_0 \in \mathcal{P}_a(\Omega)$ is such that $\rho_0 + \frac{1}{\rho_0} \in L^\infty(\Omega)$, then for $\tau > 0$ and $0 < T < \infty$,*

$$\begin{aligned}\int_{\Omega_T} \left[F'\left(\rho^h(t+\tau, x)\right) - F'\left(\rho^h(t,x)\right) \right] \left[\rho^h(t+\tau,x) - \rho^h(t,x) \right] \\ \leq \overline{\overline{M}}(\Omega, T, F, \rho_0, q, \alpha, \beta)\,\tau.\end{aligned} \tag{83}$$

`Proof:` Without loss of generality, we assume that $\frac{T}{h} \in \mathbb{N}$ and $\tau = Nh$ for some $N \in \mathbb{N}$. For simplicity, we set

$$L(h,\tau) := \int_{\Omega_T} \left[F'\left(\rho^h(t+\tau, x)\right) - F'\left(\rho^h(t,x)\right) \right] \left[\rho^h(t+\tau,x) - \rho^h(t,x) \right]$$

and

$$J(i,h,N) := \int_{\Omega} \left[F'\left(\rho^h_{i+N}(x)\right) - F'\left(\rho^h_i(x)\right) \right] \left[\rho^h_{i+N}(x) - \rho^h_i(x) \right].$$

It is straightforward to check that

$$L(h,\tau) = \sum_{i=1}^{T/h} h\, J(i,h,N). \tag{84}$$

Since $(W_c^h)^{1/q}$ does not satisfy the triangle inequality, we introduce the q-Wasserstein metric $d_q^h := (W_q^h)^{1/q}$ defined by

$$d_q^h(\rho_i^h, \rho_{i+N}^h) := \left(\int_\Omega \Big| \frac{y - S_q^h(y)}{h} \Big|^q \rho_{i+N}^h(y)\, \mathrm{d}y \right)^{1/q}, \tag{85}$$

where S_q^h denotes the $|\frac{\cdot}{h}|^q$-optimal map that pushes ρ_{i+N}^h forward to ρ_i^h. Setting $\varphi_{i,N}^h := F'\left(\rho_{i+N}^h\right) - F'\left(\rho_i^h\right)$, we obtain that

$$J(i,h,N) = \int_\Omega \left[\varphi_{i,N}^h(y) - \varphi_{i,N}^h\left(S_q^h(y)\right) \right] \rho_{i+N}^h(y)\, \mathrm{d}y.$$

Since $\rho_0 + \frac{1}{\rho_0} \in L^\infty(\Omega)$, $F \in C^2(0,\infty)$ and $\rho_i^h \nabla\left(F'(\rho_i^h)\right) = \nabla\left(P(\rho_i^h)\right) \in L^\infty(\Omega)$ (because of Proposition 2.6 - (i)), (26) gives that $\varphi_{i,N}^h \in W^{1,\infty}(\Omega)$. So, approximating $\varphi_{i,N}^h$ by $C^\infty(\Omega)$-functions and using that $\left(S_q^h\right)_\# \rho_{i+N}^h = \rho_i^h$ and the mean-value theorem, we rewrite $J(i,h,N)$ as follows:

$$J(i,h,N) = \int_\Omega \int_0^1 \left\langle \nabla\varphi_{i,N}^h\left((1-t)y + tS_q^h(y)\right), y - S_q^h(y) \right\rangle \rho_{i+N}^h(y)\, \mathrm{d}t\, \mathrm{d}y.$$

We combine Hölder's inequality and (85) to deduce that

$$\begin{aligned} &J(i,h,N) \\ &\le h\, d_q^h\left(\rho_i^h, \rho_{i+N}^h\right) \left[\int_\Omega \int_0^1 \Big| \nabla\varphi_{i+N}^h\left((1-t)y + tS_q^h(y)\right) \Big|^{q^\star} \rho_{i+N}^h(y)\, \mathrm{d}t\, \mathrm{d}y \right]^{1/q^\star}. \end{aligned} \tag{86}$$

But, observe that ρ_i^h, $\rho_{i+N}^h \le \|\rho_0\|_{L^\infty(\Omega)}$ because of Proposition 2.2, and $|\nabla\varphi_{i,N}^h|^{q^\star} \in L^\infty(\Omega)$. So, we approximate $|\nabla\varphi_{i,N}^h|^{q^\star}$ by nonnegative functions in $C_c^\infty(\mathbb{R}^d)$ and we use (31) in Proposition 2.4 to deduce that

$$\int_\Omega \Big| \nabla\varphi_{i,N}^h\left((1-t)y + t\rho_q^h(y)\right) \Big|^{q^\star} \rho_{i+N}^h(y)\, \mathrm{d}y \le \|\rho_0\|_{L^\infty(\Omega)} \int_{R^d} \Big| \nabla\varphi_{i,N}^h(y) \Big|^{q^\star} \mathrm{d}y. \tag{87}$$

We combine (84), (86) and (87) to have that

$$L(h,\tau) \le \|\rho_0\|_{L^\infty(\Omega)}^{1/q^\star} h^2 \sum_{i=1}^{T/h} d_q^h\left(\rho_i^h, \rho_{i+N}^h\right) \|\nabla\varphi_{i,N}^h\|_{L^{q^\star}(\Omega)}.$$

And since d_q^h is a metric, the triangle inequality gives that

$$L(h,\tau) \le \|\rho_0\|_{L^\infty(\Omega)}^{\frac{1}{q^\star}} h^2 \sum_{k=1}^N \sum_{i=1}^{T/h} \|\nabla\varphi_{i,N}^h\|_{L^{q^\star}(\Omega)} d_q^h\left(\rho_{i+k-1}^h, \rho_{i+k}^h\right).$$

Then we apply Hölder's inequality to the interior sum to deduce that

$$\begin{aligned} &L(h,\tau) \\ &\le \|\rho_0\|_{L^\infty(\Omega)}^{\frac{1}{q^\star}} h^{2-\frac{1}{q^\star}} \left(\sum_{i=1}^{T/h} h \|\nabla\varphi_{i,N}^h\|_{L^{q^\star}(\Omega)}^{q^\star} \right)^{1/q^\star} \sum_{k=1}^N \left[\sum_{i=1}^{T/h} d_q^h\left(\rho_{i+k-1}^h, \rho_{i+k}^h\right)^q \right]^{1/q} \end{aligned} \tag{88}$$

Because of (26) and (77), the sequences $\left(h^{1/q^\star}\left\|\nabla\left(F'(\rho_i^h)\right)\right\|_{L^{q^\star}(\Omega)}\right)_{i=1,\cdots,\frac{T}{h}}$ and $\left(h^{1/q^\star}\left\|\nabla\left(F'(\rho_{i+N}^h)\right)\right\|_{L^{q^\star}(\Omega)}\right)_{i=1,\cdots,\frac{T}{h}}$ belong to $l_{q^\star}(\Omega)$. Then we combine Hölder's inequality, Minkowski's inequality, (26) and (77) to have that

$$
\begin{aligned}
&\left(\sum_{i=1}^{T/h} h\left\|\nabla\varphi_{i,N}^h\right\|_{L^{q^\star}(\Omega)}^{q^\star}\right)^{1/q^\star}\\
&\leq\left(\sum_{i=1}^{T/h}\left(h^{1/q^\star}\left\|\nabla\left(F'(\rho_{i+N}^h)\right)\right\|_{L^{q^\star}(\Omega)}+h^{1/q^\star}\left\|\nabla\left(F'(\rho_i^h)\right)\right\|_{L^{q^\star}(\Omega)}\right)^{q^\star}\right)^{1/q^\star}\\
&\leq\left(\sum_{i=1}^{T/h} h\left\|\nabla\left(F'(\rho_{i+N}^h)\right)\right\|_{L^{q^\star}(\Omega)}^{q^\star}\right)^{1/q^\star}+\left(\sum_{i=1}^{T/h} h\left\|\nabla\left(F'(\rho_i^h)\right)\right\|_{L^{q^\star}(\Omega)}^{q^\star}\right)^{1/q^\star}\\
&\leq\frac{1}{\left\|\frac{1}{\rho_0}\right\|_{L^\infty(\Omega)}^{1/q^\star}}\left[\overline{M}\left(\Omega,T,F,\rho_0,q,\alpha\right)\right]^{1/q^\star}. \qquad (89)
\end{aligned}
$$

On the other hand, since $c(z)\geq\beta\,|\,z\,|^q$, we have that $\left(d_q^h\right)^q\leq\frac{1}{\beta}W_c^h$, and then,

$$
\sum_{k=1}^{N}\left[\sum_{i=1}^{T/h} d_q^h\left(\rho_{i+k-1}^h,\rho_{i+k}^h\right)^q\right]^{1/q}\leq\frac{1}{(\beta h)^{1/q}}\sum_{k=1}^{N}\left[\sum_{i=1}^{T/h} h\,W_c^h\left(\rho_{i+k-1}^h,\rho_{i+k}^h\right)\right]^{1/q}.
$$

We use (72) and the above inequality to deduce that

$$
\sum_{k=1}^{N}\left[\sum_{i=1}^{T/h} d_q^h\left(\rho_{i+k-1}^h,\rho_{i+k}^h\right)^q\right]^{1/q}\leq\frac{1}{\beta}\left[E_i(\rho_0)-|\,\Omega\,|F\left(\frac{1}{|\,\Omega\,|}\right)\right]^{1/q}Nh^{-1/q}. \qquad (90)
$$

We combine (88) - (90), and we use that $\tau=Nh$ to conclude that

$$
L(h,\tau)\leq\overline{\overline{\overline{M}}}(\Omega,T,F,\rho_0,q,\alpha,\beta)\,\tau. \qquad \square
$$

Lemma 3.6 *(Time-compactness)*
Assume that the assumptions of Lemma 3.4 hold. If $\rho_0\in\mathcal{P}_a(\Omega)$ is such that $\rho_0+\frac{1}{\rho_0}\in L^\infty(\Omega)$, then for $0<T<\infty$ and small $\tau>0$,

$$
\int_{\Omega_T}\left|\rho^h(t+\tau,x)-\rho^h(t,x)\right|\leq M(R,\Omega,T,F,\rho_0,\alpha,q,\beta)\sqrt{\tau}+T\Lambda(\sqrt{\tau})
$$

for some function Λ such that $\lim_{\tau\downarrow 0}\Lambda(\sqrt{\tau})=0$.

Proof: Let $R > 0$ and for fixed h, T and τ, define

$$E_R := \Big\{ t \in (0,T) : \quad \Delta_{h,\tau}(t) := \Big\| \rho^h(t) \Big\|_{L^q(\Omega)} + \Big\| \rho^h(t+\tau) \Big\|_{L^q(\Omega)}$$

$$+ \Big\| F'\left(\rho^h(t)\right) \Big\|_{W^{1,q^\star}(\Omega)} + \Big\| F'\left(\rho^h(t+\tau)\right) \Big\|_{W^{1,q^\star}(\Omega)}$$

$$+ \tfrac{1}{\tau} \int_\Omega \left[F'\left(\rho^h(t+\tau)\right) - F'\left(\rho^h(t)\right) \right] \left[\rho^h(t+\tau) - \rho^h(t) \right] > R \Big\}.$$

Because of (26), (77), (83) and the fact that $F \in C^2(0,\infty)$, we have that $(0,T) \ni t \mapsto \Delta_{h,\tau}(t)$ belongs to $L^1(0,T)$. Hence

$$| E_R | \leq \frac{M(\Omega, T, F, \rho_0, q, \alpha, \beta)}{R}. \tag{91}$$

We combine (76) and (91) to have that

$$\int_{E_R} \int_\Omega \Big| \rho^h(t+\tau, x) - \rho^h(t,x) \Big| \leq 2 \| \rho_0 \|_{L^\infty(\Omega)} | \Omega | | E_R | \leq \frac{M(\Omega, T, F, \rho_0, q, \alpha, \beta)}{R}. \tag{92}$$

On the other hand, if $t \in E_R^c := (0,T) \setminus E_R$, setting $\rho^h(t) := u_1$ and $\rho^h(t+\tau) := u_2$, we have that $\| u_i \|_{L^q(\Omega)} \leq R$, $\| F'(u_i) \|_{W^{1,q^\star}(\Omega)} \leq R$ for $i = 1, 2$, and $\int_\Omega [F'(u_2) - F'(u_1)] [u_2 - u_1] \leq R\tau$. Then Proposition 2.5 gives that

$$\int_{E_R^c} \int_\Omega | \rho^h(t+\tau, x) - \rho^h(t,x) | \leq \int_{E_R^c} \Lambda(R\tau) \leq T \Lambda(R\tau), \tag{93}$$

where $\Lambda(R\tau) := \Lambda_R(R\tau)$ is defined as in Proposition 2.5. We combine (92) - (93), and we choose $R = \frac{1}{\sqrt{\tau}}$ to conclude the proof. □

Having proved the space-compactness and time-compactness of $(\rho^h)_h$, we are now ready to show that $(\rho^h)_h$ converges strongly to ρ in $L^1(\Omega_T)$, $0 < T < \infty$, for a subsequence, where ρ is defined as in Lemma 3.3.

Proposition 3.7 *Assume that $c : \mathbb{R}^d \to [0,\infty)$ is strictly convex, of class C^1 and satisfies $c(0) = 0$ and (HC3). If $\rho_0 \in \mathcal{P}_a(\Omega)$ is such that $\rho_0 + \frac{1}{\rho_0} \in L^\infty(\Omega)$, then for $0 < T < \infty$, there is a subsequence of $(\rho^h)_{h\downarrow 0}$ which converges strongly to ρ in $L^r(\Omega_T)$ for $1 \leq r < \infty$, where ρ is defined as in Lemma 3.3.*

Proof: Fix $\delta > O$, and define $\Omega_T^{(\delta)}$ as in Lemma 3.4. Because of (76), we have that $(\rho^h)_h$ is bounded in $L^1\left(\Omega_T^{(\delta)}\right)$. Furthermore, for $\epsilon > 0$ and small $\tau > 0$ and $\eta \in (0,\delta)$, we have that $\Omega_T^{(\delta)} \subset \Omega_T^{(\eta)} \subset \Omega_T$ and then, Lemma 3.4 and Lemma 3.6 give that

$$\int_{\Omega_T^{(\delta)}} | \rho^h(t, x+\eta e) - \rho^h(t,x) | < \epsilon, \text{ and } \int_{\Omega_T^{(\delta)}} | \rho^h(t+\tau, x) - \rho^h(t,x) | < \epsilon$$

uniformly in h. We deduce that $(\rho^h)_h$ is precompact in $L^1\left(\Omega_T^{(\delta)}\right)$ (see [1], Theorem 2.21). We observe that $\lim_{\delta\to 0} | \Omega \setminus \Omega^{(\delta)} | = 0$ and then, we use the diagonal argument to obtain that $(\rho^h)_h$ converges strongly to ρ in $L^1(\Omega_T)$, for a subsequence. And since (76) gives that $(\rho^h)_h$ is bounded in $L^\infty(\Omega_T)$, we conclude that $(\rho^h)_h$ converges to ρ in $L^r(\Omega_T)$ for $1 \leq r < \infty$, up to a subsequence. □

3.3 Weak convergence of the nonlinear term

We use the energy inequality (62) to show that $\left(\mathrm{div}\{\rho^h \nabla c^\star \left[\nabla \left(F'(\rho^h)\right)\right]\}\right)_h$ converges weakly to $\mathrm{div}\{\rho\nabla c^\star \left[\nabla \left(F'(\rho)\right)\right]\}$ in Ω_T, for a subsequence. Throughout this section, $(\rho^h)_h$ denotes the (non-relabeled) subsequence of $(\rho^h)_h$ which converges to ρ in $L^r(\Omega_T)$ for $1 \leq r < \infty$ (from Proposition 3.7), and

$$\sigma^h := \nabla c^\star \left[\nabla \left(F'(\rho^h)\right)\right].$$

The next lemma shows that $(\sigma^h)_h$ is bounded in $L^q(\Omega_\infty)$ and $\left(\nabla \left(F'(\rho^h)\right)\right)_h$ converges weakly to $\nabla \left(F'(\rho)\right)$ in $L^{q^\star}(\Omega_T)$ for a subsequence.

Lemma 3.8 *Assume that $c : \mathbb{R}^d \to [0,\infty)$ is strictly convex, of class C^1 and satisfies $c(0) = 0$ and (HC3). If $\rho_0 \in \mathcal{P}_a(\Omega)$ is such that $\rho_0 + \frac{1}{\rho_0} \in L^\infty(\Omega)$, then*

$$\| \sigma^h \|^q_{L^q(\Omega_\infty)} \leq \frac{1}{\beta \left\| \frac{1}{\rho_0} \right\|_{L^\infty(\Omega)}} \left[E_i(\rho_0) - |\,\Omega\,| F\left(\frac{1}{|\,\Omega\,|}\right)\right]. \tag{94}$$

(i). Therefore, there is a subsequence of $\left(\sigma^h\right)_{h\downarrow 0}$ which converges weakly to a function σ in $L^q(\Omega_T)$, for $0 < T < \infty$.

(ii). Furthermore, there is a subsequence of $\left\{\nabla \left(F'(\rho^h)\right)\right\}_{h\downarrow 0}$ which converges weakly to $\nabla \left(F'(\rho)\right)$ in $L^{q^\star}\left((0,T)\times\Omega\right)$, for $0 < T < \infty$.

`Proof:` By (42), we have that

$$\frac{S_i^h(y) - y}{h} = \nabla c^\star \left[\nabla \left(F'(\rho_i^h(y))\right)\right] \tag{95}$$

for $i \in \mathbb{N}$, where S_i^h denotes the c_h-optimal map that pushes ρ_i^h forward to ρ_{i-1}^h. We use (26) and (95) to deduce that

$$\begin{aligned} \| \sigma^h \|^q_{L^q(\Omega_\infty)} &= \sum_{i=1}^\infty h \int_\Omega \left| \nabla c^\star \left[\nabla \left(F'(\rho_i^h(y))\right)\right] \right|^q \mathrm{d}y = \sum_{i=1}^\infty h \int_\Omega \left| \frac{S_i^h(y) - y}{h} \right|^q \mathrm{d}y \\ &\leq \frac{1}{\left\| \frac{1}{\rho_0} \right\|_{L^\infty(\Omega)}} \sum_{i=1}^\infty h \int_\Omega \left| \frac{S_i^h(y) - y}{h} \right|^q \rho_i^h(y)\, \mathrm{d}y. \end{aligned}$$

Since $c(z) \geq \beta\,|\,z\,|^q$, we obtain that

$$\| \sigma_h \|^q_{L^q(\Omega_\infty)} \leq \frac{1}{\beta \left\| \frac{1}{\rho_0} \right\|_{L^\infty(\Omega)}} \sum_{i=1}^\infty h\, W_c^h(\rho_{i-1}^h, \rho_i^h). \tag{96}$$

We combine (72) and (96) to conclude (94). (i) is a direct consequence of (94).
Now, fix $0 < T < \infty$. By Proposition 3.7, $(\rho^h)_h$ converges strongly to ρ in $L^1(\Omega_T)$, and by (76) and the fact that F' is continuous on $(0,\infty)$, $\left(F'(\rho^h)\right)_h$ is bounded in

$L^\infty(\Omega_T)$. We deduce that $\left(F'(\rho^h)\right)_h$ converges weakly to $F'(\rho)$ in $L^{q^\star}(\Omega_T)$. And, since $\left\{ \nabla\left(F'(\rho^h)\right) \right\}_h$ is bounded in $L^{q^\star}(\Omega_T)$ (because of (26) and (77)), we conclude (ii). □

The next lemma extends the energy inequality (62) to the time-space domain $(0,\infty)\times\Omega$.

Lemma 3.9 *(Energy Inequality in time-space)*
Assume that $c : \mathbb{R}^d \to [0,\infty)$ is strictly convex, of class C^1 and satisfies $c(0) = 0$ and (HC3). If $\rho_0 \in \mathcal{P}_a(\Omega)$ is such that $\rho_0 + \frac{1}{\rho_0} \in L^\infty(\Omega)$, and $t \mapsto u(t)$ is a nonnegative function in $C^2_c(\mathbb{R})$, then

$$\int_0^\infty \int_\Omega \left\langle \rho^h \nabla\left(F'(\rho^h)\right), \nabla c^\star\left[\nabla\left(F'(\rho^h)\right)\right]\right\rangle u(t) \leq \frac{1}{h}\int_0^h \int_\Omega F\left(\rho_0(x)\right) u(t) + \int_0^\infty \int_\Omega F(\rho^h)\, \partial_t^h u(t),$$

where

$$\partial_t^h u(t) := \frac{u(t+h) - u(t)}{h}.$$

`Proof:` Let T be such that $\frac{T}{h} \in \mathbb{N}$, and assume that $\operatorname{spt} u \subset [-T,T]$. We choose $G := F$ in the energy inequality (62) and we use (95) to obtain that

$$\int_\Omega \frac{F\left(\rho_i^h(y)\right) - F\left(\rho_{i-1}^h(y)\right)}{h}\, dy \leq -\int_\Omega \left\langle \nabla\left[F'\left(\rho_i^h(y)\right)\right], \nabla c^\star\left[\nabla\left(F'(\rho_i^h(y))\right)\right]\right\rangle \rho_i^h(y)\, dy,$$

for all $i \in \mathbb{N}$. Since $u \geq 0$, we deduce that

$$\sum_{i=1}^{T/h} \int_{t_{i-1}}^{t_i} \int_\Omega \frac{F\left(\rho_i^h(y)\right) - F\left(\rho_{i-1}^h(y)\right)}{h} u(t) \leq -\int_{\Omega_T} \rho^h \left\langle \nabla\left(F'(\rho^h)\right), \nabla c^\star\left[\nabla\left(F'(\rho^h)\right)\right]\right\rangle u(t). \tag{97}$$

By direct computations, the left hand side of the above inequality gives that

$$\begin{aligned}\sum_{i=1}^{T/h} \int_{t_{i-1}}^{t_i} \int_\Omega \frac{F\left(\rho_i^h(y)\right) - F\left(\rho_{i-1}^h(y)\right)}{h} u(t) &= \frac{1}{h}\int_{\Omega_T} F\left(\rho^h(t,y)\right) u(t) - \frac{1}{h}\int_{\Omega_h} F\left(\rho_0(y)\right) u(t) \\ &\quad - \frac{1}{h}\int_h^T \int_\Omega F\left(\rho^h(t-h)\right) u(t).\end{aligned}$$

We use the substitution $\tau = t - h$ in the last integral and the fact that $u(t+h) = 0$ for $t \in (T-h, T)$ to have that

$$\sum_{i=1}^{T/h} \int_{t_{i-1}}^{t_i} \int_\Omega \frac{F\left(\rho_i^h(y)\right) - F\left(\rho_{i-1}^h(y)\right)}{h} u(t) \tag{98}$$

$$= -\int_{\Omega_T} F\left(\rho^h(t,y)\right)\, \partial_t^h u(t) - \frac{1}{h}\int_{\Omega_h} F\left(\rho_0(y)\right)\, u(t).$$

We combine (97) and (98), and we let T go to ∞, to complete the proof. □

Theorem 3.10 *Assume that $c: \mathbb{R}^d \to [0,\infty)$ is strictly convex, of class C^1 and satisfies $c(0) = 0$ and (HC3), and $F: [0,\infty) \to \mathbb{R}$ is strictly convex, of class $C^2(0,\infty)$ and satisfies $F(0) = 0$ and (HF1) - (HF2). If $\rho_0 \in \mathcal{P}_a(\Omega)$ is such that $\rho_0 + \frac{1}{\rho_0} \in L^\infty(\Omega)$, and $t \mapsto u(t)$ is a nonnegative function in $C_c^2(\mathbb{R})$, then*

$$\lim_{h\downarrow 0}\int_{\Omega_\infty} \left\langle \rho^h\sigma^h, \nabla\left(F'(\rho^h)\right) \right\rangle u(t) = \int_{\Omega_\infty} \left\langle \rho\sigma, \nabla\left(F'(\rho)\right) \right\rangle u(t), \tag{99}$$

where ρ and σ are defined in Lemma 3.3 and Lemma 3.8.
Therefore, $\left(div\,(\rho^h\sigma^h)\right)_h$ converges weakly to $div\,(\rho\sigma)$ for a subsequence in $\left[C_c^2(\mathbb{R}\times\mathbb{R}^d)\right]'$, and

$$div\,(\rho\sigma) = div\left(\rho\nabla c^\star\left[\nabla\left(F'(\rho)\right)\right]\right). \tag{100}$$

`Proof:` Let $T > 0$ be such that $\operatorname{spt} u \subset [-T,T]$, and assume that $\rho(t) = \rho_0$ for $t \le 0$. Denote by $(\rho^h)_h$ the subsequence of $(\rho^h)_h$, such that

(i). $(\rho^h)_{h\downarrow 0}$ converges to ρ a.e.,

(ii). $\{\nabla\left(F'(\rho^h)\right)\}_{h\downarrow 0}$ converges weakly to $\nabla\left(F'(\rho)\right)$ in $L^{q^\star}(\Omega_T)$, and

(iii). $\{\sigma^h = \nabla c^\star\left[\nabla\left(F'(\rho^h)\right)\right]\}_{h\downarrow 0}$ converges weakly to σ in $L^q(\Omega_T)$,

as in Proposition 3.7 and Lemma 3.8. We first observe that

$$\lim_{h\downarrow 0}\int_{\Omega_T} \langle \sigma^h, \rho^h\nabla\left(F'(\rho)\right)\rangle\, u(t) = \int_{\Omega_T} \langle \sigma, \rho\nabla\left(F'(\rho)\right)\rangle\, u(t) \tag{101}$$

and

$$\lim_{h\downarrow 0}\int_{\Omega_T} \langle \rho^h\nabla c^\star\left[\nabla\left(F'(\rho)\right)\right], \nabla\left(F'(\rho^h)\right) - \nabla\left(F'(\rho)\right)\rangle\, u(t) = 0. \tag{102}$$

Indeed, since $(\rho^h)_h$ is bounded in $L^\infty(\Omega_T)$ (see (76)) and $\nabla\left(F'(\rho)\right) \in L^{q^\star}(\Omega_T)$, (i) and the dominated convergence theorem imply that $\{\rho^h\nabla\left(F'(\rho)\right)\}_{h\downarrow 0}$ converges to $\rho\nabla\left(F'(\rho)\right)$ in $L^{q^\star}(\Omega_T)$. Then we use (iii) and the fact that $u \in C_c^2(\mathbb{R})$ to conclude (101).
Because of Proposition 4.2, the convexity of c and the fact that $c(z) \ge \beta\,|\,z\,|^q$, we have that

$$\begin{aligned}
|\,\nabla c^\star(z)\,|^q \le \frac{c\left(\nabla c^\star(z)\right)}{\beta} &= \frac{1}{\beta}\left(\langle z, \nabla c^\star(z)\rangle - c^\star(z)\right) \\
&\le \frac{1}{\beta}\langle z, \nabla c^\star(z)\rangle \le M(\beta,q)\,|\,z\,|^{q^\star}.
\end{aligned}$$

We deduce that

$$\left|\nabla c^\star\left[\nabla\left(F'(\rho)\right)\right]\right|^q \le M(\beta,q)\left|\nabla\left(F'(\rho)\right)\right|^{q^\star}, \tag{103}$$

which shows that $\nabla c^\star [\nabla (F'(\rho))] \in L^q(\Omega_T)$. Then we use (i) and the dominated convergence theorem to have that $\{ \rho^h \nabla c^\star [\nabla (F'(\rho))] \}_{h\downarrow 0}$ converges to $\rho \nabla c^\star [\nabla (F'(\rho))]$ in $L^q(\Omega_T)$. We conclude (102) because of (ii).
The proof of (99) follows directly from the following three claims:

Claim 1.

$$\int_{\Omega_T} \langle \rho\sigma, \nabla (F'(\rho)) \rangle \, u(t) \le \liminf_{h\downarrow 0} \int_{\Omega_T} \left\langle \rho^h \sigma^h, \nabla \left(F'(\rho^h)\right) \right\rangle u(t).$$

`Proof:` Because $c^\star$ is convex, and u and ρ^h are nonnegative, we have that

$$\int_{\Omega_T} \rho^h \left\langle \nabla c^\star \left[\nabla \left(F'(\rho^h)\right)\right] - \nabla c^\star [\nabla (F'(\rho))], \nabla \left(F'(\rho^h)\right) - \nabla (F'(\rho)) \right\rangle u(t) \ge 0$$

and then,

$$\begin{aligned} &\liminf_{h\downarrow 0} \int_{\Omega_T} \langle \sigma^h, \rho^h \nabla (F'(\rho)) \rangle u(t) \\ &\quad \le \liminf_{h\downarrow 0} \int_{\Omega_T} \langle \rho^h \sigma^h, \nabla \left(F'(\rho^h)\right) \rangle u(t) \\ &\qquad + \limsup_{h\downarrow 0} \int_{\Omega_T} \langle \rho^h \nabla c^\star [\nabla (F'(\rho))], \nabla (F'(\rho)) - \nabla \left(F'(\rho^h)\right) \rangle u(t). \end{aligned} \tag{104}$$

We combine (101) - (104) to conclude Claim 1.

Claim 2.

$$\begin{aligned} &\limsup_{h\downarrow 0} \int_{\Omega_T} \left\langle \rho^h \sigma^h, \nabla \left(F'(\rho^h)\right) \right\rangle u(t) \\ &\quad \le \int_{\Omega} [\rho_0 F'(\rho_0) - F^\star (F'(\rho_0))] \, u(0) \\ &\qquad + \int_{\Omega_T} [\rho(t,x) F'(\rho(t,x)) - F^\star (F'(\rho(t,x)))] \, u'(t). \end{aligned}$$

`Proof:` First, we observe that

$$\lim_{h\downarrow 0} \int_{\Omega_T} F(\rho^h) \partial_t^h u(t) = \int_{\Omega_T} F(\rho) u'(t). \tag{105}$$

Indeed, it is clear that

$$\begin{aligned} &\left| \int_{\Omega_T} F(\rho^h)\partial_t^h u(t) - F(\rho) u'(t) \right| \\ &\quad \le \int_{\Omega_T} | F(\rho^h) - F(\rho) | |u'(t)| + \int_{\Omega_T} |F(\rho^h)| |\partial_t^h u(t) - u'(t)|. \end{aligned} \tag{106}$$

Because of (26) and the continuity of F, we have that $\left(F(\rho^h)\right)_h$ is bounded in $L^\infty(\Omega_T)$. We let h go to 0 in (106), and we use (i), the fact that $u \in C_c^2(\mathbb{R})$ and the Lebesgue

dominated convergence theorem to conclude (105).
Lemma 3.9 gives that

$$\limsup_{h\downarrow 0}\int_{\Omega_T}\left\langle \rho^h\sigma^h, \nabla\left(F'(\rho^h)\right)\right\rangle u(t)$$
$$\leq \liminf_{h\downarrow 0}\frac{1}{h}\int_0^h\int_\Omega F(\rho_0)u(t)+\limsup_{h\downarrow 0}\int_{\Omega_T}F(\rho^h)\partial_t^h u(t),$$

and by (105) and the continuity of u, we deduce that

$$\limsup_{h\downarrow 0}\int_{\Omega_T}\left\langle \rho^h\sigma^h, \nabla\left(F'(\rho^h)\right)\right\rangle u(t)\leq \int_\Omega F(\rho_0)u(0)+\int_{\Omega_T}F\left(\rho(t,x)\right)u'(t). \tag{107}$$

Since $F\in C^1\left((0,\infty)\right)$ is strictly convex and satisfies $F(0)=0$ and (HF1), we have that

$$F^\star\left(F'(a)\right)=aF'(a)-F(a)\quad \forall\, a>0. \tag{108}$$

We substitute (108) into (107) for $a=\rho(t,x)$ and $a=\rho_0(x)$ to conclude Claim 2.

Claim 3.

$$\int_\Omega\left[\,\rho_0F'(\rho_0)-F^\star\left(F'(\rho_0)\right)\,\right]\,u(0)$$
$$+\int_{\Omega_T}\left[\,\rho(t,x)F'\left(\rho(t,x)\right)-F^\star\left(F'\left(\rho(t,x)\right)\right)\,\right]\,u'(t)$$
$$\leq \int_{\Omega_T}\langle\rho\sigma,\nabla\left(F'(\rho)\right)\rangle\, u(t).$$

`Proof:` Set $\xi(t,x):=F'\left(\rho(t,x)\right)u(t)$ for $(t,x)\in I\!R\times\Omega$. Because of (i), (ii), (26) and the fact that $F\in C^2(0,\infty)$, we have that $F'(\rho)\in L^\infty(\Omega_T)$ and $\nabla\left(F'(\rho)\right)\in L^{q^\star}(\Omega_T)$. We approximate $F'(\rho)$ by $C^\infty(\Omega_T)$-functions in $W^{1,q^\star}(\Omega_T)$, and we use (66) with the backward derivative $\partial_t^{-h}\xi(t,x):=\frac{\xi(t,x)-\xi(t-h,x)}{h}$ and Proposition 3.2 to obtain that

$$\int_{\Omega_T}(\rho_0-\rho^h)\partial_t^{-h}\xi+\int_{\Omega_T}\langle\sigma^h,\rho^h\nabla\left(F'(\rho)\right)\rangle\, u(t)=0(h^{\epsilon(q)}),$$

where $\epsilon(q)=\min(1,q-1)$. We let h go to 0 in the subsequent equality, and we use (101) to conclude that

$$\lim_{h\downarrow 0}\int_{\Omega_T}(\rho_0-\rho^h)\,\partial_t^{-h}\xi+\int_{\Omega_T}\langle\sigma,\rho\nabla\left(F'(\rho)\right)\rangle\, u(t)=0. \tag{109}$$

Since $\operatorname{spt} u\subset[-T,T]$, we have that

$$\int_{\Omega_T}\rho_0\,\partial_t^{-h}\xi=-\frac{1}{h}\int_{-h}^0\int_\Omega\rho_0(x)\xi(t,x)$$

and then

$$\lim_{h\downarrow 0}\int_{\Omega_T}\rho_0\,\partial_t^{-h}\xi=-\int_\Omega\rho_0(x)\,\xi(0,x)=-\int_\Omega\rho_0F'(\rho_0)u(0). \tag{110}$$

We combine (109), (110) and (i) to have that

$$\int_{\Omega_T} \langle \sigma, \rho \nabla \left(F'(\rho) \right) \rangle \, u(t) = \lim_{h \downarrow 0} \int_{\Omega_T} \rho(t,x) \partial_t^{-h} \xi(t,x) + \int_\Omega \rho_0 F'(\rho_0) u(0). \tag{111}$$

By direct computations, we obtain that

$$\begin{aligned} \rho(t,x)\partial_t^{-h}\xi(t,x) &= \rho(t,x)\, F'\left(\rho(t,x)\right) \partial_t^{-h}\, u(t) \\ &\quad + \frac{1}{h}\, \rho(t,x)\, u(t-h) \left[F'\left(\rho(t,x)\right) - F'\left(\rho(t-h,x)\right) \right]. \end{aligned}$$

Since $F \in C^1\left((0,\infty)\right)$ is strictly convex, and satisfies $F(0)=0$ and (HF1), we have that

$$\left(F'(b) - F'(a) \right) b \geq F^\star\left(F'(b)\right) - F^\star\left(F'(a)\right) \quad \forall\, a,b > 0$$

and then we deduce that

$$\begin{aligned} \rho(t,x)\, \partial_t^{-h}\xi(t,x) &\geq \rho(t,x)\, F'\left(\rho(t,x)\right)\, \partial_t^{-h} u(t) \\ &\quad + \frac{1}{h}\, u(t-h) \left[F^\star\left(F'\left(\rho(t,x)\right)\right) - F^\star\left(F'\left(\rho(t-h,x)\right)\right) \right]. \end{aligned}$$

We integrate both sides of the subsequent inequality over Ω_T, and we use that $u = 0$ on $(T-h,T)$ for h small enough and $\rho(t,x) = \rho_0(x)$ for $t \in (-h,0)$ to obtain that

$$\begin{aligned} \int_{\Omega_T} \rho(t,x)\partial_t^{-h}\xi(t,x) &\geq \int_{\Omega_T} \left[\rho(t,x) F'\left(\rho(t,x)\right) - F^\star\left(F'\left(\rho(t,x)\right)\right) \right] \partial_t^{-h} u(t) \\ &\quad - \frac{1}{h} \int_0^h u(t-h) \int_\Omega F^\star\left(F'(\rho_0)\right). \end{aligned}$$

We let h go to 0 in the above inequality to deduce that

$$\begin{aligned} \lim_{h\downarrow 0} \int_{\Omega_T} \rho(t,x)\partial_t^{-h}\xi(t,x) &\geq \int_{\Omega_T} \left[\rho(t,x) F'\left(\rho(t,x)\right) - F^\star\left(F'\left(\rho(t,x)\right)\right) \right] u'(t) \\ &\quad - \int_\Omega F^\star\left(F'(\rho_0)\right) u(0). \end{aligned} \tag{112}$$

We combine (111) and (112) to conclude Claim 3.
In the end, we show that $\sigma = \nabla c^\star \left[\nabla \left(F'(\rho) \right) \right]$, which combined with Lemma 3.8 completes the proof of Theorem 3.10. Indeed, let $\epsilon > 0$, $\psi \in C^\infty(\Omega)$ and set $\omega_\epsilon(t,x) := F'\left(\rho(t,x)\right) - \epsilon\psi(x)$. It is clear that $\nabla\omega_\epsilon \in L^{q^\star}(\Omega_T)$ and

$$\left| \nabla c^\star \left(\nabla \omega_\epsilon \right) \right|^q \leq M(\beta,q)\, |\nabla \omega_\epsilon|^{q^\star}$$

as in the proof of (102). We deduce that $\nabla c^\star\left(\nabla\omega_\epsilon\right) \in L^q(\Omega_T)$. We use that $c^\star$ is convex, and ρ^h and u are nonnegative, to have that

$$\int_{\Omega_T} \rho^h \left\langle \nabla c^\star \left[\nabla \left(F'(\rho^h) \right) \right] - \nabla c^\star \left(\nabla w_\epsilon \right), \nabla \left(F'(\rho^h) \right) - \nabla w_\epsilon \right\rangle u(t) \geq 0.$$

We let h go to 0 in the above inequality to obtain that

$$\limsup_{h\downarrow 0}\int_{\Omega_T}\langle \rho^h\sigma^h, \nabla\left(F'(\rho^h)\right)\rangle\, u(t) - \liminf_{h\downarrow 0}\int_{\Omega_T}\langle \sigma^h, \rho^h\nabla w_\epsilon\rangle\, u(t)$$
$$-\liminf_{h\downarrow 0}\int_{\Omega_T}\langle \rho^h\nabla c^\star(\nabla w_\epsilon), \nabla\left(F'(\rho^h)\right) - \nabla w_\epsilon\rangle\, u(t) \geq 0. \qquad (113)$$

As in the proof of (101) and (102), we have that

$$\liminf_{h\downarrow 0}\int_{\Omega_T}\langle \sigma^h, \rho^h\nabla w_\epsilon\rangle\, u(t) = \int_{\Omega_T}\langle \sigma, \rho\nabla w_\epsilon\rangle\, u(t), \qquad (114)$$

and

$$\liminf_{h\downarrow 0}\int_{\Omega_T}\langle \rho^h\nabla c^\star(\nabla w_\epsilon), \nabla\left(F'(\rho^h)\right) - \nabla w_\epsilon\rangle\, u(t)$$
$$= \int_{\Omega_T}\langle \rho\nabla c^\star(\nabla w_\epsilon), \nabla\left(F'(\rho)\right) - \nabla w_\epsilon\rangle\, u(t). \qquad (115)$$

We combine (99) and (113) - (115) to have that

$$\int_{\Omega_T}\langle \rho\sigma - \rho\nabla c^\star(\nabla w_\epsilon), \nabla\left(F'(\rho)\right) - \nabla w_\epsilon\rangle\, u(t) \geq 0.$$

We divide the subsequent inequality by ϵ, and we let ϵ go to 0 to obtain that

$$\int_{\Omega_T}\langle \rho\sigma - \rho\nabla c^\star\left[\nabla\left(F'(\rho)\right)\right], \nabla\psi(x)\, u(t)\rangle \geq 0.$$

Choosing $-\psi$ in place of ψ, we get that

$$\int_{\Omega_T}\langle \rho\sigma - \rho\,\nabla c^\star\left[\nabla\left(F'(\rho)\right)\right], \nabla\psi(x)\, u(t)\rangle = 0.$$

And since ψ and $u \geq 0$ are arbitrary test functions, we deduce (100). This completes the proof of the theorem. □

3.4 Existence and uniqueness of solutions

Here, we state and prove the theorem of existence and uniqueness for (6).

Theorem 3.11 *(Case $V = 0$).*
Assume that $c : \mathbb{R}^d \to [0,\infty)$ is strictly convex, of class C^1 and satisfies $c(0) = 0$ and (HC3), and that $F : [0,\infty) \to \mathbb{R}$ is strictly convex, of class $C^2((0,\infty))$ and satisfies $F(0) = 0$ and (HF1) - (HF2). If $\rho_0 \in \mathcal{P}_a(\Omega)$ is such that $\rho_0 + \frac{1}{\rho_0} \in L^\infty(\Omega)$, and $V = 0$, then equation (6) has a unique weak solution $\rho : [0,\infty)\times\Omega \to [0,\infty)$ in the sense that

(i). $\rho + \frac{1}{\rho} \in L^\infty\left((0,\infty); L^\infty(\Omega)\right),\ \nabla\left(F'(\rho)\right) \in L^{q^\star}(\Omega_T)$ *for* $0 < T < \infty$, *and*

(ii). *for* $\xi \in C^2_c(\mathbb{R}\times\mathbb{R}^d)$,

$$\int_{\Omega_\infty}\left\{-\rho\frac{\partial\xi}{\partial t} + \langle \rho\,\nabla c^\star\left[\nabla\left(F'(\rho)\right)\right], \nabla\xi\rangle\right\} = \int_\Omega \rho_0(x)\,\xi(0,x)\,dx. \qquad (116)$$

`Proof:` Proposition 3.7 gives that $(\rho^h)_h$ converges to ρ a.e. for a subsequence, and since $\rho^h \geq 0$ for all h, we deduce that $\rho \geq 0$. We combine (26) and Proposition 3.7 to have that $\rho + \frac{1}{\rho} \in L^\infty((0,\infty); L^\infty(\Omega))$. We use that $\nabla(F'(\rho)) \in L^{q^\star}(\Omega_T)$ (see Lemma 3.8) to conclude (i).
Recall that (103) gives that $\nabla c^\star[\nabla(F'(\rho))] \in L^q(\Omega_T)$ for $0 < T < \infty$, and (26) and Proposition 3.7 imply that $\rho \in L^\infty(\Omega_T)$. We deduce that $\rho \nabla c^\star[\nabla(F'(\rho))] \in L^q(\Omega_T)$.
Now, fix $0 < T < \infty$, and let $\xi \in C_c^2(I\!R \times I\!R^d)$ be such that $\operatorname{spt}\xi(.,x) \subset [-T,T]$ for $x \in \Omega$. Because of Proposition 2.9 and Proposition 3.2, we have that

$$\lim_{h\downarrow 0} \int_{\Omega_T} \left\{ (\rho_0 - \rho^h)\, \partial_t^h \xi + \langle\, \rho^h \nabla c^\star \left[\nabla \left(F'(\rho^h) \right) \right], \nabla \xi \,\rangle \right\} = 0. \tag{117}$$

Lemma 3.3 gives that $(\rho^h)_h$ converges weakly to ρ in $L^1(\Omega_T)$ for a subsequence, and then we have that

$$\lim_{h\downarrow 0} \int_{\Omega_T} (\rho_0 - \rho^h)\, \partial_t^h \xi = \int_{\Omega_T} (\rho_0 - \rho) \frac{\partial \xi}{\partial t} = - \left[\int_{\Omega_T} \rho \frac{\partial \xi}{\partial t} + \int_\Omega \rho_0(x)\xi(0,x) \right]. \tag{118}$$

From Theorem 3.10, we have $\left(\operatorname{div}\{\rho^h \nabla c^\star \left(\nabla\left(F'(\rho^h)\right)\right)\}\right)_h$ converges weakly to $\operatorname{div}\{\rho \nabla c^\star(\nabla(F'(\rho)))\}$ in $\left[C_c^2(I\!R \times I\!R^d)\right]'$ for a subsequence, then we deduce that

$$\lim_{h\downarrow 0} \int_{\Omega_T} \langle \rho^h \nabla c^\star \left[\nabla\left(F'(\rho)\right)\right], \nabla\xi \rangle = \int_{\Omega_T} \langle \rho \nabla c^\star \left[\nabla\left(F'(\rho)\right)\right], \nabla\xi \rangle. \tag{119}$$

We combine (117) - (119), and we use the fact that $\operatorname{spt}\xi(.,x) \subset [-T,T]$ to conclude (116).
Here, we prove uniqueness of solutions to (6) when $\frac{\partial \rho}{\partial t} \in L^1((0,T)\times\Omega)$, for $0 < T < \infty$. Using arguments in [17], it is easy to extend the proof to the general case. In fact, assumption (2) imposed in [4] would not be required here. The convexity of $c^\star$, that is $\langle \nabla c^\star(z_1) - \nabla c^\star(z_2), z_1 - z_2 \rangle \geq 0$ for z_1, $z_2 \in I\!R^d$, suffices to extend the proof.
Let $T > 0$, and assume that ρ_1 and ρ_2 are weak solutions of (6) with the same initial data, such that $N \leq \rho_j \leq M$ a.e. and $\frac{\partial \rho_j}{\partial t} \in L^1(\Omega_T)$, $j = 1,2$. Since $\nabla(F'(\rho_j)) \in L^{q^\star}(\Omega_T)$ and

$$\left| \nabla c^\star \left[\nabla\left(F'(\rho_j)\right) \right] \right|^q \leq M(\beta,q) \left| \nabla\left(F'(\rho_j)\right) \right|^{q^\star},$$

we have that $\nabla c^\star[\nabla(F'(\rho_j))] \in L^q(\Omega_T)$. For $\delta > 0$, we define

$$\Omega_T \ni (t,x) \mapsto \xi_\delta(t,x) := \varphi_\delta\left(F'(\rho_1(t,x)) - F'(\rho_2(t,x))\right)$$

where

$$\varphi_\delta(\tau) := \begin{cases} 0 & \text{if} \quad \tau \leq 0 \\ \frac{\tau}{\delta} & \text{if} \quad 0 \leq \tau \leq \delta \\ 1 & \text{if} \quad \tau \geq \delta. \end{cases}$$

Using a smooth approximation of ξ_δ as a test function in the differential equations satisfied by ρ_1 and ρ_2, and passing to the limit, we have that

$$\int_{\Omega_T} \xi_\delta\, \partial_t(\rho_1 - \rho_2) = - \int_{\Omega_T} \langle \rho_1 \nabla c^\star \left[\nabla\left(F'(\rho_1)\right)\right] - \rho_2 \nabla c^\star\left[\nabla\left(F'(\rho_2)\right)\right], \nabla\xi_\delta \rangle,$$

which reads as

$$\int_{\Omega_T} \xi_\delta\, \partial_t(\rho_1 - \rho_2)$$
$$= -\frac{1}{\delta}\int_{\Omega^{(1,2)}_{T,\delta}} \rho_1 \langle \nabla c^\star\left[\nabla\left(F'(\rho_1)\right)\right] - \nabla c^\star\left[\nabla\left(F'(\rho_2)\right)\right], \nabla\left(F'(\rho_1) - F'(\rho_2)\right)\rangle$$
$$-\frac{1}{\delta}\int_{\Omega^{(1,2)}_{T,\delta}} (\rho_1 - \rho_2)\langle \nabla c^\star\left[\nabla\left(F'(\rho_2)\right)\right], \nabla\left(F'(\rho_1) - F'(\rho_2)\right)\rangle,$$

where $\Omega^{(1,2)}_{T,\delta} := \Omega_T \cap [\,0 < F'(\rho_1) - F'(\rho_2) < \delta\,]$. Because $c^\star$ is convex, the first term on the right hand side of the above equality is nonpositive. And since $F \in C^1(0,\infty)$ is strictly convex and satisfies (HF1), and $N \le \rho_1,\, \rho_2 \le M$ a.e., we have a.e. on $\Omega^{(1,2)}_{T,\delta}$ that

$$|\,\rho_1 - \rho_2\,| = \Big|\left[(F^\star)' \circ F'\right](\rho_1) - \left[(F^\star)' \circ F'(\rho_2)\right]\Big| \le \delta \sup_{\tau \in [\,F'(N), F'(M)\,]} (F^\star)''(\tau).$$

We deduce that

$$\int_{\Omega_T} \xi_\delta\, \partial_t(\rho_1 - \rho_2)$$
$$\le \sup_{\tau \in [\,F'(N), F'(M)\,]} (F^\star)''(\tau) \int_{\Omega^{(1,2)}_{T,\delta}} \Big|\langle \nabla c^\star\left[\nabla\left(F'(\rho_2)\right)\right], \nabla\left(F'(\rho_1) - F'(\rho_2)\right)\rangle\Big|.$$

We let δ go to 0 in the subsequent inequality, and we use that $\varphi_\delta \to I\!\!I_{[0,\infty)}$, $|\,\Omega^{(1,2)}_{T,\delta}\,| \to 0$, and $[\,F'(\rho_1) - F'(\rho_2) \ge 0\,] = [\,\rho_1 - \rho_2 \ge 0\,]$, to have that

$$\int_{\Omega_T} \partial_t\left[\,(\rho_1 - \rho_2)^+\,\right] \le 0,$$

which reads as

$$\int_\Omega [\,\rho_1(T) - \rho_2(T)\,]^+ \le \int_\Omega [\,\rho_1(0) - \rho_2(0)\,]^+ = 0$$

for $0 < T < \infty$. Interchanging ρ_1 and ρ_2 in the above argument, we conclude that $\rho_1 = \rho_2$. □

Theorem 3.12 *(General case).*
Assume that $V : \bar\Omega \to [0,\infty)$ is convex and of class C^1, and that $c : I\!\!R^d \to [0,\infty)$ is strictly convex, of class C^1 and satisfies $c(0) = 0$ and (HC3). Assume that $F : [0,\infty) \to I\!\!R$ is strictly convex, of class $C^2(0,\infty)$ and satisfies $F(0) = 0$ and (HF1) - (HF2). If $\rho_0 \in \mathcal{P}_a(\Omega)$ is such that $\rho_0 + \frac{1}{\rho_0} \in L^\infty(\Omega)$, then (6) has a unique weak solution $\rho : [0,\infty) \times \Omega \to [0,\infty)$ in the sense that

(i). $\rho + \frac{1}{\rho} \in L^\infty\left((0,\infty); L^\infty(\Omega)\right)$, $\nabla\left(F'(\rho)\right) \in L^{q^\star}(\Omega_T)$ *for* $0 < T < \infty$, *and*

(ii). *for* $\xi \in C^2_c(I\!\!R \times I\!\!R^d)$,

$$\int_{\Omega_\infty} \Big\{ -\rho\frac{\partial \xi}{\partial t} + \langle \rho\, \nabla c^\star\left[\nabla\left(F'(\rho) + V\right)\right], \nabla \xi\rangle \Big\} = \int_\Omega \rho_0(x)\, \xi(0,x)\, dx. \qquad (120)$$

`Proof:` The proof of the uniqueness of solutions is similar to that of Theorem 3.11. Here, we only prove existence of solutions to (6). Let $\xi \in C_c^2(I\!R \times I\!R^d)$ be such that $\operatorname{spt}\xi(\cdot,x) \subset [-T,T]$ for $x \in \Omega$ and for some $0 < T < \infty$. Following the arguments in the previous sections where the energy functional $E_i(\rho)$ is replaced by

$$E(\rho) := E_i(\rho) + \int_\Omega \rho\, V \,\mathrm{d}x,$$

and the minimization problem (P) defined by (16) is replaced by

$$(P^V): \quad \inf\{\, hW_c^h(\rho_0,\rho) + E(\rho): \quad \rho \in \mathcal{P}_a(\Omega)\,\},$$

we have, as in Proposition 2.9, that

$$\Big| \int_{\Omega_T} (\rho_0 - \rho^h)\, \partial_t^h \xi \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} \langle\, \rho^h \nabla c^\star \Big[\nabla \Big(F'(\rho^h) + V \Big) \Big], \nabla\xi \,\rangle \,\mathrm{d}x\,\mathrm{d}t \Big|$$
$$\leq \frac{1}{2} \sup_{[0,T]\times\bar\Omega} \Big| D^2\xi(t,x) \Big| \sum_{i=1}^{T/h} \int_{\Omega\times\Omega} |x-y|^2 \, d\gamma_i^h(x,y) \tag{121}$$

and, as in Proposition 3.2, that

$$\sum_{i=1}^{T/h} \int_{\Omega\times\Omega} |\, x - y\,|^2 \,\mathrm{d}\gamma_i^h(x,y) \leq M(\Omega, T, F, \rho_0, q, \beta)\, h^{\epsilon(q)}. \tag{122}$$

We let h go to 0 in (121), and we use (122) to deduce that

$$\lim_{h\downarrow 0} \int_{\Omega_T} \Big\{ (\rho_0 - \rho^h)\, \partial_t^h \xi + \langle\, \rho^h \nabla c^\star \Big[\nabla \Big(F'(\rho^h) + V \Big) \Big], \nabla\xi \,\rangle \Big\} = 0. \tag{123}$$

The following claim suffices to conclude Theorem 3.12.

Claim. For $0 < T < \infty$, the estimates

$$\|\, \rho^h \,\|_{L^\infty((0,\infty);L^\infty(\Omega))} \leq \|\, \rho_0 \,\|_{L^\infty(\Omega)}, \tag{124}$$

$$\int_{\Omega_T} \rho^h \Big| \nabla \Big(F'(\rho^h) \Big) \Big|^{q^\star} \leq M(\Omega, T, F, \rho_0, V, q, \alpha), \tag{125}$$

and the energy inequality in time-space

$$\int_{\Omega_\infty} \langle \rho^h \nabla \Big(F'(\rho^h) + V \Big), \nabla c^\star \Big[\nabla \Big(F'(\rho^h) + V \Big) \Big] \rangle\, u(t) \tag{126}$$
$$\leq \frac{1}{h} \int_{\Omega_h} [\, F(\rho_0) + \rho_0 V \,]\; u(t) + \int_{\Omega_\infty} \Big[F(\rho^h) + \rho^h V \Big]\; \partial_t^h\, u(t)$$

hold for nonnegative functions u in $C_c^2(I\!R)$.

Indeed, because of (124), there exists $\rho : [0,\infty) \times \Omega \to [0,\infty)$ such that *(iii).* $(\rho^h)_h$ converges to ρ weakly in $L^1(\Omega_T)$ for a subsequence.

As a consequence,

$$\lim_{h\downarrow 0}\int_{\Omega_T}(\rho_0-\rho^h)\,\partial_t^h\,\xi=\int_{\Omega_T}(\rho_0-\rho)\,\frac{\partial\xi}{\partial t}. \tag{127}$$

Using (124) and (125), we deduce the space-compactness and time-compactness of $(\rho^h)_h$ in $L^1(\Omega_T)$, as in the case where $V=0$. Hence,

(iv). $(\rho^h)_h$ converges strongly to ρ in $L^1(\Omega_T)$ for a subsequence.

Then, we use *(iv)* and (126), and we follow the lines of the proof of Theorem 3.10 where we use $F'(\rho^h)+V$ in place of $F'(\rho^h)$, and $F(\rho^h)+\rho^h V$ in place of $F(\rho^h)$, to conclude that

(v). $\left(\operatorname{div}\{\rho^h\nabla c^\star\left[\nabla\left(F'(\rho^h)+V\right)\right]\}\right)_h$ converges weakly to $\operatorname{div}\{\rho\nabla c^\star\left[\nabla\left(F'(\rho)+V\right)\right]\}$ in $\left[C_c^2(I\!R\times I\!R^d)\right]'$ for a subsequence.

Hence,

$$\lim_{h\downarrow 0}\int_{\Omega_T}\langle\rho^h\nabla c^\star\left[\nabla\left(F'(\rho^h)+V\right)\right],\nabla\xi\rangle=\int_{\Omega_T}\langle\rho\nabla c^\star\left[\nabla\left(F'(\rho)+V\right)\right],\nabla\xi\rangle. \tag{128}$$

We combine (123), (127) and (128) to conclude (120).
As in Theorem 3.11, *(i)* follows directly from (124), (125) and the Maximum/Minimum principle of Proposition 2.2 for $\nabla V\neq 0$.

`Proof of the Claim:` (124) is a direct consequence of the Maximum principle of Proposition 2.2 for $\nabla V\neq 0$.
As in the case $V=0$, we have, because of Proposition 2.6 and the Maximum/Minimum principle of Proposition 2.2, that $P(\rho_i^h)\in W^{1,\infty}(\Omega)$ and $\nabla\left(F'(\rho_i^h)\right)\in L^\infty(\Omega)$. Then choosing $G:=F$ in Theorem 2.8, the (internal) energy inequality (62) read as

$$\int_\Omega F(\rho_{i-1}^h)-\int_\Omega F(\rho_i^h)\geq\int_\Omega\langle\nabla\left(F'(\rho_i^h)\right),S_i^h(y)-y\rangle\,\rho_i^h(y)\,\mathrm{d}y,$$

where S_i^h is the c_h-optimal map that pushes ρ_i^h forward to ρ_{i-1}^h. We use that $(S_i^h)_\#\rho_i^h=\rho_{i-1}^h$ and $V\in C^1(\Omega)$ is convex to deduce the *potential energy inequality*

$$\int_\Omega\rho_{i-1}^h\,V-\int_\Omega\rho_i^h\,V\geq\int_\Omega\langle\,\nabla V,S_i^h(y)-y\,\rangle\,\rho_i^h(y)\,\mathrm{d}y.$$

We add both of the subsequent inequalities, and we use the Euler-Lagrange equation of (P^V) that is,

$$\frac{S_1^h(y)-y}{h}=\nabla c^\star\left[\nabla(F'(\rho_1^h(y))+V(y))\right]\quad\text{for a.e. }\;y\in\Omega \tag{129}$$

(where S_1^h is the c_h-optimal map that pushes ρ_1^h forward to ρ_0), to deduce the *free energy inequality*

$$E(\rho_{i-1}^h)-E(\rho_i^h)\geq h\int_{\Omega_T}\langle\nabla\left(F'(\rho_i^h)+V\right),\nabla c^\star\left[\nabla\left(F'(\rho_i^h)+V\right)\right]\rangle\,\rho_i^h, \tag{130}$$

for $i \in \mathbb{N}$. We sum (130) over i, and we use that V and $\rho^h_{T/h}$ are nonnegative, and Jensen's inequality, to have that

$$h \int_{\Omega_T} \langle \nabla \left(F'(\rho^h) + V \right), \nabla c^\star \left[\nabla \left(F'(\rho^h) + V \right) \right] \rangle \, \rho^h \leq E(\rho_0) - |\,\Omega\,| \, F \left(\frac{1}{|\,\Omega\,|} \right).$$

We conclude, as in the proof of (77), that

$$\int_{\Omega_T} \rho^h \left| \nabla \left(F'(\rho^h) + V \right) \right|^{q^\star} \leq \overline{M}(\Omega, T, F, \rho_0, q, \alpha). \tag{131}$$

On the other hand, because of (124) and the fact that $V \in C^1(\bar{\Omega})$, we have that

$$\left\| (\rho^h)^{1/q^\star} \nabla V \right\|_{L^{q^\star}(\Omega_T)} \leq \| \rho_0 \|^{1/q^\star}_{L^\infty(\Omega)} \| \nabla V \|_{L^\infty(\Omega)}. \tag{132}$$

We combine (131) and (132), to conclude (125).
The proof of (126) follows the lines of the proof of Lemma 3.9 where we use the free energy inequality (130) in place of the internal energy inequality (62). □

Remark 3.13 *(Existence of solutions to equation (6) for a wider class of ρ_0).* *Assume that c and F satisfy the assumptions in Theorem 3.12. We extend the existence Theorem 3.12 to a wider class of initial probability densities ρ_0, e.g. $\rho_0 \in L^\infty(\Omega)$ and $\frac{1}{\rho_0} \notin L^\infty(\Omega)$, or $\rho_0 \in L^p(\Omega)$ where $p \geq q$. For simplicity, we assume that $V = 0$.*

Case 1: $\rho_0 \in L^\infty(\Omega)$ and $\frac{1}{\rho_0} \notin L^\infty(\Omega)$.
From Proposition 1.4.2 [2], consider a sequence $(\rho_{0,\delta})_\delta$ in $\mathcal{P}_a(\Omega)$ such that

$$\begin{cases} \eta_\delta \leq \rho_{0,\delta} \leq \| \rho_0 \|_{L^\infty(\Omega)} \text{ a.e., where } \; 0 < \eta_\delta \leq \delta \\ \\ E_i(\rho_{0,\delta}) \leq E_i(\rho_0) \\ \\ \rho_{0,\delta} \to \rho_0 \text{ in } L^1(\Omega), \text{ as } \delta \downarrow 0. \end{cases} \tag{133}$$

Define the approximate solution ρ^h_δ to (6) by

$$\rho^h_\delta := \begin{cases} \rho_{0,\delta} & \text{if} \quad t = 0 \\ \\ \rho^h_{i,\delta} & \text{if} \quad t \in ((i-1)h, ih] \end{cases}$$

where $\rho^h_{i,\delta}$ is the unique minimizer of

$$(P_{i,\delta}) : \quad \inf \{ \, h W^h_c \left(\rho^h_{i-1,\delta}, \rho \right) + E_i(\rho) : \; \rho \in \mathcal{P}_a(\Omega) \}.$$

Since $\rho_{0,\delta} + \frac{1}{\rho_{0,\delta}} \in L^\infty(\Omega)$, we have as before, that

$$\begin{cases} \| \rho^h_\delta \|_{L^\infty((0,\infty);L^\infty(\Omega))} \leq \| \rho_{0,\delta} \|_{L^\infty(\Omega)}, \\ \\ \int_{\Omega_T} \rho^h_\delta \left| \nabla \left(F'(\rho^h_\delta) \right) \right|^{q^\star} \leq \overline{M}(\Omega, T, F, \rho_{0,\delta}, q, \alpha) \end{cases} \tag{134}$$

and

$$\int_{\Omega_T} (\rho_0 - \rho_\delta^h)\, \partial_t^h \xi + \int_{\Omega_T} \langle\, \rho_\delta^h \nabla c^\star \left[\nabla \left(F'(\rho_\delta^h) \right) \right], \nabla \xi \,\rangle = 0(h^{\epsilon(q)}), \tag{135}$$

where $\epsilon(q) := \min(1, q-1)$, $\xi \in C^2(\mathbb{R} \times \mathbb{R}^d)$, and

$$\overline{M}(\Omega, T, F, \rho_{0,\delta}, q, \alpha) := M(\alpha, q) \left(E_i(\rho_{0,\delta}) - |\,\Omega\,| F\left(\frac{1}{|\,\Omega\,|}\right) + \alpha T |\,\Omega\,|\, \|\, \rho_{0,\delta}\, \|_{L^\infty(\Omega)} \right).$$

We introduce a convex function $H : [0, \infty) \to \mathbb{R}$ satisfying the assumption

$(\mathbf{HH1})$: $\quad H \in C^1[0, \infty) \cap C^2(0, \infty)$ and $H''(x) = x^{1/q^\star} F''(x)$, $\forall\, x > 0$.

Combining (133), (134) and (HH1), we have that

$$\begin{cases} \|\, \rho_\delta^h \,\|_{L^\infty((0,\infty); L^\infty(\Omega))} \le \|\, \rho_0 \,\|_{L^\infty(\Omega)}, \\ \\ \left\| \nabla \left(H'(\rho_\delta^h) \right) \right\|_{L^{q^\star}(\Omega_T)}^{q^\star} \le \overline{M}(\Omega, T, F, \rho_0, q, \alpha). \end{cases} \tag{136}$$

We deduce that there exists $\rho : [0, \infty) \times \Omega \to [0, \infty)$ such that $(\rho_\delta^h)_{h,\delta}$ converges to ρ in $L^1(\Omega_T)$ for a subsequence, and $\{\tilde{\sigma}_\delta^h := (\rho_\delta^h)^{1/q} \nabla c^\star \left[\nabla \left(F'(\rho_\delta^h) \right) \right] \}_{h,\delta}$ converges weakly to $\rho^{1/q} \nabla c^\star \left[\nabla \left(F'(\rho) \right) \right]$ in $L^q(\Omega_T)$ for a subsequence, as (h, δ) goes to $(0, 0)$. Then we let (h, δ) go to $(0, 0)$ in (135) to conclude that $\rho \in L^\infty\left((0, \infty); L^\infty(\Omega)\right)$ is a weak solution of (6), as in Theorem 3.11, with the exception that we do not require that $\frac{1}{\rho} \in L^\infty\left((0, \infty); L^\infty(\Omega)\right)$. □

Case 2: $\rho_0 \in L^p(\Omega)$ with $p \ge q$, and $E_i(\rho_0) < \infty$.
Using Corollary 1.4.3 [2], we approximate ρ_0 by a sequence $(\rho_{0,\delta})_\delta$ in $\mathcal{P}_a(\Omega)$ such that

$$\begin{cases} \eta_\delta \le \rho_{0,\delta} \le \epsilon_\delta \text{ a.e., where } 0 < \eta_\delta \le \delta \text{ and } \epsilon_\delta \ge \frac{1}{\delta} \\ \\ E_i(\rho_{0,\delta}) \le E_i(\rho_0) \\ \\ \rho_{0,\delta} \to \rho_0 \text{ in } L^p(\Omega) \text{ as } \delta \downarrow 0. \end{cases}$$

Define ρ_δ^h as in Case 1. Since $\rho_0 \notin L^\infty(\Omega)$, we cannot obtain (136) from (134) as in Case 1. Here, we take advantage of the fact that $\rho_0 \in L^p(\Omega)$ and $E_i(\rho_0) < \infty$, as follows:

(i). We choose $G(x) := \frac{x^p}{p}$, $x > 0$, in the (internal) energy inequality (62), and we observe that $\langle \nabla \rho^{p-1}, \nabla c^\star [\nabla (F'(\rho))] \rangle \ge 0$ for $\rho \in \mathcal{P}_a(\Omega)$, to have that

$$\|\, \rho_\delta^h \,\|_{L^\infty((0,\infty); L^p(\Omega))} \le \|\, \rho_0 \,\|_{L^p(\Omega)}. \tag{137}$$

As a consequence, there exists a function $\rho : [0, \infty) \times \Omega \to [0, \infty)$ such that $(\rho_\delta^h)_{h,\delta}$ converges weakly to ρ in $L^p(\Omega_T)$, $0 < T < \infty$.

(ii). Next, we choose $G := F$ in (62) to control the spatial derivatives of ρ_δ^h as

$$\int_{\Omega_T} \rho_\delta^h \left| \nabla \left(F'(\rho_\delta^h) \right) \right|^{q^\star} \le M(\alpha, q) \left(E_i(\rho_0) - |\,\Omega\,| F\left(\frac{1}{|\,\Omega\,|}\right) + \alpha |\,\Omega_T\,|^{1/p^\star} \|\, \rho_0 \,\|_{L^p(\Omega)} \right). \tag{138}$$

We combine (137),(138), and we use (HH1) to deduce, as in the previous sections, that $(\rho_\delta^h)_{h,\delta}$ converges strongly to ρ in $L^1(\Omega_T)$, for a subsequence. We conclude, as in case 1, that ρ is a weak solution of (6) in the sense that $\rho \in L^\infty((0,\infty); L^p(\Omega))$, $\rho \nabla c^\star [\nabla (F'(\rho))] \in L^1(\Omega_T)$ for $0 < T < \infty$ and (116) holds. □

Some examples of energy density functions satisfying (HF1)- HF2) and (HH1) are $F(x) = \sum_{i=1}^n A_i F_i(x)$, where $F_i(x) \in \{x \ln(x), \frac{x^m}{m-1}\}$ with $m > 1$ or $\max\left(\frac{1}{q}, 1 - \frac{1}{d}\right) \leq m < 1$, and $A_i > 0$. For examples, for the fast diffusion equation $\frac{\partial \rho}{\partial t} = \Delta \rho^m$, this corresponds to the range $1 - \frac{1}{d} \leq m < 1$ if $d \geq 2$, and $\frac{1}{2} \leq m < 1$ if $d \leq 2$; for $\frac{\partial \rho}{\partial t} = \Delta_p \rho^n$, we have $n \geq \frac{d-(p-1)}{d(p-1)}$ if $d \geq p$, and $n \geq \frac{1}{p(p-1)}$ if $d \leq p$; and in particular, for the p-Laplacian $\frac{\partial \rho}{\partial t} = \Delta_p \rho$, we require $\frac{2d+1}{d+1} \leq p \leq d$ or $p \geq \max\left(d, \frac{1+\sqrt{5}}{2}\right)$.

4 Appendix

In Proposition 4.1 we collect results of previous authors used in this work, and in Proposition 4.2 we establish intermediate results needed in the previous sections. Proposition 4.1 is due to Cordero [7] and Otto [16] . For its proof, we refer to these references. A sketch of proof of this proposition can also be found in [2], sections 5.1 and 5.2.

Proposition 4.1 *Let $\rho_0, \rho_1 \in \mathcal{P}_a(\Omega)$ and assume that $c : \mathbb{R}^d \to [0,\infty)$ is strictly convex and satisfies $c, c^\star \in C^2(\mathbb{R}^d)$. Denote by S the c-optimal map that pushes ρ_1 forward to ρ_0, and define the interpolant map S_t and the interpolant measure μ_{1-t} by*

$$S_t := (1-t)id + tS \quad and \quad \mu_{1-t} := (S_t)_\# \rho_1,$$

for $t \in [0,1]$. Then,

(i). S_t is injective, and μ_{1-t} is absolutely continuous with respect to Lebesgue. Moreover, there exists a subset K of Ω of full measure for $\mu_1 := \rho_1(y)dy$ such that, for $y \in K$ and $t \in [0,1]$,

(ii). $\nabla S(y)$ is diagonalizable with positive eigenvalues.

(iii). The pointwise Jacobian $det(\nabla S)$ satisfies

$$0 \neq \rho_1(y) = \rho_{1-t}(S_t(y)) \; det\left[(1-t)\, id + t\nabla S(y)\right],$$

where ρ_{1-t} is the density function of μ_{1-t}.
In addition, if $\rho_1 > 0$ a.e., then

(iv). the pointwise divergence $div(S)$ is integrable on Ω, and

$$\int_\Omega div(S(y) - y)\, \xi(y)\, dy \leq -\int_\Omega \langle S(y) - y, \nabla \xi \rangle \, dy,$$

for $\xi \geq 0$ in $C_c^\infty(\mathbb{R}^d)$.

The following estimates are needed in the previous sections.

Proposition 4.2 *Assume that* $c : \mathbb{R}^d \to [0,\infty)$ *is strictly convex, of class* C^1 *and satisfies* $c(0)=0$ *and (HC2). Then*

$$\langle\, z, \nabla c^\star(z) \,\rangle \geq c^\star(z) \geq 0, \;\; \forall\, z \in \mathbb{R}^d. \tag{139}$$

In addition, if $c(z) \geq \beta\,|\,z\,|^q$ *for some* $\beta > 0$ *and* $q > 1$*, then*

$$\langle\, z, \nabla c^\star(z) \,\rangle \leq M(\beta, q)\,|\,z\,|^{q^\star}, \tag{140}$$

where $M(\beta, q)$ *is a constant which only depends on* β *and* q*.*

`Proof:` Since c is strictly convex, differentiable and satisfies (HC2), we have that $c^\star \in C^1(\mathbb{R}^d)$ is convex. Then,

$$\langle z, \nabla c^\star\,(z)\rangle = c^\star(z) + c\,(\nabla c^\star(z)) \geq c^\star(z). \tag{141}$$

Because $c(0) = 0$ and 0 minimizes c, we have that $c^\star(0) = 0$ and 0 minimizes $c^\star$. We conclude that $c^\star(z) \geq 0$, which proves (139).
Now, assume that $c(z) \geq \beta\,|\,z\,|^q$. Since $c^\star \in C^1(\mathbb{R}^d)$ is convex and nonnegative, we have that

$$\langle\, z,\, \nabla c^\star(z) \,\rangle \leq c^\star(2z) - c^\star(z) \leq c^\star(2z). \tag{142}$$

Moreover, because $c(z) \geq \beta\,|\,z\,|^q$, we have that

$$c^\star(2z) \leq M(\beta, q)\,|\,z\,|^{q^\star}. \tag{143}$$

We combine (142) and (143) to conclude (140). □

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