The mother of many geometric inequalities

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Abstract

A general inequality –established in [1]– relating the relative total energy of probability densities, their Wasserstein distance and their production entropy functional, is shown to easily imply most known geometrical –Gaussian and Euclidean– inequalities. Some of the implications are known but included here for pedagogical reasons.

1 Introduction

Let $F : [0, \infty) \to \mathbb{R}$ be a convex function, V a real functional on $\mathbb{R}^n$ and let $\Omega \subset \mathbb{R}^n$ be open, bounded and convex. The set of probability densities over Ω is denoted by $\mathcal{P}_a(\Omega) = \{\rho : \Omega \to \mathbb{R};\, \rho \geq 0 \text{ and } \int_\Omega \rho(x)dx = 1\}$. The associated *Free Energy Functional* is defined on $\mathcal{P}_a(\Omega)$ as $H_V^F(\rho) := \int_{R^n}(F(\rho) + \rho V)dx$, which is the sum of the *Internal Energy* $H^F(\rho) := \int_{R^n} F(\rho)dx$, and the *Potential Energy* $H_V(\rho) := \int_{R^n} \rho V dx$. Let $H_V^F(\rho|\bar{\rho}) := H_V^F(\rho) - H_V^F(\bar{\rho})$ denote the relative energy between two densities ρ and $\bar{\rho}$. In [1], we established the following general inequality relating the relative total energy of probability densities, their Wasserstein distance and their production entropy functional.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and convex, let $F : [0, \infty) \to \mathbb{R}$ be a differentiable function on $(0, \infty)$ such that $F(0) = 0$ and $x \mapsto x^n F(x^{-n})$ be convex and non-increasing, and let $P_F(x) := xF'(x) - F(x)$ be its associated pressure function. Then, for any strictly convex C^1-function $c : \mathbb{R}^n \to \mathbb{R}$ such that $\lim_{|x|\to\infty} \frac{c(x)}{|x|} = \infty$, and any C^2-potential $V : \mathbb{R}^n \to \mathbb{R}$ with $D^2V \geq \lambda$ (where $\lambda \in \mathbb{R}$ is not necessarily positive),*

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we have for all probability density functions ρ_0 *and* ρ_1 *on* Ω*, satisfying* $supp\,\rho_0 \subset \Omega$*,* $\rho_0 > 0$ *a.e. on* Ω *and* $P_F(\rho_0) \in W^{1,\infty}(\Omega)$*,*

$$H^F_{V+c}(\rho_0|\rho_1) + \frac{\lambda}{2} W_2^2(\rho_0, \rho_1) \leq \int_\Omega \rho_0 c^\star\left(-\nabla(F' \circ \rho_0 + V)\right)\, dx + H^{-nP_F}_{c+\nabla V.x}(\rho_0), \tag{1}$$

where W_2 *is the Wasserstein distance and where* c^* *denotes the Legendre conjugate of* c *defined by* $c^*(y) = \sup_{z\in R^n}\{y \cdot z - c(z)\}$.

Furthermore, equality holds in (1) whenever $\rho_0 = \rho_1 = \rho_\infty$ *where the latter satisfies*

$$\nabla\left(F'\left(\rho_\infty(x) + V(x)\right)\right) = -\nabla c(x) \quad a.e. \tag{2}$$

In particular, we have for any probability density ρ *such that* $P_F(\rho) \in W^{1,\infty}(\Omega)$*,*

$$H^{F+nP_F}_{V-x\cdot\nabla V}(\rho) + \frac{\lambda}{2} W_2^2(\rho, \rho_\infty) \leq \int_\Omega \rho c^\star\left(-\nabla(F' \circ \rho + V)\right)\, dx - H^{P_F}(\rho_\infty) + C_\infty \tag{3}$$

where C_∞ *is the unique constant such that*

$$F'(\rho_\infty) + V + c = C_\infty \text{ while } \int_\Omega \rho_\infty = 1. \tag{4}$$

We shall see that this inequality easily implies most known geometric inequalities. It provides a direct and unified way for computing best constants as well as the extremals where they are attained.

The term $H^{-nP_F}_{c+\nabla V.x}(\rho_0)$ should be seen as an error term in (1). It can be integrated in the entropy term which proves useful in the Gaussian case. If V is convex, then λ can be taken equal to 0 and the Wasserstein distance disappears from the equation. We then have the identity $V(x) - x \cdot \nabla V(x) = -V^*(\nabla V(x)$ in such a way that a correcting "moment" appears in the inequality:

$$H^{F+nP_F}_{-V^*(\nabla V)}(\rho) \leq \int_\Omega \rho c^\star\left(-\nabla(F' \circ \rho + V)\right)\, \mathrm{d}x - H^{P_F}(\rho_\infty) + C_\infty. \tag{5}$$

Also note that the pressure P_F is always positive which means that we can do away with the term $H^{P_F}(\rho_\infty)$ on the right hand side. Finally, the case $V = 0$ amply covers the Euclidean case where the general inequality becomes the remarkably simple:

$$H^{F+nP_F}(\rho) \leq \int_\Omega \rho c^\star\left(-\nabla(F' \circ \rho)\right)\, \mathrm{d}x + C_\infty. \tag{6}$$

2 Generalized HWI inequalities

We first deduce the following useful inequality that is relevant for the Gaussian case.

Corollary 2.1 *Under the above hypothesis on* Ω *and* F*, let* $U : I\!R^n \to I\!R$ *be a* C^2*-function with* $D^2U \geq \mu I$ *where* $\mu \in I\!R$*. Then for any* $\sigma > 0$*, we have for all probability densities* ρ_0 *and* ρ_1 *on* Ω*, satisfying* $supp\,\rho_0 \subset \Omega$*, and* $P_F(\rho_0) \in W^{1,\infty}(\Omega)$*,*

$$H^F_U(\rho_0|\rho_1) + \frac{1}{2}(\mu - \frac{1}{\sigma}) W_2^2(\rho_0, \rho_1) \leq \frac{\sigma}{2} \int_\Omega \rho \left|\nabla\left(F' \circ \rho_0 + U\right)\right|^2 dx. \tag{7}$$

Proof: Use (1) with $c(x) = \frac{1}{2\sigma}|\,x\,|^2$ and $U = V + c$, to obtain

$$\begin{aligned} &H_U^F(\rho_0) - H_U^F(\rho_1) + \frac{1}{2}(\mu - \frac{1}{\sigma})W_2^2(\rho_0, \rho_1) \\ &\quad \le -H^{nP_F}_{-c-\nabla(U-c)\cdot x}(\rho_0) + \int_\Omega \rho_0 c^* \left(-\nabla\left(F' \circ \rho_0 + U - c\right)\right)\, \mathrm{d}x. \end{aligned} \tag{8}$$

By elementary computations, we have

$$\begin{aligned} &\int_\Omega \rho c^* \left(-\nabla\left(F' \circ \rho + U - c\right)\right)\, \mathrm{d}x \\ &\quad = \frac{\sigma}{2}\int_\Omega \rho \Big|\,\nabla\left(F' \circ \rho + U\right)\,\Big|^2 \mathrm{d}x + \frac{1}{2\sigma}\int_\Omega \rho|\,x\,|^2\,\mathrm{d}x - \int_\Omega \rho x \cdot \nabla\left(F' \circ \rho\right)\, \mathrm{d}x - \int_\Omega \rho x \cdot \nabla U\,\mathrm{d}x, \end{aligned}$$

and

$$-H^{nP_F}_{-c-\nabla(U-c)\cdot x}(\rho) \;=\; -H^{nP_F}(\rho) + \int_\Omega \rho x \cdot \nabla U\,\mathrm{d}x - \frac{1}{2\sigma}\int_\Omega |\,x\,|^2 \rho\,\mathrm{d}x.$$

By combining the last 2 identities, we can rewrite the right hand side of (8) as

$$\begin{aligned} &-H^{nP_F}_{-c-\nabla(U-c)\cdot x}(\rho) + \int_\Omega \rho c^* \left(-\nabla(F' \circ \rho + U - c)\right)\, \mathrm{d}x \\ &\quad = \frac{\sigma}{2}\int_\Omega \rho|\,\nabla\left(F' \circ \rho + U\right)\,|^2\,\mathrm{d}x - \int_\Omega \rho x \cdot \nabla\left(F' \circ \rho\right)\,\mathrm{d}x - \int_\Omega nP_F(\rho)\,\mathrm{d}x \\ &\quad = \frac{\sigma}{2}\int_\Omega \rho|\,\nabla\left(F' \circ \rho + U\right)\,|^2, \mathrm{d}x + \int_\Omega \mathrm{div}\,(\rho x)F'(\rho)\,\mathrm{d}x - \int_\Omega nP_F(\rho)\,\mathrm{d}x \\ &\quad = \frac{\sigma}{2}\int_\Omega \rho\Big|\,\nabla\left(F' \circ \rho + U\right)\,\Big|^2\,\mathrm{d}x + n\int_\Omega \rho F'(\rho)\,\mathrm{d}x + \int_\Omega x \cdot \nabla F(\rho)\,\mathrm{d}x - \int_\Omega nP_F(\rho)\,\mathrm{d}x \\ &\quad = \frac{\sigma}{2}\int_\Omega \rho\Big|\,\nabla\left(F' \circ \rho + U\right)\,\Big|^2 \mathrm{d}x + \int_\Omega x \cdot \nabla F(\rho)\,\mathrm{d}x + n\int_\Omega F \circ \rho\,\mathrm{d}x \\ &\quad = \frac{\sigma}{2}\int_\Omega \rho\Big|\,\nabla\left(F' \circ \rho + U\right)\,\Big|^2\,\mathrm{d}x. \end{aligned} \tag{9}$$

Inserting (9) into (8), we conclude the proof.

If U is uniformly convex (i.e., $\mu > 0$) inequality (7) yields the following inequality obtained by Cordero et al. in [4]

Corollary 2.2 *(*Generalized Log Sobolev inequality*) Under the above hypothesis on Ω and F, let $U : I\!R^n \to I\!R$ be a C^2-function with $D^2U \ge \mu I$ where $\mu > 0$. Then for all probability densities ρ_0 and ρ_1 on Ω, satisfying supp $\rho_0 \subset \Omega$, and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$, we have*

$$H_U^F(\rho_0|\rho_1) \le \frac{1}{2\mu}\int_\Omega |\nabla(F' \circ \rho_0 + U)|^2 \rho_0\, dx. \tag{10}$$

One can also deduce the following:

Corollary 2.3 *(*Generalized Talagrand Inequality*) Under the above hypothesis on Ω and F, let $U : I\!R^n \to I\!R$ be a C^2-function with $D^2U \ge \mu I$ where $\mu > 0$. Then for all probability densities ρ_0 and ρ_1 on Ω, satisfying supp $\rho_0 \subset \Omega$, and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$, we have*

$$W_2(\rho|\rho_U) \le \sqrt{\frac{\mu}{2} H_U^F(\rho|\rho_U)}, \tag{11}$$

where ρ_U is the probability density satisfying

$$\nabla\left(F'(\rho_U)+U\right)=0 \quad a.e. \tag{12}$$

For that, it is sufficient to take $\rho_0=\rho_U$ in (7).

We now deduce the following HWI inequalities first established by Otto-Villani [7] in the case of the classical entropy $F(x)=x\ln x$.

Corollary 2.4 (Generalized HWI-inequality) *Under the above hypothesis on Ω and F, let $U:\mathbb{R}^n\to\mathbb{R}$ be a C^2-function with $D^2U\geq\mu I$ where $\mu\in\mathbb{R}$. Then we have for all probability densities ρ_0 and ρ_1 on Ω, satisfying $\operatorname{supp}\rho_0\subset\Omega$, and $P_F(\rho_0)\in W^{1,\infty}(\Omega)$,*

$$H_U^F(\rho_0|\rho_1)\leq W_2(\rho_0,\rho_1)\sqrt{I(\rho_0|\rho_U)}-\frac{\mu}{2}W_2(\rho_0,\rho_1)^2 \tag{13}$$

where

$$I(\rho_0|\rho_U)=\frac{\sigma}{2}\int_\Omega\rho\Big|\nabla\left(F'\circ\rho_0+U\right)\Big|^2dx,$$

and

$$\nabla\left(F'(\rho_U)+U\right)=0 \quad a.e. \tag{14}$$

Proof: It is sufficient to rewrite (7) as

$$H_U^F(\rho_0|\rho_1)+\frac{\mu}{2}W_2^2(\rho_0,\rho_1)\leq\frac{1}{2\sigma}W_2^2(\rho_0,\rho_1)+\frac{\sigma}{2}I(\rho_0|\rho_U), \tag{15}$$

then minimize the right hand side over the variable $\sigma>0$. The minimum is obviously achieved at $\bar{\sigma}=\frac{W_2(\rho_0,\rho_1)}{\sqrt{I(\rho_0|\rho_U)}}$.

3 Gaussian Inequalities

Corollary 2.1 applied to $F(x)=x\ln x$ yields the following extension of Gross' Log Sobolev inequality established by Otto-Villani [7]. For any function U on $\mathbb{R}^n$, denote by σ_U the integral $\int_{R^n}e^{-U}\,dx$, and by ρ_U the normalized function $\frac{e^{-U}}{\sigma_U}$.

Corollary 3.1 (Otto-Villani's HWI inequality) *Let $U:\mathbb{R}^n\to\mathbb{R}$ be a C^2-function with $D^2U\geq\mu I$ where $\mu\in\mathbb{R}$. Then for any $\sigma>0$, the following holds for any nonnegative function f such that $f\rho_U\in W^{1,\infty}(\mathbb{R}^n)$ and $\int_{R^n}f\rho_U\,dx=1$,*

$$\int_{R^n}f\ln(f)\,\rho_U\,dx+\frac{1}{2}(\mu-\frac{1}{\sigma})W_2^2(f\rho_U,\rho_U)\leq\frac{\sigma}{2}\int_{R^n}\frac{|\nabla f|^2}{f}\rho_U\,dx. \tag{16}$$

Corollary 3.2 (Original Gross Log Sobolev inequality) *If $\mu>0$ (i.e., U is uniformly convex) then for any nonnegative function f such that $f\rho_U\in W^{1,\infty}(\mathbb{R}^n)$ and $\int_{R^n}f^2\rho_U\,dx=1$, we have*

$$\int_{R^n}f^2\ln(f^2)\,\rho_U\,dx\leq\frac{1}{\mu}\int_{R^n}|\nabla f|^2\,\rho_U\,dx. \tag{17}$$

Talagrand's inequality applied to the standard Gaussian density γ and to an appropriate restriction yields

Corollary 3.3 *(*Concentration of measure inequality*) For any ϵ-neighborhood B_ϵ of a measurable set B in $\mathbb{R}^n$, we have*

$$\gamma(B_\epsilon) \geq 1 - e^{-\frac{1}{2}\left(\epsilon - \sqrt{2\ln\frac{1}{\gamma(B)}}\right)^2}. \tag{18}$$

Indeed, if γ_A denotes the normalized standard Gaussian measure restricted to a given measurable set A, then

$$\epsilon \leq W_2(\gamma_B; \gamma_{\mathbb{R}^n \setminus B_\epsilon}) \leq \sqrt{2\ln\frac{1}{\gamma(B)}} + \sqrt{2\ln\frac{1}{1-\gamma(B_\epsilon)}}, \tag{19}$$

which yields (18).

4 Euclidean Log Sobolev Inequalities

The folowing optimal Euclidean p-Log Sobolev inequality was established by Beckner [2] in the case where $p = 1$, by Del Pino- Dolbeault [5] for $1 < p < n$ and independently by Gentil for all $p > 1$.

Corollary 4.1 *(*General Euclidean Log-Sobolev inequality*) Let $\Omega \subset \mathbb{R}^n$ be open bounded and convex, and let $c : \mathbb{R}^n \to \mathbb{R}$ be a Young functional such that its conjugate $c^\star$ is p-homogeneous for some $p > 1$. Then,*

$$\int_{\mathbb{R}^n} \rho \ln \rho \, dx \leq \frac{n}{p} \ln\left(\frac{p}{ne^{p-1}\sigma_c^{p/n}} \int_{\mathbb{R}^n} \rho c^\star\left(-\frac{\nabla \rho}{\rho}\right) dx\right), \tag{20}$$

for all probability density functions ρ on $\mathbb{R}^n$, such that $\operatorname{supp}\rho \subset \Omega$ and $\rho \in W^{1,\infty}(\mathbb{R}^n)$. Moreover, equality holds in (20) if $\rho(x) = K_\lambda e^{-\lambda^q c(x)}$ for some $\lambda > 0$, where $K_\lambda = \left(\int_{\mathbb{R}^n} e^{-\lambda^q c(x)} \, dx\right)^{-1}$ and q is the conjugate of p ($\frac{1}{p} + \frac{1}{q} = 1$).

Proof: Use $F(x) = x\ln(x)$ in (5). Note that here $P_F(x) = x$ which means that $H^{P_F}(\rho) = 1$ for any $\rho \in \mathcal{P}_a(\mathbb{R}^n)$. So, $\rho_\infty(x) = \frac{e^{-c(x)}}{\sigma_c}$. We then have for $\rho \in \mathcal{P}_a(\mathbb{R}^n)$

$$\int_\Omega \rho \ln \rho \, \mathrm{d}x \leq \int_{\mathbb{R}^n} \rho c^\star\left(-\frac{\nabla \rho}{\rho}\right) \mathrm{d}x - n - \ln\left(\int_{\mathbb{R}^n} e^{-c(x)} \, \mathrm{d}x\right). \tag{21}$$

with equality when $\rho = \rho_\infty$.

Now assume that $c^\star$ is p-homogeneous and set $\Gamma_\rho^c = \int_{\mathbb{R}^n} \rho c^\star\left(-\frac{\nabla\rho}{\rho}\right) \mathrm{d}x$. Using $c_\lambda(x) := c(\lambda x)$ in (21), we get for $\lambda > 0$ that

$$\int_{\mathbb{R}^n} \rho \ln \rho \, \mathrm{d}x \leq \int_{\mathbb{R}^n} \rho c^\star\left(-\frac{\nabla \rho}{\lambda\rho}\right) \mathrm{d}x + n\ln\lambda - n - \ln \sigma_c, \tag{22}$$

for all $\rho \in \mathcal{P}_a(I\!R^n)$ satisfying $\operatorname{supp}\rho \subset \Omega$ and $\rho \in W^{1,\infty}(\Omega)$. Equality holds in (22) if $\rho_\lambda(x) = \left(\int_{R^n} e^{-\lambda^q c(x)}\,\mathrm{d}x\right)^{-1} e^{-\lambda^q c(x)}$. Hence

$$\int_{R^n} \rho \ln \rho \,\mathrm{d}x \leq -n - \ln \sigma_c + \inf_{\lambda>0} \left(G_\rho(\lambda)\right),$$

where

$$G_\rho(\lambda) = n \ln(\lambda) + \frac{1}{\lambda^p} \int_{R^n} \rho c^\star \left(-\frac{\nabla \rho}{\rho}\right) = n\ln(\lambda) + \frac{\Gamma_\rho^c}{\lambda^p}.$$

The infimum of $G_\rho(\lambda)$ over $\lambda > 0$ is attained at $\bar{\lambda}_\rho = \left(\frac{p}{n}\Gamma_\rho^c\right)^{1/p}$. Hence

$$\begin{aligned} \int_{R^n} \rho \ln \rho \,\mathrm{d}x &\leq G_\rho(\bar{\lambda}_\rho) - n - \ln(\sigma_c) \\ &= \frac{n}{p} \ln \left(\frac{p}{n}\Gamma_\rho^c\right) + \frac{n}{p} - n - \ln(\sigma_c) \\ &= \frac{n}{p} \ln \left(\frac{p}{n e^{p-1} \sigma_c^{p/n}} \Gamma_\rho^c\right), \end{aligned}$$

for all probability densities ρ on $I\!R^n$, such that $\operatorname{supp}\rho \subset \Omega$, and $\rho \in W^{1,\infty}(I\!R^n)$.

Corollary 4.2 *(*Optimal Euclidean p-Log Sobolev inequality*)*

$$\int_{R^n} |\, f\,|^p \ln(|\, f\,|^p)\, dx \leq \frac{n}{p} \ln \left(C_p \int_{R^n} |\,\nabla f\,|^p\, dx\right), \tag{23}$$

holds for all $p \geq 1$, and for all $f \in W^{1,p}(I\!R^n)$ such that $\|\, f\,\|_p = 1$, where

$$C_p := \begin{cases} \left(\frac{p}{n}\right)\left(\frac{p-1}{e}\right)^{p-1} \pi^{-\frac{p}{2}} \left[\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n}{q}+1)}\right]^{\frac{p}{n}} & \text{if} \quad p > 1, \\ \\ \frac{1}{n\sqrt{\pi}} \left[\Gamma(\frac{n}{2}+1)\right]^{\frac{1}{n}} & \text{if} \quad p = 1, \end{cases} \tag{24}$$

and q is the conjugate of p ($\frac{1}{p} + \frac{1}{q} = 1$).
For $p > 1$, equality holds in (23) for $f(x) = K e^{-\lambda^q \frac{|\, x - \bar{x}\,|^q}{q}}$ for some $\lambda > 0$ and $\bar{x} \in I\!R^n$, where $K = \left(\int_{R^n} e^{-(p-1)|\,\lambda x\,|^q}\, dx\right)^{-1/p}$.

Proof: First assume that $p > 1$, and set $c(x) = (p-1)|\,x\,|^q$ and $\rho = |\,f\,|^p$ in (20), where $f \in C_c^\infty(I\!R^n)$ and $\|\,f\,\|_p = 1$. We have that $c^\star(x) = \frac{|\,x\,|^p}{p^p}$, and then, $\Gamma_\rho^c = \int_{R^n} |\,\nabla f\,|^p\,\mathrm{d}x$. Therefore, (20) reads as

$$\int_{R^n} |\, f\,|^p \ln(|\, f\,|^p)\,\mathrm{d}x \leq \frac{n}{p} \ln \left(\frac{p}{n e^{p-1} \sigma_c^{p/n}} \int_{R^n} |\,\nabla f\,|^p\,\mathrm{d}x\right). \tag{25}$$

Now it suffices to note that

$$\sigma_c := \int_{R^n} e^{-(p-1)|\,x\,|^q}\,\mathrm{d}x = \frac{\pi^{\frac{n}{2}} \Gamma\left(\frac{n}{q}+1\right)}{(p-1)^{\frac{n}{q}} \Gamma\left(\frac{n}{2}+1\right)}. \tag{26}$$

To prove the case where $p = 1$, it is sufficient to apply the above to $p_\epsilon = 1 + \epsilon$ for some arbitrary $\epsilon > 0$. Note that

$$C_{p\epsilon} = \left(\frac{1+\epsilon}{n}\right)\left(\frac{\epsilon}{e}\right)^{\epsilon} \pi^{-\frac{1+\epsilon}{2}} \left[\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n\epsilon}{1+\epsilon}+1)}\right]^{\frac{1+\epsilon}{n}}.$$

So that when ϵ go to 0, we have

$$\lim_{\epsilon \to 0} C_{p_\epsilon} = \frac{1}{n\sqrt{\pi}}\left[\Gamma\left(\frac{n}{2}+1\right)\right]^{\frac{1}{n}} = C_1.$$

5 Gagliardo-Nirenberg and Sobolev Inequalities

Corollary 5.1 *(*Gagliardo-Nirenberg*) Let $1 < p < n$ and $r \in \left(0, \frac{np}{n-p}\right]$ such that $r \neq p$. Set $\gamma := \frac{1}{r} + \frac{1}{q}$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then, for any $f \in W^{1,p}(\mathbb{R}^n)$ we have*

$$\|f\|_r \le C(p,r)\|\nabla f\|_p^{\theta}\, \|f\|_{r\gamma}^{1-\theta}, \tag{27}$$

where θ is given by

$$\frac{1}{r} = \frac{\theta}{p^*} + \frac{1-\theta}{r\gamma}, \tag{28}$$

$p^* = \frac{np}{n-p}$ and where the best constant $C(p,r) > 0$ can be obtained by scaling.

Proof: Apply (5) with $F(x) = \frac{x^\gamma}{\gamma-1}$, where $1 \neq \gamma \ge 1 - \frac{1}{n}$, which follows from the fact that $p \neq r \in \left(0, \frac{np}{n-p}\right]$. Now, for this value of γ, the function F satisfies the conditions of Theorem 1. Let $c(x) = \frac{r\gamma}{q}|\,x\,|^q$ so that $c^*(x) = \frac{1}{p(r\gamma)^{p-1}}\Big|\,x\,\Big|^p$.

Inequality (5) then gives

$$\left(\frac{1}{\gamma-1}+n\right)\int_{R^n}|\,f\,|^{r\gamma} \le \frac{r\gamma}{p}\int_{R^n}|\,\nabla f\,|^p - H^{P_F}(\rho_\infty) + C_\infty. \tag{29}$$

where $\rho_\infty = h_\infty^r$ satisfies

$$-\nabla h_\infty(x) = x|\,x\,|^{q-2} h^{\frac{r}{p}}(x) \;\text{ a.e.}, \tag{30}$$

and where C_∞ insures that $\int h_\infty^r = 1$. The constants on the right hand side of (29) are not easy to calculate, so one can obtain θ and the best constant by a standard scaling procedure. Namely, write (29) as

$$\frac{r\gamma}{p}\frac{\|\nabla f\|_p^p}{\|f\|_r^p} - \left(\frac{1}{\gamma-1}+n\right)\frac{\|f\|_{r\gamma}^{r\gamma}}{\|f\|_r^{r\gamma}} \ge C, \tag{31}$$

for some constant C. Then apply it to $f_\lambda(x) = f(\lambda x)$ for $\lambda > 0$. A minimization over λ gives the required constant.

The case where $\gamma = 1 - \frac{1}{n}$ gives the standard Sobolev inequality.

Corollary 5.2 *Let $1 < p < n$, then we have for any $f \in W^{1,p}(\mathbb{R}^n)$,*

$$\|f\|_{p^*} \leq C(p,n)\|\nabla f\|_p \tag{32}$$

for some constant $C(p,n) > 0$.

By letting $p \to 1$, one then gets the isoperimetric inequality: For any closed subset of $\mathbb{R}^n$, with σ denoting surface measure and $|\cdot|$ Lebesgue measure.

$$\sigma(\partial A) \geq n|B|^{\frac{1}{n}}|A|^{\frac{n-1}{n}}. \tag{33}$$

Similar results can be established in the presence of an additional convolution operator.

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