Geometric inequalities via a duality between certain quasilinear PDEs and Fokker-Planck equations

M. Agueh*, N. Ghoussoub† and X. Kang‡

December 18, 2002

Pacific Institute for the Mathematical Sciences
and
Department of Mathematics, The University of British Columbia, Vancouver, B. C.
V6T 1Z2, Canada

Abstract

Motivated by a paper of Cordero-Nazaret-Villani, we proceed to describe how the Monge-Kantorovich theory for mass transport, leads to a remarkable correspondence between ground state solutions of certain quasilinear (or semi-linear) equations of the form $\mathrm{div}\{\nabla V^*(-\nabla f)\} - (G \circ \psi)'(f) = \psi'(f)$ on some domain Ω of $\mathbb{R}^n$ and stationary solutions of Fokker-Planck equations $\frac{\partial u}{\partial t} = \mathrm{div}\{u\nabla(F'(u) + V)\}$, where V is an appropriate convex function, V^* its Fenchel conjugate and where F, G and ψ are related functions. This duality implies most known geometric inequalities –including a general HWI inequality for any displacement convex functional–and yields convergence rates for -generalized- entropies of degenerate nonlinear Fokker-Planck equations to their equilibria i.e., without assumptions of uniform convexity on confinement potentials. It also gives a direct and unified way for computing best constants in geometric inequalities and the extremals where they are attained. In particular, an optimal Euclidean p-Log Sobolev inequality is established for any $p \geq 1$, extending corresponding results of Beckner ($p = 1$) and Del Pino-Dolbeault ($1 < p < n$). A forthcoming paper will deal with the dynamic aspect of the above duality, with inequalities in the presence of an additional convolution operator as well as with rates of convergence to equilibrium of Fokker-Planck type equations in the whole space.

---

*This paper was done while this author held a postdoctoral fellowship at the University of British Columbia.

†The three authors were partially supported by a grant from the Natural Science and Engineering Research Council of Canada.

‡This paper is part of this author's PhD's thesis under the supervision of N. Ghoussoub.

1 Introduction

Let $F : [0, \infty) \to \mathbb{R}$ be a convex function, V a real functional on $\mathbb{R}^n$ and let $\Omega \subset \mathbb{R}^n$ be open, bounded and convex. The set of probability densities over Ω is denoted by $\mathcal{P}_a(\Omega) = \{\rho : \Omega \to \mathbb{R};\ \rho \geq 0 \text{ and } \int_\Omega \rho(x)dx = 1\}$. The associated *Free Energy Functional* is then defined on $\mathcal{P}_a(\Omega)$ as:

$$H_V^F(\rho) := \int_\Omega (F(\rho) + \rho V)dx,$$

which is the sum of the internal energy

$$H^F(\rho) := \int_\Omega F(\rho)dx,$$

and the potential energy

$$H_V(\rho) := \int_\Omega \rho V dx.$$

Motivated by the recent work of [6], we shall establish yet another inequality relating the free energy functional to the entropy production functional associated to a Fokker-Planck equation. However, our main goal here is to establish a remarkable correspondence between Fokker-Planck evolution equations and certain quasilinear or semi-linear equations which appear as Euler-Lagrange equations of the entropy production functionals. More precisely, assume that F is differentiable on $(0, \infty)$, that $F(0) = 0$ and that $x \mapsto x^n F(x^{-n})$ is convex and non-increasing, and let $P_F(x) := xF'(x) - F(x)$ be its associated pressure function. We show that, for any strictly convex C^1-function $c : \mathbb{R}^n \to \mathbb{R}$ such that $\lim_{|x|\to\infty} \frac{c(x)}{|x|} = \infty$, and any convex potential V,

$$-H_{V+c}^F(\rho_1) \leq -H_{V-x\cdot\nabla V}^{F+nP_F}(\rho_0) + \int_\Omega \rho_0 c^\star\left(-\nabla(F' \circ \rho_0 + V)\right) dx, \tag{1}$$

for any probability density $\rho_0 \in W^{1,\infty}(\Omega)$ with support in Ω, and any $\rho_1 \in \mathcal{P}_a(\Omega)$. Here c^* denotes the Legendre conjugate of c defined by $c^*(y) = \sup_{z\in R^n}\{y \cdot z - c(z)\}$. Moreover, equality holds whenever $\rho_0 = \rho_1 = \rho_\infty$ where ρ_∞ is a probability density on Ω such that $\nabla(F'(\rho_\infty) + V + c) = 0$.

This inequality leads to a remarkable duality between ground state solutions of certain quasilinear (or semi-linear) equations of the form

$$\operatorname{div}\{\nabla c^*(-\nabla f)\} - (G \circ \psi)'(f) = \psi'(f) \tag{2}$$

and stationary solutions of (non-linear) Fokker-Planck equations:

$$\frac{\partial u}{\partial t} = \operatorname{div}\{u\nabla(F'(u) + c)\} \tag{3}$$

where c^* is the Legendre transform of c, $G(x) = (1-n)F(x) + nxF'(x)$ and where ψ satisfies $|\psi^{\frac{1}{p}}(F' \circ \psi)'| = K$, for some constant K that we choose equal to 1 for simplicity. Here we have assumed that c^* is p-homogeneous, that is $c^\star(\lambda x) = |\,\lambda\,|^p c^*(x)$, for all $\lambda \in I\!R$ and for all $x \in I\!R^n$.

Behind this correspondence lies a non-trivial "change of variable" that is given by the solution of the Monge transport problem. It essentially maps the solutions of the evolution equation associated to (2) to those of the Fokker-Planck equations (3). The full theory will be developed in a forthcoming paper. In this note, we shall only deal with the stationary case which will follow from a variational duality between the Energy Functional

$$I(f) = \int_\Omega [c^*(-\nabla f(x)) - G(f(x))]\,dx \tag{4}$$

whose L^2-Euler-Lagrange equations on the manifold $\{f \in C_0^\infty(\Omega);\ \int_\Omega \psi(f(x))dx = 1\}$ is essentially equation (2) (modulo Lagrange multipliers), and the Free Energy functional

$$J(\rho) = -\int_\Omega [F(\rho(y)) + c(y)\rho(y)]dy \tag{5}$$

on the set of probability densities $\mathcal{P}_a(\Omega)$, whose gradient flow with respect to the Wasserstein distance is precisely the evolution equation (3).

Under appropriate hypothesis on F and c, one gets that

$$\sup\{J(\rho);\ \int_\Omega \rho(x)dx = 1\} \le \inf\{I(f);\ \int_\Omega \psi(f(x))dx = 1\}, \tag{6}$$

with equality occurring whenever there exists $\bar{f}$ (and $\bar{\rho} = \psi(\bar{f})$) that satisfies the first order equation:

$$-(F' \circ \psi)'(\bar{f})\nabla \bar{f}(x) = \nabla c(x) \text{ a.e.} \tag{7}$$

In this case, the extrema are achieved at $\bar{f}$ (resp. $\bar{\rho} = \psi(\bar{f})$) which also satisfies (2) (resp., is a stationary solution of (3)).

A typical example is the correspondence between the "Yamabe" equation

$$-\Delta f = |f|^{2^*-2} f \text{ on } I\!R^n, \tag{8}$$

where $2^* = \frac{2n}{n-2}$ is the critical Sobolev exponent, and the non-linear Fokker-Planck equation

$$\frac{\partial u}{\partial t} = \Delta u^{1-\frac{1}{n}} + \operatorname{div}(x.u), \tag{9}$$

which –after appropriate scaling– reduces to the fast diffusion equation:

$$\frac{\partial u}{\partial t} = \Delta u^{1-\frac{1}{n}}. \tag{10}$$

The correspondence was motivated by the work of [6] where mass transport is used to establish Sobolev-type inequalities. Solutions of (8) can be obtained by minimizing the energy functional on the unit sphere of L^{2^*}, that is:

$$\inf\Big\{\Big(\frac{n-1}{n-2}\Big)^2 \int_{R^n} |\nabla f|^2 dx;\ f \in C_0^\infty(I\!R^n),\ \int_{R^n} |f|^{2^*} dx = 1\Big\}. \tag{11}$$

Using mass transport, they show that the above infimum is equal to the supremum of the functional

$$J(\rho) = n \int_{R^n} \rho(x)^{\frac{n-1}{n}} dx - \frac{1}{2} \int_{R^n} |x|^2 \rho(x) dx \tag{12}$$

over the space of probability densities.

Cordero et al. also deal with the Gagliardo-Nirenberg inequalities and obtain best constant results that Del Pino-Dolbeault had obtained earlier by carefully analyzing porous media evolution equations [8]. The link between the two methods becomes much clearer via the above correspondence.

This duality seems to be at the heart of many geometric inequalities and also allows the derivation of associated dual inequalities. Indeed, assuming that $D^2V \geq \lambda I$ where $\lambda \in I\!R$ is not necessarily positive, inequality (1) is a special case of a more general one. Namely, we have for all probability density functions ρ_0 and ρ_1 on Ω, satisfying $\operatorname{supp} \rho_0 \subset \Omega$, and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$,

$$H^F_{V+c}(\rho_0|\rho_1) + \frac{\lambda}{2} W_2^2(\rho_0, \rho_1) \leq \int_\Omega c^* \left(-\nabla \left(F' \circ \rho_0 + V\right)\right) \rho_0 \, \mathrm{d}x + H^{-nP_F}_{c+x.\nabla V}(\rho_0), \tag{13}$$

where W_2 is the Wasserstein distance, and

$$H^F_{V+c}(\rho_0/\rho_1) := H^F_{V+c}(\rho_0) - H^F_{V+c}(\rho_1).$$

In particular, we have for any probability density ρ with support containing in Ω and such that $P_F(\rho) \in W^{1,\infty}(\Omega)$,

$$H^{F+nP_F}_{V-x\cdot\nabla V}(\rho) + \frac{\lambda}{2} W_2^2(\rho, \rho_\infty) \leq \int_\Omega \rho c^\star \left(-\nabla(F' \circ \rho + V)\right) \mathrm{d}x - H^{P_F}(\rho_\infty) + C_\infty \tag{14}$$

where C_∞ is the unique constant such that

$$F'(\rho_\infty) + V + c = C_\infty \text{ while } \int_\Omega \rho_\infty = 1. \tag{15}$$

We shall see that this inequality easily implies most known geometric inequalities. It provides a direct and unified way for computing best constants as well as the extremals where they are attained.

The term $H^{-nP_F}_{c+\nabla V.x}(\rho_0)$ should be seen as an error term in (13). It can be integrated in the entropy production-type term - that is, the integral term in (13) - which proves useful in the Gaussian case, hence leading to the following generalized HWI inequality essentially established by Otto-Villani [12] in the case of the classical entropy $F(x) = x \ln x$: For any U such that $D^2U \geq \mu I$ with $\mu \in I\!R$ and any $\sigma > 0$,

$$H^F_U(\rho_0|\rho_1) + \frac{1}{2}(\mu - \frac{1}{\sigma}) W_2^2(\rho_0, \rho_1) \leq \frac{\sigma}{2} \int_\Omega |\nabla(F' \circ \rho_0 + U)|^2 \rho_0 \, \mathrm{d}x, \tag{16}$$

In the case where U is a uniformly convex confinement potential (i.e. $\mu > 0$) one then gets - by setting $\sigma = \frac{1}{\mu}$ in (16) - the generalized Gross Log Sobolev inequality

$$H^F_U(\rho_0|\rho_1) \leq \frac{1}{2\mu} \int_\Omega |\nabla(F' \circ \rho_0 + U)|^2 \rho_0 \, \mathrm{d}x, \tag{17}$$

and the generalized Talagrand inequality: for any probability density ρ on Ω, we have

$$W_2(\rho|\rho_U) \leq \sqrt{\frac{2}{\mu} H_U^F(\rho|\rho_U)}, \tag{18}$$

where ρ_U is the probability density satisfying

$$\nabla\left(F'(\rho_U) + U\right) = 0 \quad \text{a.e.} \tag{19}$$

If V is now simply convex, then λ can be taken equal to 0 and the Wasserstein distance disappears from the equation. Furthermore, if V is strictly convex and grows super-linearly as $|\,x\,| \to \infty$, we have the identity $V(x) - x \cdot \nabla V(x) = -V^*(\nabla V(x)$ in such a way that a correcting "moment" appears in the inequality:

$$H^{F+nP_F}_{-V^*(\nabla V)}(\rho) \leq \int_\Omega \rho c^\star\left(-\nabla(F' \circ \rho + V)\right) \mathrm{d}x - H^{P_F}(\rho_\infty) + C_\infty. \tag{20}$$

Also note that the pressure P_F is always non-negative which means that we can do away with the term $H^{P_F}(\rho_\infty)$ on the right hand side. This will prove useful in inequality (24). Finally, the case $V = 0$ amply covers the Euclidean case where the general inequality becomes the remarkably simple:

$$H^{F+nP_F}(\rho) \leq \int_\Omega \rho c^\star\left(-\nabla(F' \circ \rho)\right) \mathrm{d}x - H^{P_F}(\rho_\infty) + C_\infty. \tag{21}$$

Similar results can be established in the presence of an additional convolution operator.

We shall apply this to various functionals to revisit various inequalities and to determine their corresponding best constants. In particular, we show in section (3) how it readily implies various Log-Sobolev inequalities, the Gagliardo-Nirenberg inequalities, and in particular the Sobolev inequalities. For example, it yields the optimal Euclidean p-Log Sobolev inequality for any $p > 1$. We note that the case where $p = 1$ was established by Beckner in [3], and for $1 < p < n$ by Del-Pino and Dolbeault[1].

In section (4), we show how the above inequality also yields convergence rates to equilibria for -generalized- entropies of degenerate nonlinear Fokker-Planck equations, in the absence of confinement potentials. Here it is worth noticing that even in the absence of a potential V, our energy estimate (1) introduces naturally a non-trivial confinement potential in the game via the Young functional. Often, this can be used to cover degenerate cases. Indeed, recall that the *generalized relative Fisher information* of ρ with respect to ρ_1, *measured against the Young functional* c^* is defined as

$$I_{c^*}(\rho|\rho_1) := \int_\Omega \nabla(F'(\rho) - F'(\rho_1)) \cdot \nabla c^\star\left(\nabla(F'(\rho) - F'(\rho_1))\right) \rho \,\mathrm{d}x, \tag{22}$$

and when $\nabla\left(F'(\rho_1)\right) = 0$, it reduces to

$$I_{c^*}(\rho) := \int_\Omega \nabla\left(F'(\rho)\right) \cdot \nabla c^\star\left(\nabla\left(F'(\rho)\right)\right) \rho \,\mathrm{d}x. \tag{23}$$

[1]We have been informed by J. Dolbeault that the case where $p > n$ has also been established recently and independently by I. Gentil who used the Prékopa-Leindler inequality and the Hopf-Lax semi-group associated to the Hamilton-Jacobi equation.

Then, we see that whenever c is a q-homogenous convex Young functional for some $q > 1$ and if p is its conjugate, then for all probability densities ρ_0 and ρ_1 on Ω such that $\operatorname{supp}\rho_0 \subset \Omega$ and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$, we have

$$H^F(\rho_0|\rho_1) \leq \frac{p}{(p-1)^{1/q}}[H_c(\rho_1)]^{1-\frac{1}{p}}[I_{c^*}(\rho_0)]^{\frac{1}{p}}. \tag{24}$$

This will allow us to study the rate of convergence to equilibrium of degenerate Fokker-Planck equations of the form

$$\begin{cases} \frac{\partial \rho}{\partial t} = \operatorname{div}\Big\{\rho\nabla c^*\left[\nabla\left(F'(\rho)+V\right)\right]\Big\} & \text{in} \quad (0,\infty)\times\Omega \\ \rho\nabla c^*\left[\nabla\left(F'(\rho)+V\right)\right]\cdot\nu = 0 & \text{on} \quad (0,\infty)\times\partial\Omega \\ \rho(t=0)=\rho_0 & \text{in} \quad \{0\}\times\Omega \end{cases} \tag{25}$$

on bounded domains Ω when the confinement potential V is zero. In a forthcoming paper, we investigate the case where the equations have a confinement potential that is not uniformly convex.

2 A Remarkable Duality

Our approach is based on the recent advances in the theory of mass transport as developed by Brenier [4], McCann-Gangbo [9] and many others. For a survey, see Villani [13]. Here is a brief summary of the needed results.

Fix a non-negative C^1, strictly convex function $d : I\!R^n \to I\!R$ such that $d(0) = 0$. Given two probability measures μ and ν on $I\!R^n$, the minimum cost for transporting μ onto ν is given by

$$W_d(\mu,\nu) := \inf_{\gamma\in\Gamma(\mu,\nu)} \int_{R^n\times R^n} d(x-y)d\gamma(x,y), \tag{26}$$

where $\Gamma(\mu,\nu)$ is the set of Borel probability measures with marginals μ and ν, respectively. If $d(x) = |\,x\,|^2$, then $W_d = W_2^2$, where W_2 is the Wasserstein distance. We say that a Borel map $T : I\!R^n \to I\!R^n$ pushes μ forward to ν, if $\mu(T^{-1}(B)) = \nu(B)$ for any Borel set $B \subset I\!R^n$. The map T is then said to be d-optimal if

$$W_d(\mu,\nu) = \int_{R^n} d(x-Tx)d\mu(x) = \inf_S \int_{R^n} d(x-Sx)d\mu(x), \tag{27}$$

where the infimum is taken over all Borel maps $S : I\!R^n \to I\!R^n$ that push μ forward to ν.

For quadratic cost functions $d(z) = \frac{1}{2}|z|^2$, Brenier [4] characterized the optimal transport map T as the gradient of a convex function. An analogous result holds for general cost functions d, provided convexity is replaced by an appropriate notion of d-concavity. See [9], [5] for details.

Let's call *Young function* any non-negative C^1, strictly convex function $c : I\!R^n \to I\!R$ such that $c(0) = 0$ and $\lim_{|\,x\,|\to\infty} \frac{c(x)}{|\,x\,|} = \infty$.

Here and after, $\operatorname{supp}\rho$ denotes the support of $\rho \in \mathcal{P}_a(\Omega)$, that is, the closure of the set $\{x \in \Omega;\ \rho(x) \neq 0\}$. Here is our starting point:

Theorem 2.1 *Let $\Omega \subset I\!\!R^n$ be open, bounded and convex, let $F : [0,\infty) \to I\!\!R$ be differentiable on $(0,\infty)$ such that $F(0) = 0$ and $x \mapsto x^n F(x^{-n})$ be convex and non-increasing, and let $P_F(x) := xF'(x) - F(x)$ be its associated pressure function. Then, for any Young function $c : I\!\!R^n \to I\!\!R$, and any $V : I\!\!R^n \to I\!\!R$ with $D^2V \geq \lambda$ with $\lambda \in I\!\!R$, we have for all probability density functions ρ_0 and ρ_1 on Ω, satisfying $\operatorname{supp}\rho_0 \subset \Omega$, $\rho_0 > 0$ a.e. on Ω and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$,*

$$H^F_{V+c}(\rho_0|\rho_1) + \frac{\lambda}{2}W_2^2(\rho_0,\rho_1) \leq \int_\Omega \rho_0 c^\star\left(-\nabla(F'\circ\rho_0 + V)\right)\, dx + H^{-nP_F}_{c+\nabla V.x}(\rho_0). \tag{28}$$

Furthermore, equality holds in (28) whenever $\rho_0 = \rho_1 = \rho_\infty$ where the latter satisfies:

$$\nabla\left(F'(\rho_\infty) + V + c\right) = 0 \quad a.e. \tag{29}$$

In particular, we have for any probability density ρ on Ω with $\operatorname{supp}\rho \subset \Omega$ and $P_F(\rho) \in W^{1,\infty}(\Omega)$,

$$H^{F+nP_F}_{V-x\cdot\nabla V}(\rho) + \frac{\lambda}{2}W_2^2(\rho,\rho_\infty) \leq \int_\Omega \rho c^\star\left(-\nabla(F'\circ\rho + V)\right)\, dx - H^{P_F}(\rho_\infty) + C_\infty \tag{30}$$

where C_∞ is unique constant such that

$$F'(\rho_\infty) + V + c = C_\infty \text{ while } \int_\Omega \rho_\infty = 1. \tag{31}$$

Proof: If T is the optimal map that pushes $\rho_0 \in \mathcal{P}_a(\Omega)$ forward to $\rho_1 \in \mathcal{P}_a(\Omega)$ - for the quadratic cost function $d(x) = |\,x\,|^2$ -, define a path of probability densities joining them, by letting ρ_t be the push-forward measure of ρ_0 by the map $T_t = (1-t)I + tT$, where I is the identity map.

Under the above assumptions on F, it turns out (see McCann [10]) that the function $t \mapsto H^F(\rho_t)$ is convex on $[0,1]$, which essentially leads to the following inequality for the internal energy:

$$H^F(\rho_1) - H^F(\rho_0) \geq [\frac{d}{dt}H^F(\rho_t)]_{t=0}. \tag{32}$$

As noted in [7], the fact that $V : I\!\!R^n \to I\!\!R$ is uniformly convex with constant $\lambda \in I\!\!R$, implies the following inequality for the corresponding potential energy:

$$H_V(\rho_1) - H_V(\rho_0) \geq [\frac{d}{dt}H_V(\rho_t)]_{t=0} + \frac{\lambda}{2}W_2^2(\rho_0,\rho_1). \tag{33}$$

By combining (32) and (33), one gets the following inequality which seems to be the "mother" of many geometric inequalities.

$$-H^F_V(\rho_1) + \frac{\lambda}{2}W_2^2(\rho_0,\rho_1) \leq -H^F_V(\rho_0) + \int_\Omega (I-T).\nabla(F'\circ\rho_0 + V)\rho_0, \tag{34}$$

This inequality essentially describes the evolution of a generalized entropy functional along optimal transport. In the case where $V = 0$, it was first obtained by Otto [11] for the Tsallis entropy functionals and by Agueh [1] in general. The case of a nonzero potential V was included in [2], [7]. Actually, one can also add another convolution operator as noted in [7]. In case $\lambda > 0$ (i.e., if V is uniformly convex), Cordero et al. [7] obtain Gaussian Log-Sobolev inequalities by simply using Young's inequality (with convex function $\lambda\frac{|x|^2}{2}$ and its Fenchel conjugate) applied to the last term $(I-T).\nabla(F'\circ\rho_0+V)$, then noting that $W_2^2(\rho_0,\rho_1)=\int_{R^n}|x-Tx|^2\rho_0$ to conclude that

$$-H_V^F(\rho_1)\le -H_V^F(\rho_0)+\frac{1}{2\lambda}\int_\Omega|\nabla(F'\circ\rho_0+V)|^2\rho_0. \tag{35}$$

In the case where λ is not necessarily positive, one can still proceed with a slightly different application of Young's inequality after evaluating $\int_\Omega I.\nabla(F'(\rho_0)+V)\rho_0$. An additional advantage is that here the Young function need not be related to the cost of the Wasserstein distance. Indeed, since $\rho_0\nabla(F'\circ\rho_0)=\nabla(P_F\circ\rho_0)$, we integrate by part in $\int_\Omega\langle\rho_0\nabla(F'\circ\rho_0),x\rangle\,\mathrm{d}x$ and obtain

$$-H_V^F(\rho_1)+\frac{\lambda}{2}W_2^2(\rho_0,\rho_1)\le -H^{F+nP_F}_{V-\nabla V.x}(\rho_0)+\int_\Omega\langle-\nabla\left(F'(\rho_0)+V\right),T\rangle\rho_0 dx. \tag{36}$$

Now, use Young's inequality with a convex function c to get:

$$\langle-\nabla\left(F'\left(\rho_0(x)+V(x)\right)\right),T(x)\rangle\le c\left(T(x)\right)+c^\star\left(-\nabla\left(F'\left(\rho_0(x)+V(x)\right)\right)\right), \tag{37}$$

and deduce that

$$-H_V^F(\rho_1)+\frac{\lambda}{2}W_2^2(\rho_0,\rho_1)\le -H^{F+nP_F}_{V-\nabla V.x}(\rho_0)+\int_\Omega\rho_0c^\star\left(-\nabla\left(F'(\rho_0+V)\right)\right)+\int_\Omega c(T)\rho_0\,\mathrm{d}x.$$

Finally, use again that T pushes ρ_0 forward to ρ_1, to rewrite the second integral on the right hand side as $\int_\Omega c(y)\rho_1(y)dy$ to obtain (28).
Now, we set $\rho_0=\rho_1:=\rho$ in (36). We have that $T=\mathrm{id}$, and equality holds in (36). Therefore, equality holds in (28) whenever equality holds in (37), where $T(x)=x$. This occurs when (29) is satisfied.
(30) is straightforward when choosing $\rho_0:=\rho$ and $\rho_1:=\rho_\infty$ in (28), where ρ_∞ satisfies (29).

Note that actually, the assumption $\rho_0>0$ a.e. is not needed in (28) because all the terms in (28) remain unchanged when replacing ρ_0 by $\rho_0\chi_{[\rho_0>0]}$, where $\chi_{[\rho_0>0]}$ denotes the characteristic function of the set $[\rho_0>0]$.

In the rest of this section, we apply the above theorem to obtain a general duality theory, in the case when the confinement potential $V=0$.

Corollary 2.2 *Let $\Omega\subset I\!R^n$ be open, bounded and convex, let $F:[0,\infty)\to I\!R$ be differentiable on $(0,\infty)$ such that $F(0)=0$ and $x\mapsto x^nF(x^{-n})$ be convex and non-increasing. Let $\psi: I\!R\to[0,\infty)$ differentiable be chosen in such a way that $\psi(0)=0$ and $|\psi^{\frac{1}{p}}(F'\circ\psi)'|=K$ where $p>1$, and K is chosen to be 1 for simplicity. Then, for* *any Young function c, such that its Legendre transform c^* is p-homogeneous, we have the following inequality:*

$$\sup\{-\int_\Omega F(\rho)+c\rho;\rho\in\mathcal{P}_a(\Omega)\}\leq\inf\{\int_\Omega c^*(-\nabla f)-G_F\circ\psi(f);f\in C_0^\infty(\Omega),\int_\Omega\psi(f)=1\} \tag{38}$$

where $G_F(x) := (1-n)F(x)+nxF'(x)$. Furthermore, equality holds in (38) if there exists $\bar{f}$ (and $\bar{\rho}=\psi(\bar{f})$) that satisfies

$$-(F'\circ\psi)'(\bar{f})\nabla\bar{f}(x)=\nabla c(x)\ \ a.e. \tag{39}$$

Moreover, $\bar{f}$ solves

$$\begin{array}{ll}\operatorname{div}\{\nabla c^*(-\nabla f)\}-(G_F\circ\psi)'(f)=\lambda\psi'(f) & \text{in } \Omega\\ \nabla c^*(-\nabla f)\cdot\nu=0 & \text{on } \partial\Omega,\end{array} \tag{40}$$

for some $\lambda\in\mathbb{R}$.

Proof: Assume that c^* is p-homogeneous, and let $Q''(x)=x^{\frac{1}{q}}F''(x)$. Let

$$J(\rho):=-\int_\Omega[F(\rho(y))+c(y)\rho(y)]dy$$

and

$$\tilde{J}(\rho):=-\int_\Omega(F+nP_F)(\rho(x))dx+\int_\Omega c^*(-\nabla(Q'(\rho(x)))dx.$$

Equation (28) (where we use $V=0$ and then $\lambda=0$) then becomes

$$J(\rho_1)\leq\tilde{J}(\rho_0) \tag{41}$$

for all probability densities ρ_0,ρ_1 on Ω such that $\operatorname{supp}\rho_0\subset\Omega$ and $P_F(\rho_0)\in W^{1,\infty}(\Omega)$. If $\bar{\rho}$ satisfies

$$-\nabla(F'(\bar{\rho}(x)))=\nabla c(x)\text{ a.e.},$$

then equality holds in (41), and $\bar{\rho}$ is an extremal of the variational problems

$$\sup\{J(\rho);\ \rho\in\mathcal{P}_a(\Omega)\}=\inf\{\tilde{J}(\rho);\rho\in\mathcal{P}_a(\Omega),\operatorname{supp}\rho\subset\Omega,P_F(\rho)\in W^{1,\infty}(\Omega)\}.$$

In particular, $\bar{\rho}$ is a solution of

$$\begin{array}{ll}\operatorname{div}\{\rho\nabla(F'(\rho)+c)\}=0 & \text{in } \Omega\\ \rho\nabla(F'(\rho)+c)\cdot\nu=0 & \text{on } \partial\Omega.\end{array} \tag{42}$$

Suppose now $\psi:\mathbb{R}\to[0,\infty)$ differentiable, $\psi(0)=0$ and that $\bar{f}\in C_0^\infty(\Omega)$ satisfies $-(F'\circ\psi)'(\bar{f})\nabla\bar{f}(x)=\nabla c(x)$ a.e. Then equality holds in (41), and $\bar{f}$ and $\bar{\rho}=\psi(\bar{f})$ are extremals of the following variational problems

$$\inf\{I(f);\ f\in C_0^\infty(\Omega),\int_\Omega\psi(f)=1\}=\sup\{J(\rho);\rho\in\mathcal{P}_a(\Omega)\}$$

where

$$I(f) = \tilde{J}(\psi(f)) = -\int_\Omega [F \circ \psi + nP_F \circ \psi](f) + \int_\Omega c^*(-\nabla(Q' \circ \psi(f))).$$

If now ψ is such that $|\,\psi^{\frac{1}{p}}(F' \circ \psi)'\,| = 1$, then $|\,(Q' \circ \psi)'\,| = 1$ and

$$I(f) = -\int_\Omega [F \circ \psi + nP_F \circ \psi](f) + \int_\Omega c^*(-\nabla f)),$$

because c^* is p-homogeneous. The Euler-Lagrange equation of the variational problem

$$\inf\Big\{ \int_\Omega c^*(-\nabla(f)) - [F \circ \psi + nP_F \circ \psi](f);\ \int_\Omega \psi(f) = 1 \Big\}$$

becomes

$$\begin{array}{ll} \operatorname{div}\{\nabla c^*(-\nabla f)\} - (G_F \circ \psi)'(f) = \lambda \psi'(f) & \text{in } \Omega \\ \nabla c^*(-\nabla f) \cdot \nu = 0 & \text{on } \partial\Omega \end{array} \tag{43}$$

where $\lambda \in I\!R$ is a Lagrange multiplier, and $G(x) = (1-n)F(x) + nxF'(x)$.

Many important inequalities follow from Corollary 2.2. First, we apply it to the functions $F(x) = x \ln x$, $\psi(x) = |\,x\,|^p$, $c(x) = (p-1)|\,\mu x\,|^q$, where $\mu > 0$ and $c^*(x) = \frac{1}{p}|\,\frac{x}{\mu}\,|^p$ and $\frac{1}{p} + \frac{1}{q} = 1$. We note here that the condition $|\,\psi^{\frac{1}{p}}(F' \circ \psi)\,| = K$ holds for $K = p$. We obtain the following:

Corollary 2.3 *Let $p > 1$ and let q be its conjugate ($\frac{1}{p} + \frac{1}{q} = 1$). For all $f \in W^{1,p}(I\!R^n)$, such that $\|\, f\,\|_p = 1$, any probability density ρ such that $\int_{R^n} \rho(x)|x|^q dx < \infty$, and any $\mu > 0$, we have*

$$J_\mu(\rho) \le I_\mu(f), \tag{44}$$

where

$$J_\mu(\rho) := -\int_{R^n} \rho \ln(\rho)\ dy - (p-1)\int_{R^n} |\,\mu y\,|^q \rho(y)\, dy,$$

and

$$I_\mu(f) := -\int_{R^n} |\,f\,|^p \ln(|\,f\,|^p) + \int_{R^n} \Big|\,\frac{\nabla f}{\mu}\,\Big|^p - n.$$

Furthermore, if $h \in W^{1,p}(I\!R^n)$ is such that $h \ge 0$, $\|\,h\,\|_p = 1$, and

$$\nabla h(x) = -\mu^q x |\,x\,|^{q-2} h(x) \quad a.e.,$$

then

$$J_\mu(h^p) = I_\mu(h).$$

Therefore, h (resp., $\rho = h^p$) is an extremum of the variational problem:

$$\sup\{\, J_\mu(\rho) : \rho \in W^{1,1}(I\!R^n),\ \|\,\rho\,\|_1 = 1\} = \inf\{\, I_\mu(f) : f \in W^{1,p}(I\!R^n), \|\,f\,\|_p = 1\}.$$

It follows that h satisfies the Euler-Lagrange equation corresponding to the constraint minimization problem, i.e., h is a solution of

$$\mu^{-p}\Delta_p f + pf|\,f\,|^{p-2}\ln(|\,f\,|) = \lambda f|\,f\,|^{p-2}, \tag{45}$$

where λ is a Lagrange multiplier. On the other hand, $\rho = h^p$ is a stationary solution of the evolution equation:

$$\frac{\partial u}{\partial t} = \Delta u + \mathrm{div}(p\mu^q|x|^{q-2}xu). \tag{46}$$

We can also apply Corollary 2.2 to recover the duality associated to the Gagliardo-Nirenberg inequalities obtained recently in [6].

Corollary 2.4 *Let $1 < p < n$, and $r \in \left(0, \frac{np}{n-p}\right]$ such that $r \neq p$. Set $\gamma := \frac{1}{r} + \frac{1}{q}$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then, for $f \in W^{1,p}(I\!R^n)$ such that $\|\,f\,\|_r = 1$, for any probability density ρ and for all $\mu > 0$, we have*

$$J_\mu(\rho) \leq I_\mu(f) \tag{47}$$

where

$$J_\mu(\rho) := -\frac{1}{\gamma - 1}\int_{R^n} \rho^\gamma - \frac{r\gamma\mu^q}{q}\int_{R^n} |\,y\,|^q \rho(y)(y)\,dy,$$

and

$$I_\mu(f) := -\left(\frac{1}{\gamma-1} + n\right)\int_{R^n} |\,f\,|^{r\gamma} + \frac{r\gamma}{p\mu^p}\int_{R^n} |\,\nabla f\,|^p.$$

Furthermore, if $h \in W^{1,p}(I\!R^n)$ is such that $h \geq 0$, $\|\,h\,\|_r = 1$, and

$$\nabla h(x) = -\mu^q x|\,x\,|^{q-2} h^{\frac{r}{p}}(x) \quad a.e.,$$

then

$$J_\mu(h^r) = I_\mu(h).$$

Therefore, h (resp., $\rho = h^r$) is an extremum of the variational problems

$$\sup\{\,J_\mu(\rho) : \rho \in W^{1,1}(I\!R^n),\ \|\,\rho\,\|_1 = 1\} = \inf\{\,I_\mu(f) : f \in W^{1,p}(I\!R^n), \|\,f\,\|_r = 1\}.$$

Proof: Again, the proof follows from Corollary 2.2, by using now $\psi(x) = |\,x\,|^r$ and $F(x) = \frac{x^\gamma}{\gamma - 1}$, where $1 \neq \gamma \geq 1 - \frac{1}{n}$, which follows from the fact that $p \neq r \in \left(0, \frac{np}{n-p}\right]$. Indeed, for this value of γ, the function F satisfies the conditions of Corollary 2.2. The Young function is now $c(x) = \frac{r\gamma}{q}|\,\mu x\,|^q$, that is, $c^*(x) = \frac{1}{p(r\gamma)^{p-1}}\left|\,\frac{x}{\mu}\,\right|^p$, and the condition $|\,\psi^{\frac{1}{p}}(F' \circ \psi)'\,| = K$ holds with $K = r\gamma$.
Moreover, if $h \geq 0$ satisfies (39), which is here,

$$-\nabla h(x) = \mu^q x|\,x\,|^{q-2} h^{\frac{r}{p}}(x) \ \text{ a.e.},$$

then h is extremal in the minimization problem defined in Corollary 2.4.

As above, we also note that h satisfies the Euler-Lagrange equation corresponding to the constraint minimization problem, that is, h is a solution of

$$\mu^{-p}\Delta_p f + \left(\frac{1}{\gamma - 1} + n\right) f|\, f\,|^{r\gamma-2} = \lambda f|\, f\,|^{r-2}, \tag{48}$$

where λ is a Lagrange multiplier. On the other hand, $\rho = h^r$ is a stationary solution of the evolution equation:

$$\frac{\partial u}{\partial t} = \Delta u^\gamma + \mathrm{div}(r\gamma\mu^q|x|^{q-2}xu). \tag{49}$$

We end this section with the following generalized degenerate Log Sobolev inequality. It will be useful for the study of convergence to equilibrium in section (4).

Proposition 2.1 *Let $\Omega \subset I\!\!R^n$ be open, bounded and convex, let $F : [0,\infty) \to I\!\!R$ be differentiable on $(0,\infty)$ such that $F(0) = 0$ and $x \mapsto x^n F(x^{-n})$ be convex and non-increasing, and let $P_F(x) := xF'(x) - F(x)$ be its associated pressure function. Assume c is an even and q-homogenous Young function for some $q > 1$. Then for all probability densities ρ_0 and ρ_1 on Ω such that $\mathrm{supp}\, \rho_0 \subset \Omega$, $P_F(\rho_0) \in W^{1,\infty}(\Omega)$ and $H_c(\rho_1) \neq 0$, we have*

$$H^F(\rho_0|\rho_1) \leq \frac{p}{(p-1)^{1/q}}[H_c(\rho_1)]^{1-\frac{1}{p}}[I_{c^*}(\rho_0)]^{\frac{1}{p}}. \tag{50}$$

where q is the conjugate of p, $I_{c^*}(\rho_0)$ is defined by (23), and $H^F(\rho_0/\rho_1) := H^F(\rho_0) - H^F(\rho_1)$.

Proof: Set $V = 0$, and then $\lambda = 0$ in (28) to obtain for all probability density functions ρ_0 and ρ_1 on Ω, satisfying $\mathrm{supp}\, \rho_0 \subset \Omega$, and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$,

$$-H_c^F(\rho_1) \leq -H^{F+nP_F}(\rho_0) + \int_\Omega \rho_0 c^\star\left(-\nabla(F' \circ \rho_0)\right) \mathrm{d}x. \tag{51}$$

For $\lambda > 0$, apply (51) to $c_\lambda(x) = c(\lambda x)$ to get:

$$H^F(\rho_0|\rho_1) \leq \int_\Omega \rho_0\, c_\lambda^*\left(-\nabla\left(F' \circ \rho_0\right)\right) \mathrm{d}x + H_{c_\lambda}(\rho_1) - H^{nP_F}(\rho_0)$$

We use that c is even, that

$$c_\lambda^*(x) \leq x.\nabla c_\lambda^*(x) \;\text{ for }\; x > 0,$$

that c is q-homogenous and that ∇c^* is $(p-1)$-homogenous to get

$$H^F(\rho_0|\rho_1) \leq \frac{1}{\lambda^p} I_{c^*}(\rho_0) + \lambda^q H_c(\rho_1) - H^{nP_F}(\rho_0).$$

Since the pressure is nonnegative, we get that

$$H^F(\rho_0|\rho_1) \leq \inf\left\{\frac{1}{\lambda^p} I_{c^*}(\rho_0) + \lambda^q H_c(\rho_1); \lambda > 0\right\}.$$

The infimum is attained at

$$\bar{\lambda} = \left(\frac{pI_{c^*}(\rho_0)}{qH_c(\rho_1)}\right)^{\frac{1}{p+q}},$$

which means that

$$H^F(\rho_0|\rho_1) \leq H_c(\rho_1)^{\frac{p}{p+q}} \left[\left(\frac{q}{p}\right)^{\frac{p}{p+q}} + \left(\frac{p}{q}\right)^{\frac{q}{p+q}}\right] I_{c^*}(\rho_0)^{\frac{q}{p+q}}.$$

In other words,

$$H^F(\rho_0|\rho_1) \leq \frac{p}{(p-1)^{1/q}}[H_c(\rho_1)]^{1-\frac{1}{p}}[I_{c^*}(\rho_0)]^{\frac{1}{p}}.$$

3 Optimal Geometric Inequalities

We now use Theorem 2.1 to establish various new and old inequalities. We start with the following general HWI-inequality.

Corollary 3.1 *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and convex, let $F : [0,\infty) \to \mathbb{R}$ be differentiable on $(0,\infty)$ such that $F(0) = 0$ and $x \mapsto x^n F(x^{-n})$ be convex and non-increasing, and let $P_F(x) := xF'(x) - F(x)$ be its associated pressure function. Let $U : \mathbb{R}^n \to \mathbb{R}$ be a C^2-function with $D^2U \geq \mu I$ where $\mu \in \mathbb{R}$. Then for any $\sigma > 0$, we have for all probability density functions ρ_0 and ρ_1 on Ω, satisfying supp $\rho_0 \subset \Omega$, and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$,*

$$H_U^F(\rho_0|\rho_1) + \frac{1}{2}(\mu - \frac{1}{\sigma})W_2^2(\rho_0,\rho_1) \leq \frac{\sigma}{2}\int_\Omega \rho_0 \Big|\, \nabla\left(F' \circ \rho_0 + U\right)\, \Big|^2 dx. \tag{52}$$

Proof: Use (28) with $c(x) = \frac{1}{2\sigma}|\, x\,|^2$ and $U = V + c$, to obtain

$$\begin{aligned} &H_U^F(\rho_0) - H_U^F(\rho_1) + \frac{1}{2}(\mu - \frac{1}{\sigma})W_2^2(\rho_0,\rho_1) \\ &\quad \leq -H^{nP_F}_{-c-\nabla(U-c)\cdot x}(\rho_0) + \int_\Omega \rho_0 c^*\left(-\nabla\left(F' \circ \rho_0 + U - c\right)\right) \mathrm{d}x. \end{aligned} \tag{53}$$

By elementary computations, we have

$$\begin{aligned} &\int_\Omega \rho_0 c^*\left(-\nabla\left(F' \circ \rho_0 + U - c\right)\right) \mathrm{d}x \\ &\quad = \frac{\sigma}{2}\int_\Omega \rho_0 \Big|\, \nabla\left(F' \circ \rho_0 + U\right) - \frac{x}{\sigma}\,\Big|^2 \mathrm{d}x \\ &\quad = \frac{\sigma}{2}\int_\Omega \rho_0 \Big|\, \nabla\left(F' \circ \rho_0 + U\right)\,\Big|^2 \mathrm{d}x + \frac{1}{2\sigma}\int_\Omega \rho_0 |\, x\,|^2 \mathrm{d}x - \int_\Omega \rho_0 x \cdot \nabla\left(F' \circ \rho_0\right) \mathrm{d}x \\ &\qquad - \int_\Omega \rho_0 x \cdot \nabla U \,\mathrm{d}x, \end{aligned}$$

and

$$
\begin{aligned}
-H^{nP_F}_{-c-\nabla(U-c)\cdot x}(\rho_0) &= -H^{nP_F}(\rho_0) + \int_\Omega \frac{|\,x\,|^2}{2\sigma}\rho_0\,\mathrm{d}x + \int_\Omega \rho_0\cdot x\nabla U\,\mathrm{d}x - \frac{1}{\sigma}\int_\Omega |\,x\,|^2\rho_0\,\mathrm{d}x \\
&= -H^{nP_F}(\rho_0) + \int_\Omega \rho_0 x\cdot\nabla U\,\mathrm{d}x - \frac{1}{2\sigma}\int_\Omega |\,x\,|^2\rho_0\,\mathrm{d}x.
\end{aligned}
$$

By combining the last 2 identities, we can rewrite the right hand side of (53) as

$$
\begin{aligned}
&-H^{nP_F}_{-c-\nabla(U-c)\cdot x}(\rho_0) + \int_\Omega \rho_0 c^*\left(-\nabla(F'\circ\rho_0+U-c)\right)\,\mathrm{d}x \\
&= \frac{\sigma}{2}\int_\Omega \rho_0|\,\nabla\left(F'\circ\rho_0+U\right)\,|^2\,\mathrm{d}x - \int_\Omega \rho_0 x\cdot\nabla\left(F'\circ\rho_0\right)\,\mathrm{d}x - \int_\Omega nP_F(\rho_0)\,\mathrm{d}x \\
&= \frac{\sigma}{2}\int_\Omega \rho_0|\,\nabla\left(F'\circ\rho_0+U\right)\,|^2,\mathrm{d}x + \int_\Omega \mathrm{div}\,(\rho_0 x)F'(\rho_0)\,\mathrm{d}x - \int_\Omega nP_F(\rho_0)\,\mathrm{d}x \\
&= \frac{\sigma}{2}\int_\Omega \rho_0\Big|\,\nabla\left(F'\circ\rho_0+U\right)\,\Big|^2\,\mathrm{d}x + n\int_\Omega \rho_0 F'(\rho_0)\,\mathrm{d}x + \int_\Omega x\cdot\nabla F(\rho_0)\,\mathrm{d}x \\
&\qquad - \int_\Omega nP_F(\rho_0)\,\mathrm{d}x \\
&= \frac{\sigma}{2}\int_\Omega \rho_0\Big|\,\nabla\left(F'\circ\rho_0+U\right)\,\Big|^2\mathrm{d}x + \int_\Omega x\cdot\nabla F(\rho_0)\,\mathrm{d}x + n\int_\Omega F\circ\rho_0\,\mathrm{d}x \\
&= \frac{\sigma}{2}\int_\Omega \rho_0\Big|\,\nabla\left(F'\circ\rho_0+U\right)\,\Big|^2\,\mathrm{d}x,
\end{aligned} \tag{54}
$$

where we use an integration by part to get the 2nd and 5th equalities. Inserting (54) into (53), we conclude the proof.

If U is uniformly convex (i.e., $\mu > 0$) inequality (52) - where we use $\sigma = \frac{1}{\mu}$ - yields the following *Generalized Log Sobolev inequality* inequality obtained by Cordero et al. in [7]: For all probability densities ρ_0 and ρ_1 on Ω, satisfying $\mathrm{supp}\,\rho_0 \subset \Omega$, and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$, we have

$$
H^F_U(\rho_0|\rho_1) \leq \frac{1}{2\mu}\int_\Omega |\nabla(F'\circ\rho_0+U)|^2\rho_0\,\mathrm{d}x, \tag{55}
$$

as well as the *Generalized Talagrand Inequality*: for any probability density ρ on Ω, we have

$$
W_2(\rho,\rho_U) \leq \sqrt{\frac{2}{\mu}H^F_U(\rho|\rho_U)}, \tag{56}
$$

where ρ_U is the probability density satisfying

$$
\nabla\left(F'(\rho_U)+U\right) = 0 \quad \text{a.e.} \tag{57}
$$

For that, it is sufficient to take $\rho_0 = \rho_U$ in (52), and then let σ go to ∞.

We now deduce the following HWI inequalities first established by Otto-Villani [12] in the case of the classical entropy $F(x) = x\ln x$.

Corollary 3.2 *(*Generalized HWI-inequality*) Under the above hypothesis on Ω and F, let $U : \mathbb{R}^n \to \mathbb{R}$ be a C^2-function with $D^2U \geq \mu I$ where $\mu \in \mathbb{R}$. Then we have for all probability densities ρ_0 and ρ_1 on Ω, satisfying $supp\,\rho_0 \subset \Omega$, and $P_F(\rho_0) \in W^{1,\infty}(\Omega)$,*

$$H_U^F(\rho_0|\rho_1) \leq W_2(\rho_0,\rho_1)\sqrt{I(\rho_0|\rho_U)} - \frac{\mu}{2}W_2(\rho_0,\rho_1)^2 \tag{58}$$

where

$$I(\rho_0|\rho_U) = \frac{\sigma}{2}\int_\Omega \rho_0 \Big| \nabla\left(F'\circ\rho_0 + U\right)\Big|^2 dx,$$

and

$$\nabla\left(F'(\rho_U) + U\right) = 0 \quad a.e. \tag{59}$$

Proof: It is sufficient to rewrite (52) as

$$H_U^F(\rho_0|\rho_1) + \frac{\mu}{2}W_2^2(\rho_0,\rho_1) \leq \frac{1}{2\sigma}W_2^2(\rho_0,\rho_1) + \frac{\sigma}{2}I(\rho_0|\rho_U), \tag{60}$$

then minimize the right hand side over the variable $\sigma > 0$. The minimum is obviously achieved at $\bar\sigma = \frac{W_2(\rho_0,\rho_1)}{\sqrt{I(\rho_0|\rho_U)}}$.

Corollary 3.1 applied to $F(x) = x\ln x$ yields the following inequality established by Otto-Villani [12]. For any function U on $\mathbb{R}^n$, denote by σ_U the integral $\int_{R^n} e^{-U}\,\mathrm{d}x$, and by ρ_U the normalized function $\frac{e^{-U}}{\sigma_U}$. If $D^2U \geq \mu I$ for $\mu \in \mathbb{R}$, then for any $\sigma > 0$, the following holds for any nonnegative function f such that $f\rho_U \in W^{1,\infty}(\mathbb{R}^n)$ and $\int_{R^n} f\rho_U\,\mathrm{d}x = 1$,

$$\int_{R^n} f\ln(f)\,\rho_U \mathrm{d}x + \frac{1}{2}(\mu - \frac{1}{\sigma})W_2^2(f\rho_U,\rho_U) \leq \frac{\sigma}{2}\int_{R^n}\frac{|\nabla f|^2}{f}\rho_U \mathrm{d}x, \tag{61}$$

and in particular, the original Gross Log Sobolev inequality: That is for any nonnegative function f such that $f\rho_U \in W^{1,\infty}(\mathbb{R}^n)$ and $\int_{R^n} f^2\rho_U\,\mathrm{d}x = 1$, we have

$$\int_{R^n} f^2\ln(f^2)\,\rho_U \mathrm{d}x \leq \frac{2}{\mu}\int_{R^n}|\nabla f|^2\rho_U \mathrm{d}x. \tag{62}$$

(61) is a straightforward consequence of (52) when choosing $F(x) = x\ln x$, $\rho_0 = f\rho_U$ and $\rho_1 = \rho_U$. (62) follows easily from (61) by setting $\sigma = \frac{1}{\mu}$, and then changing f to f^2.

Corollary 3.3 *(*General Euclidean Log-Sobolev inequality*)*
Let $\Omega \subset \mathbb{R}^n$ be open bounded and convex, and let $c : \mathbb{R}^n \to \mathbb{R}$ be a Young functional such that its conjugate $c^\star$ is p-homogeneous for some $p > 1$. Then,

$$\int_{R^n}\rho\ln\rho\,dx \leq \frac{n}{p}\ln\left(\frac{p}{ne^{p-1}\sigma_c^{p/n}}\int_{R^n}\rho c^\star\left(-\frac{\nabla\rho}{\rho}\right)\,dx\right), \tag{63}$$

for all probability density functions ρ on $\mathbb{R}^n$, such that $supp\,\rho \subset \Omega$ and $\rho \in W^{1,\infty}(\mathbb{R}^n)$. Here, $\sigma_c := \int_{R^n} e^{-c}\,dx$. Moreover, equality holds in (63) if $\rho(x) = K_\lambda e^{-\lambda^q c(x)}$ for some $\lambda > 0$, where $K_\lambda = \left(\int_{R^n} e^{-\lambda^q c(x)}\,dx\right)^{-1}$ and q is the conjugate of p ($\frac{1}{p} + \frac{1}{q} = 1$).

Proof: Use $F(x) = x\ln(x)$ in (21). Note that here $P_F(x) = x$ which means that $H^{P_F}(\rho) = 1$ for any $\rho \in \mathcal{P}_a(I\!\!R^n)$. So, $\rho_\infty(x) = \frac{e^{-c(x)}}{\sigma_c}$. We then have for $\rho \in \mathcal{P}_a(I\!\!R^n) \cap W^{1,\infty}(I\!\!R^n)$ such that $\operatorname{supp}\rho \subset \Omega$,

$$\int_\Omega \rho \ln \rho \,\mathrm{d}x \le \int_{R^n} \rho c^\star \left(-\frac{\nabla\rho}{\rho}\right) \mathrm{d}x - n - \ln\left(\int_{R^n} e^{-c(x)}\,\mathrm{d}x\right), \tag{64}$$

with equality when $\rho = \rho_\infty$.

Now assume that $c^\star$ is p-homogeneous and set $\Gamma_\rho^c = \int_{R^n} \rho c^\star\left(-\frac{\nabla\rho}{\rho}\right)\mathrm{d}x$. Using $c_\lambda(x) := c(\lambda x)$ in (64), we get for $\lambda > 0$ that

$$\int_{R^n} \rho \ln \rho \,\mathrm{d}x \le \int_{R^n} \rho c^\star \left(-\frac{\nabla\rho}{\lambda\rho}\right) \mathrm{d}x + n\ln\lambda - n - \ln\sigma_c, \tag{65}$$

for all $\rho \in \mathcal{P}_a(I\!\!R^n)$ satisfying $\operatorname{supp}\rho \subset \Omega$ and $\rho \in W^{1,\infty}(\Omega)$. Equality holds in (65) if $\rho_\lambda(x) = \left(\int_{R^n} e^{-\lambda^q c(x)}\,\mathrm{d}x\right)^{-1} e^{-\lambda^q c(x)}$. Hence

$$\int_{R^n} \rho\ln\rho\,\mathrm{d}x \le -n - \ln\sigma_c + \inf_{\lambda>0}\left(G_\rho(\lambda)\right),$$

where

$$G_\rho(\lambda) = n\ln(\lambda) + \frac{1}{\lambda^p}\int_{R^n} \rho c^\star\left(-\frac{\nabla\rho}{\rho}\right) = n\ln(\lambda) + \frac{\Gamma_\rho^c}{\lambda^p}.$$

The infimum of $G_\rho(\lambda)$ over $\lambda > 0$ is attained at $\bar\lambda_\rho = \left(\frac{p}{n}\Gamma_\rho^c\right)^{1/p}$. Hence

$$\begin{aligned}
\int_{R^n} \rho\ln\rho\,\mathrm{d}x &\le G_\rho(\bar\lambda_\rho) - n - \ln(\sigma_c) \\
&= \frac{n}{p}\ln\left(\frac{p}{n}\Gamma_\rho^c\right) + \frac{n}{p} - n - \ln(\sigma_c) \\
&= \frac{n}{p}\ln\left(\frac{p}{ne^{p-1}\sigma_c^{p/n}}\Gamma_\rho^c\right),
\end{aligned}$$

for all probability densities ρ on $I\!\!R^n$, such that $\operatorname{supp}\rho \subset \Omega$, and $\rho \in W^{1,\infty}(I\!\!R^n)$.

Corollary 3.4 *(Optimal Euclidean p-Log Sobolev inequality)*

$$\int_{R^n} |\, f\,|^p \ln(|\, f\,|^p)\, dx \le \frac{n}{p}\ln\left(C_p \int_{R^n} |\,\nabla f\,|^p\, dx\right), \tag{66}$$

holds for all $p \ge 1$, and for all $f \in W^{1,p}(I\!\!R^n)$ such that $\|\, f\,\|_p = 1$, where

$$C_p := \begin{cases} \left(\frac{p}{n}\right)\left(\frac{p-1}{e}\right)^{p-1} \pi^{-\frac{p}{2}} \left[\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n}{q}+1)}\right]^{\frac{p}{n}} & \text{if } \ p > 1, \\[2ex] \frac{1}{n\sqrt{\pi}}\left[\Gamma(\frac{n}{2}+1)\right]^{\frac{1}{n}} & \text{if } \ p = 1, \end{cases} \tag{67}$$

and q is the conjugate of p ($\frac{1}{p} + \frac{1}{q} = 1$).

For $p > 1$, equality holds in (66) for $f(x) = Ke^{-\lambda^q \frac{|\,x-\bar x\,|^q}{q}}$ for some $\lambda > 0$ and $\bar x \in I\!\!R^n$, where $K = \left(\int_{R^n} e^{-(p-1)|\,\lambda x\,|^q}\, dx\right)^{-1/p}$.

Proof: First assume that $p > 1$, and set $c(x) = (p-1)|\,x\,|^q$ and $\rho = |\,f\,|^p$ in (63), where $f \in C_c^\infty(I\!R^n)$ and $\|\,f\,\|_p = 1$. We have that $c^\star(x) = \frac{|\,x\,|^p}{p^p}$, and then, $\Gamma_\rho^c = \int_{R^n} |\,\nabla f\,|^p\,\mathrm{d}x$. Therefore, (63) reads as

$$\int_{R^n} |\,f\,|^p \ln(|\,f\,|^p)\,\mathrm{d}x \le \frac{n}{p} \ln\left(\frac{p}{ne^{p-1}\sigma_c^{p/n}} \int_{R^n} |\,\nabla f\,|^p\,\mathrm{d}x\right). \tag{68}$$

Now it suffices to note that

$$\sigma_c := \int_{R^n} e^{-(p-1)|\,x\,|^q}\,\mathrm{d}x = \frac{\pi^{\frac{n}{2}}\Gamma\left(\frac{n}{q}+1\right)}{(p-1)^{\frac{n}{q}}\Gamma\left(\frac{n}{2}+1\right)}. \tag{69}$$

To prove the case where $p = 1$, it is sufficient to apply the above to $p_\epsilon = 1+\epsilon$ for some arbitrary $\epsilon > 0$. Note that

$$C_{p\epsilon} = \left(\frac{1+\epsilon}{n}\right)\left(\frac{\epsilon}{e}\right)^{\epsilon} \pi^{-\frac{1+\epsilon}{2}} \left[\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n\epsilon}{1+\epsilon}+1)}\right]^{\frac{1+\epsilon}{n}},$$

so that when ϵ go to 0, we have

$$\lim_{\epsilon\to 0} C_{p_\epsilon} = \frac{1}{n\sqrt{\pi}}\left[\Gamma\left(\frac{n}{2}+1\right)\right]^{\frac{1}{n}} = C_1.$$

Corollary 3.5 *(*Gagliardo-Nirenberg*)*
Let $1 < p < n$ and $r \in \left(0, \frac{np}{n-p}\right)$ such that $r \neq p$. Set $\gamma := \frac{1}{r} + \frac{1}{q}$, where $\frac{1}{p}+\frac{1}{q} = 1$. Then, for any $f \in W^{1,p}(I\!R^n)$ we have

$$\|f\|_r \le C(p,r)\|\nabla f\|_p^\theta \,\|f\|_{r\gamma}^{1-\theta}, \tag{70}$$

where θ is given by

$$\frac{1}{r} = \frac{\theta}{p^*} + \frac{1-\theta}{r\gamma}, \tag{71}$$

$p^* = \frac{np}{n-p}$ and where the best constant $C(p,r) > 0$ can be obtained by scaling.

Proof: Let $F(x) = \frac{x^\gamma}{\gamma-1}$, where $1 \neq \gamma > 1 - \frac{1}{n}$, which follows from the fact that $p \neq r \in \left(0, \frac{np}{n-p}\right)$. Now, for this value of γ, the function F satisfies the conditions of Theorem 2.1. Let $c(x) = \frac{r\gamma}{q}|\,x\,|^q$ so that $c^*(x) = \frac{1}{p(r\gamma)^{p-1}}\Big|\,x\,\Big|^p$. Inequality (21) then gives for all $f \in C_c^\infty(I\!R^n)$ such that $\|\,f\,\|_r = 1$,

$$\left(\frac{1}{\gamma-1}+n\right)\int_{R^n} |\,f\,|^{r\gamma} \le \frac{r\gamma}{p}\int_{R^n} |\,\nabla f\,|^p - H^{P_F}(\rho_\infty) + C_\infty. \tag{72}$$

where $\rho_\infty = h_\infty^r$ satisfies

$$-\nabla h_\infty(x) = x|\,x\,|^{q-2} h^{\frac{r}{p}}(x) \;\text{ a.e.,} \tag{73}$$

and where C_∞ insures that $\int h_\infty^r = 1$. The constants on the right hand side of (72) are not easy to calculate, so one can obtain θ and the best constant by a standard scaling procedure. Namely, write (72) as

$$\frac{r\gamma}{p}\frac{\|\nabla f\|_p^p}{\|f\|_r^p} - \left(\frac{1}{\gamma-1}+n\right)\frac{\|f\|_{r\gamma}^{r\gamma}}{\|f\|_r^{r\gamma}} \geq H^{P_F}(\rho_\infty) - C_\infty =: C, \tag{74}$$

for some constant C. Then apply it to $f_\lambda(x) = f(\lambda x)$ for $\lambda > 0$. A minimization over λ gives the required constant.

The case where $\gamma = 1 - \frac{1}{n}$ - that is, $r = p^* := \frac{np}{n-p}$ - gives the standard Sobolev inequality: If $1 < p < n$, then for any $f \in W^{1,p}(\mathbb{R}^n)$,

$$\|f\|_{p^*} \leq C(p,n)\|\nabla f\|_p \tag{75}$$

for some constant $C(p,n) > 0$.
Note here that the scaling argument cannot be used to compute $C(p,n)$ since $\| \nabla f_\lambda \|_p^p = \lambda^{p-n}\| \nabla f \|_p^p$ and $\| f_\lambda \|_r^p = \lambda^{p-n}\| f \|_r^p$ scale the same way. Instead, one can proceed directly from (72) and (73) to have that

$$C(p,n) = \left(\frac{p^*(n-1)}{np\,[H^{P_F}(\rho_\infty) - C_\infty]}\right)^{1/p},$$

where $\rho_\infty = \left(\frac{p^*}{nq}| x |^q - \frac{C_\infty}{n-1}\right)^{-n}$. Then using that ρ_∞ is a probabily density, one finds easily that

$$C_\infty = (1-n)\left[\int_{R^n}\left(\frac{p^*}{nq}| x |^q + 1\right)^{-n} \mathrm{d}x\right]^{p/n}.$$

Similar results can be established in the presence of an additional convolution operator.

4 Trend to equilibrium

We now use our approach to study the asymptotic behaviour of solutions of degenerate but fairly general Fokker-Planck equations of the form stated in (25) when the confinement potential V is zero. Here Ω is an open bounded and convex subset of $\mathbb{R}^n$.
The long-time existence of the solution to (25) was established recently by Agueh in [1]. Also, when V is uniformly c-convex with constant $\lambda > 0$, Agueh [2] used the entropy dissipation method to show –modulo regularizing the solutions of (25)– an exponential decay in relative entropy H_V^F and in the c-Wasserstein cost functional W_c (26) for the convergence to the equilibrium state ρ_∞, $\nabla(F'(\rho_\infty) + V) = 0$. The rate of convergence is $p\lambda^{p-1}$ when $c(z) = \frac{|z|^q}{q}$ $(\frac{1}{p}+\frac{1}{q}=1)$ and is equal to 1 if c is not necessarily homogeneous, but $\lambda \geq 1$. His proof uses the generalized Log-Sobolev inequality established in [7] which only holds when V is uniformly c-convex with constant $\lambda > 0$. If $\Omega = \mathbb{R}^n$ and $V = 0$, his arguments can be extended to self-similar solutions of (76) at least when $c(z) = \frac{| z |^q}{q}$ for $q \geq 2$, and $F(x) = x\ln x$ or $\frac{x^\gamma}{\gamma-1}$ for $1 \neq \gamma \geq 1 - \frac{1}{n}$. Details and complete

proofs will be provided in a forthcoming paper. But in a bounded domain Ω, the proof in [2] does not hold when V is merely convex (i.e., when $\lambda = 0$ or even when $V = 0$) and therefore does not extend to equations of the form:

$$\begin{cases} \frac{\partial \rho}{\partial t} = \mathrm{div}\Big\{\rho \nabla c^* \left[\nabla \left(F'(\rho)\right)\right]\Big\} & \text{in} \quad (0,\infty)\times\Omega \\ \rho \nabla c^* \left[\nabla \left(F'(\rho)\right)\right]\cdot \nu = 0 & \text{on} \quad (0,\infty)\times\partial\Omega \\ \rho(t=0) = \rho_0 & \text{in} \quad \{0\}\times\Omega \end{cases} \tag{76}$$

when Ω is bounded. Here, we show again that inequality (28) - or precisely, estimate (50) - can be used as a substitute to cover this degenerate case. One way to see it is that our new energy estimate introduces to the game –via the Young functional c– a convenient non-trivial confinement potential.

Proposition 4.1 *Let $\Omega \subset I\!R^n$ be open, bounded and convex, $F : [0,\infty) \to I\!R$ be strictly convex, differentiable on $(0,\infty)$ such that $F(0) = 0$ and $x \mapsto x^n F(x^{-n})$ be convex and non-increasing. Let $c : I\!R^n \to I\!R$ be a q-homogeneous Young functional for some $q > 1$, and denotes by p the conjugate of q. Assume that the initial probability density $\rho_0 \in \mathcal{P}_a(\Omega)$ has finite energy $H^F(\rho_0)$. Then (modulo regularization), solutions $\rho = \rho(t,x)$ of (76) decay algebraically to the equilibrium solution $\rho_\infty = \frac{1}{|\Omega|}$ at a rate $t^{-\frac{1}{p-1}}$:*

$$H^F\left(\rho(t)|\rho_\infty\right) \leq \left(\left[H^F(\rho_0|\rho_\infty)\right]^{1-p} + \alpha(p-1)t\right)^{-\frac{1}{p-1}}, \tag{77}$$

where

$$\alpha = \frac{1}{p^p}\left(\frac{p-1}{H_c(\rho_\infty)}\right)^{p-1},$$

and $H^F(\rho(t)/\rho_\infty) = H^F(\rho(t)) - H^F(\rho_\infty)$.

`Proof:` Since F is strictly convex, the internal energy functional $H^F(\rho)$ (which is here the Lyapunov functional associated with (76) has a unique minimizer ρ_∞ which satisfies $\nabla\left(F'(\rho_\infty)\right) = 0$ in Ω, that is, $\rho_\infty = \frac{1}{|\Omega|}$ because Ω is bounded. Differentiating with respect to time $H^F\left(\rho(t)\right)$ - modulo regularizing solutions of (76) - along solutions ρ of (76), we have the following free energy dissipation equation:

$$\frac{d}{dt}H^F\left(\rho(t)|\rho_\infty\right) = -I_{c^*}\left(\rho(t)\right), \tag{78}$$

where $I_{c^*}(\rho)$ is defined by (23). Combining (78) with the generalized degenerate Log-Sobolev inequality (50), we have for any solution ρ of (76) that

$$\frac{d}{dt}H^F\left(\rho(t)|\rho_\infty\right) \leq -\left(\frac{(p-1)^{1/q}}{p\left(H_c(\rho_\infty)\right)^{1-\frac{1}{p}}}\right)^p \left[H_F\left(\rho(t)|\rho_\infty\right)\right]^p,$$

which reads as

$$\frac{d}{dt} H^F\left(\rho(t)|\rho_\infty\right) \leq -\frac{1}{p^p}\left(\frac{p-1}{H_c(\rho_\infty)}\right)^{p-1}\left[H_F\left(\rho(t)|\rho_\infty\right)\right]^p. \tag{79}$$

Integrating (79) over $[0,t]$, and using that $p>1$, we obtain that

$$\left[H^F\left(\rho(t)|\rho_\infty\right)\right]^{p-1} \leq \frac{1}{\left[H^F\left(\rho_0|\rho_\infty\right)\right]^{1-p}+\alpha(p-1)t},$$

where α is defined as in Proposition 4.1. This proves (77).

One can apply Proposition 4.1 to various equations that can be written in the form (76).

Examples:

- If $c(z)=\frac{|z|^2}{2}$ in which case (76) is the heat equation $\frac{\partial\rho}{\partial t}=\Delta\rho$ when $F(x)=x\ln x$, and the porous-media or the fast diffusion equation $\frac{\partial\rho}{\partial t}=\Delta\rho^\gamma,\ 1\neq\gamma\geq 1-\frac{1}{n}$ when $F(x)=\frac{x^\gamma}{\gamma-1}$, Proposition 4.1 gives an algebraic decay of solutions of these equations to the equilibrium solution $\rho_\infty=\frac{1}{|\Omega|}$ at the rate t^{-1}.

- If $c(z)=\frac{|z|^q}{q},\ q>1$, and p is the conjugate of q ($\frac{1}{p}+\frac{1}{q}=1$) in which case (76) reads as $\frac{\partial\rho}{\partial t}=\Delta_p\rho^{\frac{1}{p-1}}$ when $F(x)=x\ln x$, and $\frac{\partial\rho}{\partial t}=\Delta_p\rho^m,\ m\geq\frac{n-(p-1)}{n(p-1)}$ when $F(x)=\frac{nx^\gamma}{\gamma(\gamma-1)}$ and $1\neq\gamma:=m+\frac{p-2}{p-1}$, Proposition 4.1 shows an algebraic decay to the equilibrium solution $\rho_\infty=\frac{1}{|\Omega|}$ at the rate $t^{-\frac{1}{p-1}}$.

References

[1] M. Agueh. *Existence of solutions to degenerate parabolic equations via the Monge-Kantorovich theory*. Preprint, 2002.

[2] M. Agueh. *Existence of solutions to degenerate parabolic equations via the Monge-Kantorovich theory*. Ph.D Thesis, Georgia Tech, Summer 2002.

[3] W. Beckner. *Geometric asymptotics and the logarithmic Sobolev inequality*. Forum Math. **11** (1999), No. 1, 105-137.

[4] Y. Brenier. *Polar factorization and monotone rearrangement of vector-valued functions*. Comm. Pure Appl. Math. 44, 4 (1991), 375 - 417.

[5] L. Caffarelli. *Allocation maps with general cost function, in Partial Differential Equations and Applications (P. Marcellini, G. Talenti and E. Vesintin, eds)*. pp. 29 - 35. Lecture notes in Pure and Appl. Math., 177. Decker, New-York, 1996.

[6] D. Cordero-Erausquin, B. Nazaret, and C. Villani. *A mass-transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities*. Preprint 2002.

[7] D. Cordero-Erausquin, W. Gangbo, and C. Houdré. *Inequalities for generalized entropy and optimal transportation.* To appear in Proceedings of the Workshop: Mass transportation Methods in Kinetic Theory and Hydrodynamics.

[8] M. Del Pino, and J. Dolbeault. *The optimal euclidean L^p-Sobolev logarithmic inequality.* To appear in J. Funct. Anal. (2002).

[9] W. Gangbo and R. McCann. *The geometry of optimal transportation.* Acta Math. 177, 2, (1996), 113 - 161.

[10] R. McCann. *A convexity principle for interacting gases*. Adv. Math 128, 1, (1997), 153 - 179.

[11] F. Otto. Doubly degenerate diffusion equations as steepest descent. Preprint. Univ. Bonn, (1996).

[12] F. Otto and C. Villani. *Generalization of an inequality by Talagrand, and links with the logarithmic Sobolev inequality.* J. Funct. Anal. 173, 2 (2000), 361 - 400.

[13] C. Villani. *Topics in Optimal Transportation*, Garduate Studies in Math, 58, AMS (2003).