# The worst-case running time of the random simplex algorithm is exponential in the height

Andrei Z. Broder[1,2] Martin E. Dyer[2,3] Alan M. Frieze[2,4]
Prabhakar Raghavan[2,5] Eli Upfal[6]

May 23, 1995

**Abstract**

The random simplex algorithm for linear programming proceeds as follows: at each step, it moves from a vertex $v$ of the polytope to a randomly chosen neighbor of $v$, the random choice being made from those neighbors of $v$ that improve the objective function. We exhibit a polytope defined by $n$ constraints in three dimensions with height $O(\log n)$, for which the expected running time of the random simplex algorithm is $\Omega(n/\log n)$.

**Keywords:** Linear programming, simplex algorithm, randomized algorithm

We consider linear programming problems defined by $n$ constraints involving $d$ variables. The constraints define a polytope in $d$ dimensions; we seek a point of the polytope that maximizes an objective function that is linear in the variables. It is known that the optimum, if bounded, occurs at a vertex of this polytope. The classic *simplex* algorithm begins at a vertex of the polytope and, at each step proceeds to a neighboring vertex that improves the objective function. The simplex algorithm uses a deterministic rule to choose among the neighbors with improving objective function at each step. For virtually all such known rules, there are polytopes for which the simplex algorithm requires time exponential in $d$ [5].

The *random simplex* algorithm [1, 2] is the following: at each step, go to a neighbor chosen uniformly at random from all neighbors that improve the objective function. It is not known whether the expected number of steps for the random simplex algorithm is sub-exponential in $d$.

---

[1]Digital Systems Research Center, Palo Alto, CA 94301.

[2]This work was done while the author was visiting the Weizmann Institute of Science, Israel.

[3]School of Computer Studies, Leeds University, Leeds LS2 9JT. Supported in part by the ESPRIT working group RAND.

[4]Department of Mathematics, Carnegie Mellon University, Pittsburgh, PA 15213. Supported in part by NSF Grant CCR9224935.

[5]IBM T.J. Watson Research Center, Yorktown Heights, NY 10598.

[6]Department of Applied Mathematics, The Weizmann Institute of Science, Rehovot, Israel, and IBM Almaden Research Center, San Jose, CA 95120. Work at the Weizmann Institute supported by the N.D. Cohen Professorial Chair of Computer Science, a grant from the Israeli Academy of Science, and a grant from the MINERVA foundation.

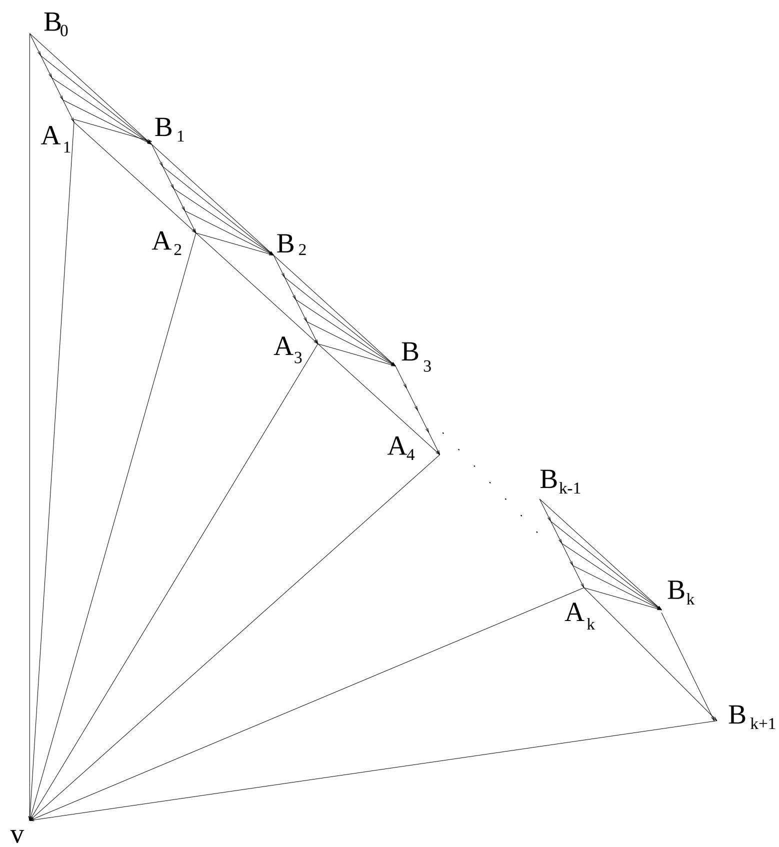


Figure 1: Graph of polytope

Without loss of generality, we restrict our attention to polytopes for which the optimum occurs at a unique vertex. We may associate a directed graph with the polytope, in which there is one node for each vertex of the polytope. There is an arc directed from each node to each neighbor that improves the objective function. The distance from a node in this graph to the optimal vertex is called the *height* of the node.

One route to establishing a polynomial bound might be to show that for every instance the expected number of steps is polynomial in the maximum height of any node. Currently the best-known upper bound on the height is $n^{\log_2 2d}$ [3], but a variant of the celebrated Hirsch conjecture states that in fact the height is $n-d$ (Todd [6] in fact shows that it cannot be exactly $n-d$, but the height could still be very close to $n-d$). If it were possible to show that the expected number of steps for the random simplex algorithm is polynomial in the height, one could then focus on improving the upper bound on height. We show here that this particular approach is doomed, by exhibiting a polytope in three dimensions with height $O(\log n)$ on which the expected number of vertices visited by the random simplex algorithm is $\Omega(n/\log n)$. However, it should be stressed that this does not preclude the possibility that the expected number of steps is polynomial in the number of variables and constraints in the instance. Our result only shows that an upper bound polynomial in the height alone will not suffice.

Consider the planar graph shown in Figure 1. This is the edge-vertex graph of a three-

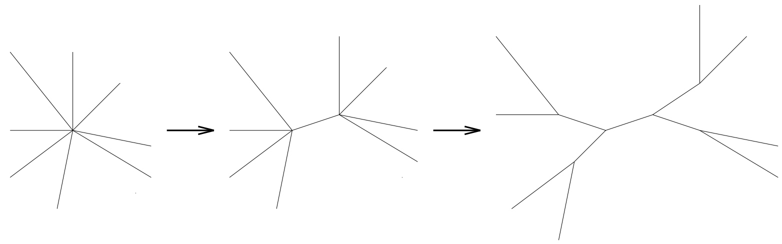

Figure 2: Pulling apart degenerate vertices

dimensional polytope. Imagine an arrow on each arc pointing downwards in the figure; the arrow denotes the direction in which the objective function improves. The reader may think of the direction of improving objective function as "vertically downwards". Optimality is attained at the vertex denoted $v$. For a constant $c > 1$, the polytope is composed of a cascade of $n/\log_2 n$ "traps", each resembling a parallelogram, one triangle of which contains a "fan". Each fan has $c \log_2 n$ vertices; thus the length of the path running down the right edge of the figure is $\Omega(n/\log_2 n)$.

It may not be immediately apparent that the graph of Figure 1 can be realized as a three-dimensional polytope with an objective function which induces the required ordering. There is a single plane "bottom" face containing $v$ and the bounding polygon in Figure 1. The remainder of the polytope is above this plane. To construct it we first build the polytope which results from deleting the fans in Figure 1. It is easy to do this inductively by adding pairs of points, starting with a tetrahedron $P_0$ having vertices $vA_1B_0B_1$ with $v$ far below $B_0B_1$ in the bottom plane and $A_1$ above the plane near $B_0$ and $B_1$. Then, to construct $P_{k+1}$ from $P_k$ for $k \geq 0$, we add two vertices. We put $B_{k+2}$ in the bottom plane near $B_{k+1}$ so that it forms a convex polygon with the other $B$'s, then we place $A_{k+1}$ near $B_{k+2}$ so that it forms new edges with $A_k$, $B_k$, $B_{k+2}$ and $v$, and the edge $A_kB_{k+1}$ disappears. It is clear that this can be done by placing the new vertices appropriately with respect to the existing faces. Now the fans are created by "slightly pulling" the edge $B_iA_{i+1}$ for $i = 1, 2, \ldots$ so that it forms a convex polygon in the plane $A_iB_iA_{i+1}$ whose vertices "see" only $B_{i+1}$. For $i = 0$ we do the same but with $A_0 = v$. This completes the construction.

The role of the fans is to repeatedly foil the algorithm from getting to the bottom-most vertex of any fan (where it would have a constant probability of proceeding to $v$). Instead, the fans repeatedly deflect the algorithm to the path running down the right edge of the figure.

The algorithm is started at the highest neighbor of $A_0$. With probability $(1 - n^{-c})$ it fails to reach the bottom of the first fan and proceeds to the next, and so on. There is a probability $(1 - n^{-c+1})$ that the algorithm never reaches the bottom-most vertex of any of the fans, and thus visits all the $\Omega(n/\log n)$ vertices of the path running down the right edge of the figure.

The reader may object that the polytope of Figure 1 is highly degenerate, particularly at the vertex $v$. However, it is easy to transform each high degree vertex into a balanced tree with all interior vertices of degree 3. The construction consists of repeatedly "pulling apart" the faces meeting at the vertex. It is illustrated in Figure 2. This modification will at most double the path lengths in the example.

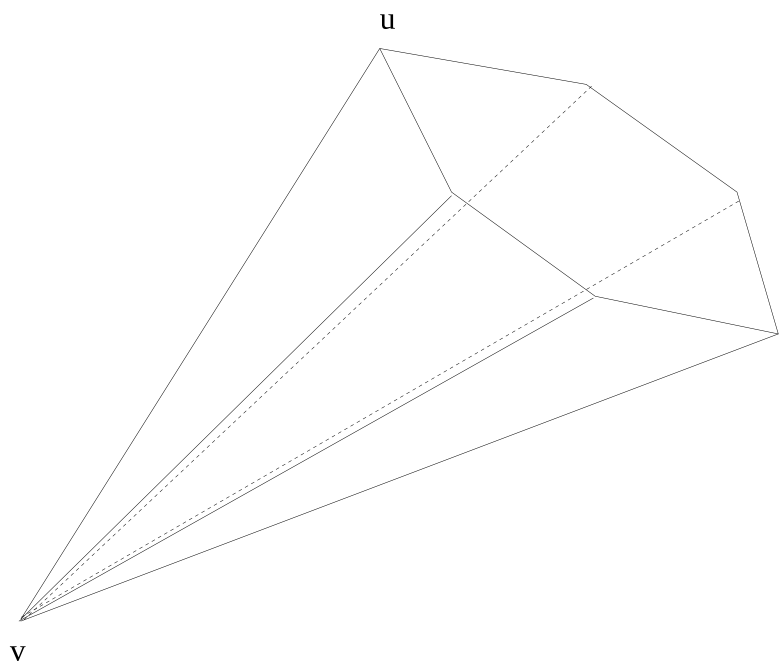


Figure 3: Example for Kalai's algorithm

An even simpler example shows that a similar dependence holds for Kalai's algorithm [4]. It is the tilted cone with an $n$-sided polygon as base as shown in Figure 3. The algorithm is started at $u$. It chooses a random facet incident on $u$ and recursively optimizes over this facet, and then repeats. There is a constant probability that the algorithm first chooses to traverse the polygon, and hence takes $\Omega(n)$ steps, when the height of any vertex is $O(\log n)$ (it is not a constant, but rather $O(\log n)$ because we break up the point of the cone as in Figure 2.)

# References

[1] G.B. Dantzig, Linear Programming and Extensions, *Princeton University Press*, Princeton 1963.

[2] B. Gärtner and G.M. Ziegler, Randomized simplex algorithms on Klee-Minty cubes, *Proceedings of the 35th Annual IEEE Symposium on Foundations of Computer Science*, (1994) 502-510.

[3] G. Kalai and D.J. Kleitman. A quasi-polynomial bound for the diameter of graphs of polyhedra. *Bulletin of the AMS*, 26:315–316, April 1992.

[4] G. Kalai, A subexponential randomized simplex algorithm, *Proceedings of the 24th Annual ACM Symposium on Theory of Computing*, (1992) 475-482.

[5] V. Klee and G.J. Minty, How good is the Simplex Method, in *Inequalities III*, (O. Sisha, ed.), Academic Press, New York, (1972) 159-175.

[6] M. J. Todd, The monotonic bounded Hirsch conjecture is false for dimension at least 4, *Mathematics of Operations Research*, 5 (1980) 599–601.