Solving a Random Asymmetric TSP Exactly in Quasi-Polynomial Time w.h.p.

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Abstract

Let the costs $C(i,j)$ for an instance of the Asymmetric Traveling Salesperson Problem (ATSP) be independent copies of an absolutely continuous random variable C that (i) satisfies $F(x) = \mathbb{P}(C \leq x) = x + O(x^2)$ as $x \to 0$ and (ii) has an exponential tail. We describe an algorithm that solves ATSP exactly in time $e^{\log^{3+o(1)} n}$, w.h.p.

1 Introduction

Let the costs $C(i,j)$ for an instance of the Asymmetric Traveling Salesperson Problem (ATSP) be independent copies of an absolutely continuous random variable C that satisfies

(i) $F(x) = \mathbb{P}(C \leq x) = x + O(x^2)$ as $x \to 0$ and (ii) $\mathbb{P}(C \geq x) \leq \alpha e^{-\beta x}$ for constants $\alpha, \beta > 0$.

In 1971 Bellmore and Malone [4] conjectured that using the assignment problem in a branch and bound algorithm would give a polynomial expected time algorithm. Lenstra and Rinnooy Kan [17] and Zhang [21] found errors in the argument of [4]. Since then, there has been little progress on this problem, up until now. The main result of this paper is

Theorem 1 *Let the costs for ATSP $C(i,j)$ be independent copies of C. There is an algorithm that solves ATSP exactly in $e^{\log^{3+o(1)} n)}$ time, w.h.p.*

1.1 Background

Given an $n \times n$ matrix $C = (C(i,j))$ we can define two discrete optimization problems. Let S_n denote the set of permutations of $[n] = \{1, 2, \ldots, n\}$. Let $T_n \subseteq S_n$ denote the set of *cyclic* permutations i.e. those permutations whose cycle structure consists of a single cycle. The *Assignment Problem* (AP) is the problem of minimising $C(\pi) = \sum_{i=1}^{n} C(i, \pi(i))$ over all permutations $\pi \in S_n$. We let $Z_{\text{AP}} = Z_{\text{AP}}^{(C)}$ denote the optimal cost for AP. The *Asymmetric Traveling-Salesperson Problem* (ATSP) is the problem of minimising $C(\pi) = \sum_{i=1}^{n} C(i, \pi(i))$ over all permutations $\pi \in T_n$. We let $Z_{\text{ATSP}} = Z_{\text{ATSP}}^{(C)}$ denote the optimal cost for ATSP.

Alternatively, the assignment problem is that of finding a minimum cost perfect matching in the complete bipartite graph $K_{A,B}$ where $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ and the cost of edge (a_i, b_j) is $C(i, j)$.

It is evident that $Z_{\text{AP}}^{(C)} \leq Z_{\text{ATSP}}^{(C)}$. The ATSP is NP-hard, whereas the AP is solvable in time $O(n^3)$. Several authors, e.g. Balas and Toth [3], Kalczynski [14], Miller and Pekny [20], Zhang [22] have investigated whether the AP can be used effectively in a branch-and-bound algorithm to solve the ATSP and have observed that the AP gives extremely good bounds on random instances. Experiments suggest that if the costs $C(i, j)$ are independently and uniformly generated as integers in the range $[0, L]$ then as L gets larger the problem gets harder to solve. Rigorous analysis supporting this thesis was given by Frieze, Karp and Reed [11]. They showed that if $L(n) = o(n)$ then $Z_{\text{ATSP}} = Z_{\text{AP}}$ w.h.p. and that w.h.p. $Z_{\text{ATSP}} > Z_{\text{AP}}$ if $L(n)/n \to \infty$. In some sense this shows why branch and bound is effective for small L.

We implicitly study a case where $L(n)/n \to \infty$. Historically, researchers have considered the case where the costs $C(i, j)$ are independent copies of the uniform $[0, 1]$ random variable $U[0, 1]$. This model was first considered by Karp [15]. He proved the surprising result that

$$Z_{\text{ATSP}} - Z_{\text{AP}} = o(1) \text{ w.h.p.} \tag{1}$$

Since w.h.p. $Z_{\text{AP}} > 1$ we see that this rigorously explained the observed quality of the assignment bound. Karp [15] proved (1) constructively, analysing an $O(n^3)$ *patching* heuristic that transformed an optimal AP solution into a good ATSP solution. Karp and Steele [16] simplified and sharpened this analysis, and Dyer and Frieze [8] improved the error bound of through the analysis of a related more elaborate algorithm to $O\left(\frac{\log^4 n}{n \log \log n}\right)$. Frieze and Sorkin [12] reduced the error bound to

$$Z_{\text{ATSP}} - Z_{\text{AP}} \leq \frac{\alpha_1 \log^2 n}{n} \text{ w.h.p.} \tag{2}$$

One might think that with such a small gap between Z_{AP} and Z_{ATSP}, that branch and bound might run in polynomial time w.h.p. Indeed one is encouraged by the recent results of Dey, Dubey and Molinaro [7] and Borst, Dadush, Huiberts and Tiwari [5] that with a similar integrality gap, branch and bound with LP based bounds solves random multi-dimensional knapsack problems in polynomial time w.h.p. Given Theorem 1, one is tempted to side with [4]and conjecture that branch and bound can be made to run in polynomial time w.h.p.

In the analysis below, $\omega = \omega(n)$ is an arbitrary function such that $\omega \to \infty, \omega = \log^{o(1)} n$.

2 Outline Proof of Theorem 1

Let $M^* = \left\{(a_i, b_{\phi(i)}), i \in [n]\right\}$ denote the optimum matching that solves AP. Any other perfect matching of $K_{A,B}$ can be obtained from M^* by choosing a set of vertex disjoint *alternating cycles* $C_1, C_2, \ldots, C_m$ and replacing M^* by $M^* \oplus C_1 \cdots \oplus C_m$. Here an alternating cycle is one whose edges alternate between being in M^* and not in M^*. We use the notation $S \oplus T = (S \setminus T) \cup (T \setminus S)$.

The basic idea of the proof is to show that if $|M \oplus M^*| \geq \log^{2+o(1)} n$ then w.h.p. $C(M) - C(M^*) > \frac{\alpha_1 \log^2 n}{n}$ where α_1 is from (2). Given this, it does not take too long to check all possible M, close in Hamming distance to M^*, to see if M defines a tour and then determine its total cost.

3 Analysis of the Assignment Problem

3.1 M^* only has low cost edges

In this section we prove that w.h.p.,

$$\max\left\{C(i, \phi(i))\right\} \leq \gamma^* = \frac{\gamma \log n}{n} \text{ for some absolute constant } \gamma > 0. \tag{3}$$

Define the k-neighborhood of a vertex to be the k vertices nearest it, where distance is given by the matrix C. Let the k-neighborhood of a set be the union of the k-neighborhoods of its vertices. In particular, for a complete bipartite graph $K_{A,B}$ and any $S \subseteq A, T \subseteq B$,

$$N_k(S) = \{b \in B : \exists s \in S \text{ s.b. } (s,b) \text{ is one of the } k \text{ least cost edges incident with } s\}, \tag{4}$$
$$N_k(T) = \{a \in A : \exists t \in T \text{ s.t. } (a,t) \text{ is one of the } k \text{ least cost edges incident with } t\}. \tag{5}$$

Given the complete bipartite graph $K_{A,B}$, any permutation $\pi : A \to B$ has an associated matching $M_\pi = \{(a,b) : a \in A, b \in B, a = \pi(b)\}$. Given a cost matrix C and permutation π, define the digraph

$$\vec{D} = \vec{D}_{C,\pi} = (A \cup B, \vec{E}) \tag{6}$$

consisting of *backwards* matching edges and forward "short" edges:

$$\vec{E} = \{(b,a) : b \in B, a \in A, b = \pi(a)\} \cup \{(a,b) : a \in A, b \in N_{40}(a)\} \\ \cup \{(a,b) : b \in B, a \in N_{40}(b)\}. \tag{7}$$

The edges of directed paths in $\vec{D}$ are alternately forwards $X \to Y$ and backwards $Y \to X$ and so they correspond to *alternating paths* with respect to the perfect matching defined by π. Since "adding" an alternating cycle to a matching produces a new matching, finding low-cost alternating paths is key to our constructions. In particular, an alternating path's backward edges (from the old matching) will be replaced by its forward ones, and so it helps to know (Lemma 2, next) that given $x \in X, y \in Y$ we can find an alternating path from x to y with $O(\log n)$ edges. The forward edges have expected length $O(1/n)$ and we will be able to show (Lemma 4, below) that we can w.h.p. be guaranteed to find an alternating path from x to y in which the *difference* in weight between forward and backward edges is $O(\log n/n)$. It is then simple to prove the upper bound in Lemma 3. A long edge can be removed by the use of such an alternating path.

Lemma 2 *W.h.p. over random cost matrices C, for every permutation π, the (unweighted) diameter of $\vec{D} = \vec{D}_{C,\pi}$ is at most $k_0 = \lceil 3 \log_4 n \rceil$.*

Proof. This is Lemma 5 of [12]. $\square$

If we ignore the savings from edge deletions in traversing an alternating path then it follows fairly easily that

$$\max \{C(i, \phi(i))\} \leq \frac{\gamma_1 \log^2 n}{n} \text{ for some absolute constant } \gamma_1 > 0. \tag{8}$$

For a fixed i we have

$$\mathbb{P}\left(C(i,j) \geq \frac{6 \log n}{n} \text{ for } j \in [n/2]\right) \leq \left(1 - \frac{6 \log n}{n} + O\left(\frac{\log^2 n}{n^2}\right)\right)^{n/2} = n^{-(3-o(1))}.$$

It follows that w.h.p. all of the forward edges in the paths alluded to in Lemma 2 have cost at most $\frac{6 \log n}{n}$. If $x \in A$ and $y \in B$ then Lemma 2 implies that w.h.p. there is a path from x to y for which the sum of the costs of the forward edges is at most $\frac{6k_0 \log n}{n}$. So if there is a matching edge of cost greater than $\frac{6k_0 \log n}{n}$ then there is an alternating path of using at most k_0 edges that can be used to give a matching of lower cost, contradiction. This verifies (8).

We now take account of the edges removed in an alternating path and thereby remove an extra $\log n$ factor. We will need the following inequality, analogous to Lemma 4.2(b) of [10], which deals with uniform $[0, 1]$ random variables.

Lemma 3 *Suppose that $k_1 + k_2 + \cdots + k_M = K \leq a \log N, a = O(1)$, and $Y_1, Y_2, \ldots, Y_M$ are independent random variables with Y_i distributed as the k_ith minimum of N independent copies of C. If $\lambda > 1, \lambda = O(1)$ and N is large, then*

$$\mathbb{P}\left(Y_1 + \cdots + Y_M \geq \frac{\lambda a \log N}{N}\right) \leq N^{a(\alpha + \log \lambda - \theta a^{-1} \lambda)},$$

where $\theta = \frac{1}{2} \min \{1, \beta, L^{-1}\}$ where L is the hidden constant in $F(x) = x + O(x^2)$ for $x \leq 1$.

Proof. The density function $f_k(x)$ of the kth order statistic $Y_{(k)}$ satisfies

$$f_k(x) = \binom{N}{k} (x + O(x^2))^{k-1} \left(1 - \frac{x}{2}\right)^{N-k} \qquad \text{for } x \leq \frac{1}{2L}.$$

$$f_k(x) \leq \alpha \binom{N}{k} (x + O(x^2))^{k-1} e^{-\beta(N-k)x} \qquad \text{for } x > \frac{1}{2L}.$$

Let $\widehat{\beta} = \min\{\beta, 1\}$. Hence the moment generating function of $Y_{(k)}$ is given by

$$
\begin{aligned}
\mathbf{E}(e^{tY_{(k)}}) &\leq \alpha^k \binom{N}{k} \int_{x\geq 0} e^{tx}(x+O(x^2))^{k-1} e^{-(N-k)\theta x} dx \\
&\leq \alpha^k \binom{N}{k} \int_{x\geq 0} x^{k-1}(1+O(x))^{k-1} e^{-((N-k)\theta - t)x} dx \\
&\leq \alpha^k \binom{N}{k} \int_{x\geq 0} x^{k-1} e^{-((N-k)\theta - t - Lk)x} dx \\
&\leq \frac{(\alpha N)^k}{k((N-k)\theta - t - L)^k}.
\end{aligned}
$$

So, if $Y = Y_1 + \cdots + Y_M$ then

$$
\begin{aligned}
\mathbf{E}(e^{tY}) \leq \prod_{i=1}^{M} \left(\frac{(\alpha N)^{k_i}}{((N-k_i)\theta - t - L)^{k_i}} \right) = \left(\frac{\alpha N}{\theta N - t} \right)^K \prod_{i=1}^{M} \left(1 + \frac{\theta k_i + L}{(N-k_i)\theta - t - c_i} \right) \\
\sim \left(\frac{\alpha N}{\theta N - t} \right)^K = \lambda^K,
\end{aligned}
$$

if we take $t = \theta N - \alpha \lambda^{-1}$.

So,

$$
\mathbb{P}\left(Y \geq \frac{\lambda a \log N}{N} \right) \leq \mathbb{P}\left(e^{tY} \geq \exp\left\{ \frac{t \lambda a \log N}{N} \right\} \right) \lesssim \frac{\lambda^K}{N^{\theta\lambda - \alpha a}}.
$$

□

Given this lemma we can verify (3).

Lemma 4 *Equation* (3) *holds w.h.p.*

Proof. Let

$$
Z_1 = \max\left\{ \sum_{i=0}^{k} C(x_i, y_i) - \sum_{i=0}^{k-1} C(y_i, x_{i+1}) \right\}, \tag{9}
$$

where the maximum is over sequences $x_0, y_0, x_1, \ldots, x_k, y_k$ where (x_i, y_i) is one of the 40 shortest edges leaving x_i for $i = 0, 1, \ldots, k \leq k_0 = \lceil 3 \log_4 n \rceil$, and (y_i, x_{i+1}) is a backwards matching edge. Also, in the maximum we assume that all $C(\cdot, \cdot)$ are bounded above by $L = \frac{\gamma_1 \log^2 n}{n}$, see (8). We compute an upper bound on the probability that Z_1 is large. For any constant $\zeta > 0$ we have

$$
\begin{aligned}
\mathbb{P}\left(Z_1 \geq \frac{\zeta \log n}{n} \right) \lesssim \sum_{k=0}^{k_0} n^{2k+2} \frac{1}{(n-1)^{k+1}} \times \\
\int_{y=0}^{L} \left[\frac{1}{(k-1)!} \left(\frac{y \log n}{n} \right)^{k-1} \sum_{\rho_0 + \rho_1 + \cdots + \rho_k \leq 40(k+1)} q(\rho_0, \rho_1, \ldots, \rho_k; \zeta + y) \right] dy
\end{aligned}
$$

where

$$q(\rho_0, \rho_1, \ldots, \rho_k; \eta) = \mathbb{P}\left(X_0 + X_1 + \cdots + X_k \geq \frac{\eta \log n}{n}\right),$$

$X_0, X_1, \ldots, X_k$ are independent and X_j is distributed as the ρ_jth minimum of $n-1$ copies of C. (When $k = 0$ there is no term $\frac{1}{k!}\left(\frac{y \log n}{n}\right)^k$).

Explanation: We have $\leq n^{2k+2}$ choices for the sequence $x_0, y_0, x_1, \ldots, x_k, y_k$. The term $\frac{1}{(k-1)!}\left(\frac{y \log n}{n}\right)^{k-1} dy$ asymptotically bounds the probability that the sum $\Sigma = C(y_0, x_1) + \cdots + C(y_{k-1}, x_k)$, is in $\frac{\log n}{n}[y, y + dy]$. Indeed, if $C_1, C_2, \ldots, C_k$ are independent copies of C then since $y \leq L$,

$$\begin{aligned}
\mathbb{P}\left(C_1 + \cdots + C_k \in \frac{\log n}{n}[y, y+dy]\right) &= \int_{z_1+\cdots+z_k \in \frac{\log n}{n}[y,y+dy]} \prod_{i=1}^{k} \left(1 + O\left(\frac{\log^2 n}{n}\right)\right) d\mathbf{z} \\
&\sim \int_{z_1+\cdots+z_k \in \frac{\log n}{n}[y,y+dy]} 1 d\mathbf{z} = \frac{1}{(k-1)!}\left(\frac{y \log n}{n}\right)^{k-1} dy.
\end{aligned}$$

We integrate over y. $\frac{1}{n-1}$ is the probability that (x_i, y_i) is the ρ_ith shortest edge leaving x_i, and these events are independent for $0 \leq i \leq k$. The final summation bounds the probability that the associated edge lengths sum to at least $\frac{(\zeta+y)\log n}{n}$.

It follows from Lemma 3 that if ζ is sufficiently large then, for all $y \geq 0$, $q(\rho_1, \ldots, \rho_k; \zeta + y) \leq n^{-(\zeta+y)/2}$ and since the number of choices for $\rho_0, \rho_1, \ldots, \rho_k$ is at most $\binom{41k+40}{k}$ (the number of non-negative integral solutions to $x_0 + x_1 + \ldots + x_{k+1} = 40(k+1)$) we have

$$\begin{aligned}
\mathbb{P}\left(Z_1 \geq \zeta \frac{\log n}{n}\right) &\leq 2n^{2-\zeta/2} \sum_{k=0}^{k_0} \frac{\log^{k-1} n}{(k-1)!} \binom{42k}{k} \int_{y=0}^{\infty} y^{k-1} n^{-y/2} dy \\
&\leq 2n^{2-\zeta/2} \sum_{k=0}^{k_0} \frac{\log^{k-1} n}{(k-1)!} \left(\frac{42e}{\log n}\right)^k \Gamma(k) \\
&\leq 2n^{2-\zeta/2}(42e)^{k_0+1} \\
&= o(n^{-2}).
\end{aligned}$$

If $a \in A$ and $b \in B$ then Lemma 2 implies that w.h.p. there is a path of length at most k_0 from a to b and by the above, it will w.h.p. have length at most $\frac{\zeta \log n}{n}$. So if there is a matching edge of cost greater than $\frac{\zeta \log n}{n}$ there is an alternating path of length at most k_0 that can be used to give a matching of lower cost, contradiction. □

3.2 A high probability bound on $Z_{\text{ATSP}} - Z_{\text{AP}}$

We now verify (2) with our more general distribution for costs. We let the $\widehat{C}(i, j)$ be independent copies of a uniform $[0, 1]$ random variable and then let $C(i, j) = F^{-1}(\widehat{C}(i, j)$. Then we have

$$
\begin{aligned}
C(ATSP) &\leq \left(1 + O\left(\frac{\log n}{n}\right)\right) \widehat{C}(ATSP) \\
&\leq \left(1 + O\left(\frac{\log n}{n}\right)\right) \left(\widehat{C}(AP) + O\left(\frac{\log^2 n}{n}\right)\right), \qquad \text{from (2)}, \\
&\leq \left(1 + O\left(\frac{\log^2 n}{n}\right)\right) \widehat{C}(AP) \\
&\leq C(AP) + O\left(\frac{\log^2 n}{n}\right).
\end{aligned}
$$

3.3 AP as a linear program

The assignment problem AP has a linear programming formulation $\mathcal{LP}$. In the following $z_{i,j}$ indicates whether or not (a_i, b_j) is an edge of the optimal solution.

$$
\begin{aligned}
\mathcal{LP} \qquad & \text{Minimise} \sum_{(i,j)\in[n]^2} C(i,j) z_{i,j} \\
& \text{subject to} \sum_{j=1}^{n} z_{i,j} = 1, \text{ for } i = 1, 2, \ldots, n. \\
& \sum_{i=1}^{n} z_{i,j} = 1, \text{ for } j = 1, 2, \ldots, n. \\
& 0 \leq z_{i,j} \leq 1, \text{ for } (i,j) \in [n]^2.
\end{aligned}
\tag{10}
$$

This has the dual linear program:

$$
\begin{aligned}
\mathcal{DLP} \qquad & \text{Maximise} \sum_{i=1}^{n} u_i + \sum_{j=1}^{n} v_j \\
& \text{subject to } u_i + v_j \leq C(i,j), \text{ for } (i,j) \in [n]^2.
\end{aligned}
\tag{11}
$$

Proposition 5 *Condition on an optimal basis for* (10)*. We may w.l.o.g. take* $u_1 = 0$ *in* (11)*, whereupon with probability 1 the other dual variables are uniquely determined. Furthermore, the reduced costs of the* non-basic *variables* $\bar{C}(i,j) = C(i,j) - u_i - v_j$ *are independently distributed as either (i)* $C - u_i - v_j$ *if* $u_i + v_j < 0$ *or (ii)* $C - u_i - v_j$ *conditional on* $C \geq u_i + v_j$*, if* $u_i + v_j \geq 0$*.*

Proof. The $2n - 1$ dual variables are unique with probability 1 because they satisfy $2n - 1$ full rank linear equations. The only conditions on the non-basic edge costs are that $C(i,j) \geq (u_i + v_j)^+$, where $x^+ = \max\{x, 0\}$. $\square$

3.4 Trees and bases

An optimal basis of $\mathcal{LP}$ can be represented by a spanning tree T^* of $K_{A,B}$ that contains the perfect matching M^*, see for example Ahuja, Magnanti and Orlin [1], Chapter 11. The edges of such a tree are referred to as *basic* edges, when the tree in question is T^*. We have that for every optimal basis T^*,

$$C(i,j) = u_i + v_j \text{ for } (a_i, b_j) \in E(T^*) \tag{12}$$

and

$$C(i,j) \geq u_i + v_j \text{ for } (a_i, b_j) \notin E(T^*). \tag{13}$$

Lemma 6

$$|u_i|, |v_i| \leq 2\gamma^* \textit{ for } i \in [n], \textit{ w.h.p.} \tag{14}$$

Proof. For each $i \in [n]$ there is some $j \in [n]$ such that $u_i + v_j = C(i,j)$. This is because of the fact that a_i meets at least one edge of T and we assume that (12) holds. We also know that if $\mathcal{B}$ occurs then $u_{i'} + v_j \leq C(i', j)$ for all $i' \neq i$. It follows that $u_i - u_{i'} \geq C(i,j) - C(i',j) \geq -\gamma^*$ for all $i' \neq i$. Since i is arbitrary, we deduce that $|u_i - u_{i'}| \leq \gamma^*$ for all $i, i' \in [n]$. This implies that $|u_i| \leq \gamma^*$ for $i \in r$. We deduce by a similar argument that $|v_j - v_{j'}| \leq \gamma^*$ for all $j, j' \in [n]$. Now because for the optimal matching edges $(i, \phi(i)), i \in [n]$ we have $u_i + v_{\phi(i)} = C(i, \phi(i))$, we see that $|v_j| \leq 2\gamma^*$ for $j \in [n]$. $\Box$

Condition on M^* and let G_+ denote the subgraph of $K_{A,B}$ induced by the edges (a_i, b_j) for which $u_i + v_j \geq 0$, where $\mathbf{u}, \mathbf{v}$ are optimal dual variables. Let $\mathcal{T}_+$ denote the set of spanning trees of G_+ that contain the edges of M^*.

Lemma 7 *If $T \in \mathcal{T}_+$ and* (14) *holds then*

$$\mathbb{P}(T^* = T \mid \mathbf{u}, \mathbf{v}) \sim \prod_{\substack{(a_i,b_j) \in G_+ \\ C(i,j) \leq 4\gamma^*}} (1 - u_i - v_j), \tag{15}$$

which is independent of T.

Proof. Fixing $\mathbf{u}$,$\mathbf{v}$ and T fixes the lengths of the edges in T. If $(a_i, b_j) \notin E(T)$ then $\mathbb{P}(C(i,j) \geq u_i + v_j) = 1$ if $u_i + v_j < 0$ and $1 - (u_i + v_j) + O((\gamma^*)^2)$ otherwise. Also, given (14), we have that $\mathbb{P}(C(i,j) \geq u_i + v_j) = 1$ if $C(i,j) \geq 2\gamma^*$. We then note that the Chernoff bounds imply that w.h.p. $|\{(i,j) : C(i,j) \leq 4\gamma^*\}| \leq 4n^2\gamma^* = O(n \log n)$. Thus,

$$\begin{aligned}\mathbb{P}(T^* = T \mid \mathbf{u}, \mathbf{v}) &= \prod_{\substack{(a_i,b_j) \notin E(T) \\ C(i,j) \leq 4\gamma^*}} (1 - (u_i + v_j)^+ + O((\gamma^*)^2)) \prod_{(a_i,b_j) \in E(T)} (1 - u_i - v_j + O((\gamma^*)^2)) \\ &\sim \prod_{\substack{(a_i,b_j) \in G_+ \\ C(i,j) \leq 4\gamma^*}} (1 - u_i - v_j). \end{aligned} \tag{16}$$

$\square$

Thus

$$T^* \text{ is an asymptotically uniform random member of } \mathcal{T}_+. \tag{17}$$

Now let Γ_+ be the multi-graph obtained from G_+ by contracting the edges of M^* and let $\widehat{T}^*$ be the corresponding contraction of T^*.

Lemma 8 *The distribution of the tree $\widehat{T}^*$ is asymptotically equal to that of a random spanning tree of $K_n + \widehat{M}$ where $\widehat{M}$ is a matching of size at most $\lambda^* = \lambda \log^4 n$ for some constant $\lambda > 0$. ($\widehat{M}$ yields double edges, other edges occur once.)*

Proof. We have that for all $i, j \in [n]$,

$$(u_i + v_{\phi(j)}) + (u_j + v_{\phi(i)}) = (u_i + v_{\phi(i)}) + (u_j + v_{\phi(j)}) = C(i, \phi(i)) + C(j, \phi(j)) > 0.$$

So, either $u_i + v_{\phi(j)} > 0$ or $u_j + v_{\phi(i)} > 0$ which implies that Γ_+ contains the edge $\{a_i, a_j\}$. So, Γ_+ contains K_A as a subgraph.

We know from (12) and Lemma 14 that $\widehat{T}^*$ only contains edges of cost at most $2\gamma^*$. So from (17), $\widehat{T}^*$ is a random spanning tree of a graph distributed as G_{n,γ^*} plus a set of edges $\widehat{M}$. The edges $\widehat{M}$ arise from 4-cycles (C_4) where each edge has cost at most γ^*. The expected number of such cycles is $O((n\gamma^*)^4)$ and so by standard results on the number of copies of balanced graphs, we see that $|\widehat{M}| = O(\log^4 n)$ w.h.p. At this density, any copies of C_4 will be vertex disjoint w.h.p., as can easily be verified by a first moment calculation.

A random spanning tree of $G_{n,p} + \widehat{M}$, where $\widehat{M}$ is a random matching, is by symmetry, a random spanning tree of $K_n + \widehat{M}$. $\square$

We need to know that w.h.p., for each a_i, there are many b_j for which $u_i + v_j \geq 0$. We fix a tree T and condition on $T^* = T$. For $i = 1, 2, \ldots, r$ let $L_{i,+} = \{j : u_i + v_j \geq 0\}$ and let $L_{j,-} = \{i : u_i + v_j \geq 0\}$. Then let $\mathcal{A}_{i,+}$ be the event that $|L_{i,+}| \leq \eta n$ and let $\mathcal{A}_{j,-}$ be the event that $|L_{j,-}| \leq \eta n$ where η will be some small positive constant.

Lemma 9 *Fix a spanning tree T of G_r.*

$$\mathbb{P}\left(\bigcup_{i=1}^{n} (\mathcal{A}_{i,+} \cup \mathcal{A}_{j,-}) \mid T^* = T\right) = o(1). \tag{18}$$

Proof. In the following analysis T is fixed. Throughout the proof we assume that the costs $C(i, j)$ for $(a_i, b_j) \in T$ are distributed as independent copies of C, conditional on $C(i, j) \leq \gamma^*$. Equation (3) is the justification for this in that we can solve the assignment problem, only using edges of cost at most γ^*. Furthermore, in G_r, the number of edges of cost at most γ^* incident with a fixed vertex is dominated by $Bin(n, \gamma^*)$ and so w.h.p. the maximum degree of the trees we consider can be bounded by $2\gamma \log n$.

We fix s and put $u_s = 0$. The remaining values $u_i, i \neq s, v_j$ are then determined by the costs of the edges of the tree T. Let $\mathcal{B}$ be the event that $C(i,j) > u_i + v_j$ for all $(a_i, b_j) \notin E(T)$. Note that if $\mathcal{B}$ occurs then $T^* = T$.

Let $\mathcal{E}$ be the event that $|u_i|, |v_j| \leq 2\gamma^*$ for all i, j. It follows from the argument in the previous paragraph that $\mathcal{B} \subseteq \mathcal{E}$.

We now condition on the set E_T of edges (and the associated costs) of $\{(a_i, b_j) \notin E(T)\}$ such that $C(i,j) \geq 2\gamma^*$. Let $F_T = \{(a_i, b_j) \notin E(T)\} \setminus E_T$. Note that $|F_T|$ is dominated by $Bin(n^2, 2\gamma^* + O((\gamma^*)^2))$ and so $|F_T| \leq 3n^2\gamma^*$ with probability $1 - o(n^{-2})$.

Let $Y = \{C(i,j) : (a_i, b_j) \in E(T)\}$ and let $\delta_1(Y)$ be the indicator for $\mathcal{A}_{s,+} \wedge \mathcal{E}$. We write,

$$\mathbb{P}(\mathcal{A}_{s,+} \mid \mathcal{B}) = \mathbb{P}(\mathcal{A}_{s,+} \wedge \mathcal{E} \mid \mathcal{B}) = \frac{\int \delta_1(Y)\mathbb{P}(\mathcal{B} \mid Y)dC}{\int \mathbb{P}(\mathcal{B} \mid Y)dC} \tag{19}$$

Then we note that since $(a_i, b_j) \notin F_T \cup E(T)$ satisfies the condition (13),

$$\begin{aligned}
\mathbb{P}(\mathcal{B} \mid Y) &= \prod_{(a_i,b_j)\in F_T} (1 - (u_i(Y) + v_j(Y))^+ + O((\gamma^*)^2)) \\
&\lesssim \prod_{(a_i,b_j)\in F_T} (1 - (u_i(Y) + v_j(Y))^+) \\
&\leq e^{-W},
\end{aligned} \tag{20}$$

where $W = W(Y) = \sum_{(a_i,b_j)\in F_T} (u_i(Y) + v_j(Y))^+ \leq 12n^2(\gamma^*)^2 = 12\gamma^2 \log^2 n$. Then we have

$$\begin{aligned}
\int_Y \delta_1(Y)\mathbb{P}(\mathcal{B} \mid Y)\, dC &= \int_Y e^{-W}\delta_1(Y)\, dC \\
&\leq \left(\int_Y e^{-2W}\, dC\right)^{1/2} \times \left(\int_Y \delta_1(Y)^2\, dC\right)^{1/2} \\
&= e^{-\mathbf{E}(W)} \left(\int_Y e^{-2(W-\mathbf{E}(W))} dC\right)^{1/2} \times \mathbb{P}(\mathcal{A}_{s,+} \mid \mathcal{E})^{1/2} \\
&\leq e^{-\mathbf{E}(W)} e^{12\gamma^2 \log^2 n} \mathbb{P}(\mathcal{A}_{s,+} \mid \mathcal{E})^{1/2}.
\end{aligned} \tag{21}$$

$$\int \mathbb{P}(\mathcal{B} \mid Y)dC = \mathbf{E}(e^{-W}) \geq e^{-\mathbf{E}(W)}. \tag{22}$$

Let b_j be a neighbor of a_s in G_r and let $P_j = (i_1 = s, j_1, i_2, j_2, \ldots, i_k, j_k = j)$ define the path from a_s to b_j in T.

It then follows from (19),(21) and (22) that

$$\mathbb{P}(\mathcal{A}_{s,+} \mid \mathcal{B}) \leq e^{12\gamma^2 \log^2 n} \mathbb{P}(\mathcal{A}_{s,+} \mid \mathcal{E})^{1/2}. \tag{23}$$

Note that if $\mathcal{B}$ occurs and (12) holds then $T^* = T$. Let b_j be a neighbor of a_s in G_r and let $P_j = (i_1 = s, j_1, i_2, j_2, \ldots, i_k, j_k = j)$ define the path from a_s to b_j in T. Then it follows from (12) that $v_{j_l} = v_{j_{l-1}} - C(i_l, j_{l-1}) + C(i_l, j_l))$. Thus v_j is the final value S_k of a random walk

$S_t = X_0 + X_1 + \cdots + X_t, t = 0, 1, \ldots, k$, where $X_0 \geq 0$ and each $X_t, t \geq 1$ is the difference between two independent copies of C that are conditionally bounded above by γ^*. Given $\mathcal{E}$ we can assume that the partial sums S_i satisfy $|S_i| \leq 2\gamma^*$ for $i = 1, 2, \ldots, k-1$. Assume for the moment that $k \geq 4$ and let $x = u_{i_{k-3}} \in [-2\gamma^*, 2\gamma^*]$. Given x we see that there is some positive probability $p_0 = p_0(x)$ that $S_k > 0$. Indeed,

$$p_0 = \mathbb{P}(S_k > 0 \mid \mathcal{E}) = \mathbb{P}(x + Z_1 - Z_2 > 0) - \mathbb{P}(\mathcal{E}), \tag{24}$$

where $Z_1 = Z_{1,1} + Z_{1,2} + Z_{1,3}$ and $Z_2 = Z_{2,1} + Z_{2,2}$ are the sums of independent copies of C, each conditioned on being bounded above by γ^* and such that $|x + \sum_{j=1}^{t}(Z_{1,j} - Z_{2,j})| \leq 2\gamma^*$ for $t = 1, 2$ and that $|x + Z_1 - Z_2| \leq 2\gamma^*$. The absolute constant $\eta_1 = p_0(-2\gamma^*) > 0$ is such that $\min\{x \geq -2\gamma^* : p_0(x)\} \geq \eta_0$.

We now partition (most of) the neighbors of a_s into N_0, N_1, N_2 where
$N_t = \{b_j : k \geq 3, k \mod 3 = t\}$, k being the number of edges in the path P_j from a_s to b_j. Now because T has maximum degree $2\gamma \log n$, as observed at the beginning of the proof of this lemma, we know that there exists t such that $|N_t| \geq (n - (2\gamma \log n)^3)/3 \geq n/4$. It then follows from (24) that $|L_{s,+}|$ dominates $Bin(n, \eta_0)$ and then $\mathbb{P}(|L_{s,+}| \leq \eta_0 n/10) = O(e^{-\Omega(n)})$ follows from the Chernoff bounds. Similarly for $L_{1,-}$. Applying the union bound over n choices for s and applying (23) gives the lemma with $\eta = \eta_0/10$. □

3.5 Alternating paths

We now consider the the number of edges in alternating paths that consist only of basic edges. We call these *basic alternating paths.*

Lemma 10 *The expected number of basic alternating paths with k edges is at most $n^2\left(1 - \frac{\eta}{1+\eta}\right)^k$, where η is as in Lemma 9.*

Proof. Let $P = (b_{\phi(i_1)}, a_{i_1}, b_{\phi(i_2)}, a_{i_2}, \ldots, b_{\phi(i_k)}, a_{i_k})$ be a prospective basic alternating path. Then $Q = (a_{i_1}, a_{i_2}, \ldots, a_{i_k})$ must be a path in $\widehat{T}^*$. When we uncontract M^*, the edge $\left\{a_{i_t}, a_{i_{t+1}}\right\}$ arises either (i) from $(b_{\phi(i_t)}, a_{i_{t+1}}, b_{\phi(i_{t+1})})$ or (ii) from $(a_{i_t}, b_{\phi(i_{t+1})}, a_{i_{t+1}})$ and we get an alternating path only if we have the former case for $t = 1, 2, \ldots, k$.

Consider the random walk struction of a spanning tree as described in Aldous [2] and Broder [6]. We have to modify the walk so that the tree contains M^*. We do this by giving the edges of M^* a large weight $W \gg n$. This will mean that when the walk arrives at some a_i it is very likely to move to $b_{\phi(i)}$ and then back to a_i and so on. It will however eventually leave the edge $(a_i, b_{\phi(i)})$ and either leave from a_i or from $b_{\phi(i)}$. We can model this via a sequence of independent experiments where the probability of success is at most $n/(W+n)$ at odd steps at least $\eta n/(W + \eta n)$ at even steps. Here odd steps correspond to being at a_i and being in case (i) and even steps correspond to being at $b_{\phi(i)}$ and being in case (ii). Lemma 9 implies that when the walk adds an edge to the tree there is a probability of at least η that the edge arises from case (ii) above. The probability of an even success is therefore at least

$$\sum_{k\geq 1}\left(1-\frac{n}{W+n}\right)^k\left(1-\frac{\eta n}{W+n}\right)^{k-1}\cdot\frac{\eta n}{W+n}=$$
$$\left(1-\frac{n}{W+n}\right)\cdot\frac{\eta n}{W+n}\cdot\frac{1}{1-\left(1-\frac{n}{W+n}\right)\left(1-\frac{\eta n}{W+n}\right)}\sim\frac{\eta}{1+\eta}.$$

This will be independent of the addition of previous edges and so the probability we find a basic alternating path with k edges can be bounded by $n^2\left(1-\frac{\eta}{1+\eta}\right)^k$ and the lemma follows. $\square$

So,

Corollary 11 *W.h.p. the maximum length of a basic alternating path is at most* $3\eta^{-1}\log n$.

Let Z_1 denote the number of basic alternating paths. We would like to use the following result of Meir and Moon [19]: if T is a uniform random spanning tree of the complete graph K_n and $d_T(i,j)$ is the distance between $i\neq j\in[n]$ in T, then

$$\mathbb{P}(d_T(i,j)=k)=\frac{k}{n-1}\cdot\frac{n(n-1)\cdots(n-k+1)}{n^k},\qquad \text{for } 1\leq k\leq n-1.$$

The problem is that if a tree of K_A contains ℓ edges of X (see Lemma 8) then its probability of occuring in Γ_+ is inflated by 2^ℓ. On the other hand, the probability that a random tree in K_A contains ℓ given edges is at most $(2/n)^\ell$. ($2/n$ for $\ell=1$ and at most $(2/n)^\ell$ in general using negative correlation, see [18].) So, assuming $|\widehat{M}|\leq\lambda\log^4 n$ (see Lemma 8), and accounting for asymptotic uniformity, the expected number of basic alternating paths can be bounded by $1+o(1)$ times

$$n^2\sum_{k=1}^{n}\frac{k}{n-1}\cdot\left(1-\frac{\eta}{1+\eta}\right)^k\cdot\sum_{\ell\geq 0}\binom{\lambda\log^4 n}{\ell}\left(\frac{4}{n}\right)^\ell\leq\frac{2n}{\eta}.$$

Combining this with Corollary 11 we obtain

Lemma 12 *W.h.p there are at most* $m=\omega n$ *basic alternating paths, each using* $O(\log n)$ *edges.*

So, w.h.p. the matching M corresponding to the ATSP solution is derived from a collection of short basic alternating paths $P_1,P_2,\ldots,P_m$ joined by non-basic edges to create alternating cycles $C_1,C_2,\ldots,C_\ell$. Now consider an alternating cycle $C=(a_{i_1},b_{j_1},\ldots,b_{j_t},a_{i_1})$ made up from such paths by adding non-basic edges joining up the endpoints. Putting $\tilde{C}(i,j)=$

$C(i,j) - u_i - v_j$, we have that where $j_t = \phi(i_t)$ for $t = 1, 2, \ldots, k$,

$$\begin{aligned}
C(C \oplus M^*) - C(M^*) &= \sum_{k=1}^{t} (C(i_{k+1}, j_k) - C(i_k, j_k)) \\
&= \sum_{k=1}^{t} ((\tilde{C}(i_{k+1}, j_k)) + u_{i_{k+1}} + v_{j_k}) - (\tilde{C}(i_k, j_k) + u_{i_k} + v_{j_k}) \\
&= \sum_{k=1}^{t} \tilde{C}(i_{k+1}, j_k) \\
&= \sum_{\substack{k=1 \\ (i_{k+1}, j_k) \text{ non-basic}}}^{t} \tilde{C}(i_{k+1}, j_k).
\end{aligned}$$

We now estimate the expected number of sets of such cycles Z_2 that increase the cost by at most $\zeta = \frac{\alpha_1 \log^2 n}{n}$, the upper bound on the gap between AP and ATSP found in [12]. Thus, where $\tilde{C}_1, \tilde{C}_2, \ldots, \tilde{C}_k$ are independent random variables distributed as (i), (ii) of Proposition 5,

$$\mathbf{E}(Z_2) \leq \sum_{k \geq 2} \sum_{\ell=1}^{k/2} \sum_{k_1 + \cdots + k_\ell = k} \prod_{i=1}^{\ell} \binom{m}{k_i} k_i! 2^{k_i} \mathbb{P}\left(\tilde{C}_1 + \tilde{C}_2 + \cdots + \tilde{C}_k \leq \zeta\right) \tag{25}$$

$$\lesssim \sum_{k \geq 2} (2m)^k \frac{(2\zeta)^k}{k!} \sum_{\ell=1}^{k/2} \binom{k}{\ell} \tag{26}$$

$$\leq \sum_{k \geq 2} \left(\frac{8\alpha_1 e \omega \log^2 n}{k} \right)^k. \tag{27}$$

Explanation: We choose $\ell, k_1 + \cdots + k_\ell = k$ and then for each i choose k_i paths and order them and orient them in at most $\binom{m}{k_1} k_i! 2^{k_i}$ ways. To go from (25) to (26) we use

$$\begin{aligned}
\mathbb{P}\left(\tilde{C}_1 + \cdots + \tilde{C}_k \leq \zeta\right) &\leq \int_{z_1 + \cdots + z_k \leq \zeta} \prod_{i=1}^{k} \left(\frac{1 + O(z_i)}{1 - 3\gamma^*} \right) d\mathbf{z} \\
&\lesssim 2^k \int_{z_1 + \cdots + z_k \leq \zeta} 1 d\mathbf{z} = \frac{(2\zeta)^k}{k!}.
\end{aligned}$$

4 Finishing the proof of Theorem 1

It follows from (27) that w.h.p. we can restrict our attention to sets of cycles containing at most $L = \omega^2 \log^2 n$ non-basic edges in our search for an optimal solution to ATSP. Note that the basic edges of such cycles are determined by T^*. We can examine all such sets of cycles and solve ATSP in $O(n^{2L} 2^L L!)$ time. This matches the claimed running time in Theorem 1.

Although the above scheme finds the optimal tour w.h.p., it does not give a proof of optimality. A small adjustment will solve this problem. Let Z_3 denote the number of sets of disjoint paths and cycles that use L non-basic edges and with $\tilde{C}$ cost at most η. Arguing as in (25), we find that $\mathbf{E}(Z_3) = o(1)$. Now if there is a set of cycles with more than L non-basic edges and with a $\tilde{C}$ value less than η, then $Z_3 > 0$. So, by checking collections of paths and cycles that use L non-basic edges as well as the collections of cycles we also get a proof of optimality. This completes the proof of Theorem 1.

5 Summary and open questions

One can easily put the enumerative algorithm in the framework of branch and bound. At each node of the B&B tree one branches by excluding edges of M^*. So, at the top of the tree the branching factor is n and in general, at level k, it is $n-k$. W.h.p. the tree will have depth at most L.

The result of Theorem 1 does not resolve the question as to whether or not there is a branch and bound algorithm that solves ATSP w.h.p. in polynomial time. This remains an open question.

Less is known probabilistically about the symmetric TSP. Frieze [9] proved that if the costs $C(i,j) = C(j,i)$ are independent uniform $[0,1]$ then the asymptotic cost of the TSP and the cost 2F of the related 2-factor relaxation are asymptotically the same. The probabilistic bounds on $|TSP - 2F|$ are inferior to those given in [12]. Still, it is conceivable that the 2-factor relaxation or the subtour elimination constraints are sufficient for branch and bound to run in polynomial time w.h.p.

References

[1] R.K. Ahuja, T.L. Magnanti and J.B. Orlin, *Network Flows Theory, Algorithms and Applications*, Prentice-Hall, 1993.

[2] D. Aldous, The Random Walk Construction of Uniform Spanning Trees and Uniform Labelled Trees, *SIAM Journal on Discrete Mathematics* 3 (1990) 450-465.

[3] E. Balas and P. Toth, *Branch and Bound Methods*, in The Traveling Salesman Problem: A Guided Tour of Combinatorial Optimization (E.L. Lawler, J.K. Lenstra, A.H. Rinnooy Kan and D.B. Shmoys (editors)), Wiley, 1986, 361–400.

[4] M. Bellmore and J. Malone, Pathology of Traveling-Salesman subtour elimination algorithms, *Operations Research* 19 (1971) 278-307.

[5] S. Borst, D. Dadush, S. Huiberts and S. Tiwari, On the Integrality Gap of Binary Integer Programs with Gaussian Data, *Mathematical Programming* 197 (2023) 1221-1263.

[6] A. Broder, Generating random spanning trees, *Proceedings of the 30th Annual Symposium on Foundations of Computer Science* (1989) 442-447.

[7] S. Dey, Y. Dubey and M. Molinaro, Branch-and-Bound Solves Random Binary Packing IPs in Polytime

[8] M.E. Dyer and A.M. Frieze, *On patching algorithms for random asymmetric traveling salesman problems*, Mathematical Programming 46 (1990) 361–378.

[9] A.M. Frieze, On random symmetric travelling salesman problems, *Mathematics of Operations Research* 29 (2004) 878-890.

[10] A.M. Frieze and G. Grimmett, The shortest path problem for graphs with random arc-lengths, *Discrete Applied Mathematics* 10 (1985) 57-77.

[11] A.M. Frieze, R.M. Karp and B. Reed, *When is the assignment bound asymptotically tight for the asymmetric traveling-salesman problem?*, SIAM Journal on Computing 24 (1995) 484–493.

[12] A.M. Frieze and G. Sorkin, The probabilistic relationship between the assignment and asymmetric traveling salesman problems, *SIAM Journal on Computing* 36 (2007) 1435-1452.

[13] M. Held and R.M. Karp, *A dynamic programming approach to sequencing problems*, SIAM Journal of Applied Mathematics 10 (1962) 196–210.

[14] P. Kalczynski, *A Java implementation of the Branch and Bound Algorithm: The Asymmetric Traveling Salesman Problem*, Journal of Object Technology 4 (2005).

[15] R.M. Karp, *A patching algorithm for the non-symmetric traveling salesman problem*, SIAM Journal on Computing 8 (1979) 561–573.

[16] R.M. Karp and J.M. Steele, *Probabilistic analysis of heuristics*, in The traveling salesman problem: a guided tour of combinatorial optimization, E.L. Lawler, J.K. Lenstra, A.H.G. Rinnooy Kan and D.B. Shmoys Eds. (1985) 181–206.

[17] J. Lenstra and A. Rinnooy Kan, On the expected performance of branch and bound algorithms, *Operations Research* 26 (1979) 347-349.

[18] R. Lyons and Y. Peres, Probability on trees and networks, Cambridge University Press, 2017.

[19] A. Meir and J. Moon, The distance between points in random trees, *Journal of Combinatorial Theory* 8 (1970) 99-103.

[20] D. Miller and J. Pekny, Exact solution of large asymmetric traveling salesman problems, *Science* 251 (1991) 754-761.

[21] W. Zhang, A note on the complexity of the Asymmetric Traveling Salesman Problem, *Operations Research Letters* 20 (1997) 31-38.

[22] W. Zhang, Phase Transitions and Backbones of the Asymmetric Traveling Salesman problem, Journal of Artificial Intelligence 21 (2004) 471-497.