Broadcasting in Random Graphs.

Alan Frieze[*] Michael Molloy[†]

February 23, 2002

Abstract

We do a probabilistic analysis of the problem of distributing a single piece of information to the vertices of a graph G. Assuming that the input graph G is $G_{n,p}$, we prove an $O(\ln n/n)$ upper bound on the edge density needed so that with high probability the information can be broadcast in $\lceil \log_2 n \rceil$ rounds.

1 Introduction

Let $G = (V, E)$ be a graph, and for $v \in V$, let $N(v)$ denote the set of v's neighbours in G. We will study the problem of distributing a piece of information $\imath$, residing initially at one given vertex v_0, to the rest of the vertices. At each time step, any vertex knowing $\imath$ can share it with *one* of its neighbours.

[*]Department of Mathematics, Carnegie-Mellon University. Supported in part by NSF grants CCR9024935 and CCR9225008.

[†]Department of Mathematics, Carnegie-Mellon University. Supported in part by NSF grant CCR9225008.

Let $V_t, t = 0, 1, 2, \ldots$ denote the set of vertices which have $\imath$ at the beginning of step t. Thus $V_0 = \{v_0\}$.

Clearly $|V_{t+1}| \leq 2|V_t|$, and so if $|V| = n$ then it takes at least $\nu = \lceil \log_2 n \rceil$ rounds before every vertex has $\imath$. For the purposes of this paper let a graph have property $\mathcal{B}$ if is is possible to distribute a piece of information in ν rounds, from every possible starting vertex.

We will study the probability that the random graph $G_{n,p}$ has property $\mathcal{B}$. Observe first that if $c < 1$ is constant and $p \leq c \ln n/n$ then **whp**[1] $G_{n,p}$ has isolated vertices and so does not have $\mathcal{B}$. In terms of an upper threshold for p, Scheinermann and Wierman [4] and Dolan [1] showed that if $p \geq c(\ln n)^2/n$ for some constant $c > 0$ then $G_{n,p}$ has $\mathcal{B}$ **whp** Recently Gerbessiotis [3] reduced the upper bound to $c(\ln \ln n) \ln n/n$.

In this paper we give a simple proof of

Theorem 1 *There exists a constant $c > 0$ such that if $p \geq c \ln n/n$ then $G_{n,p}$ has $\mathcal{B}$* **whp**.

Proof In the proof we assume $p = 18 \ln n/n$. We define a *broadcast tree* T rooted at a vertex $v \in [n]$. The tree defines an increasing sequence of sets $\{v\} = W_0 \subset W_1 \subset \ldots \subset W_\nu = [n]$. Here $|W_t| = 2^t$ for $0 \leq t < \nu$. The edges of T consist of matchings $M_0, M_1, \ldots, M_{\nu-1}$, where M_t is a perfect matching between W_t and $W_{t+1} \setminus W_t$ for $0 \leq t < \nu - 1$, and $M_{\nu-1}$ is a matching of $W_\nu \setminus W_{\nu-1}$ into $W_{\nu-1}$.

[1]An event $\mathcal{E}_n$ is said to occur **whp** (with high probability) if $\mathbf{Pr}(\mathcal{E}_n) = 1 - o(1)$ as $n \longrightarrow \infty$.

Given a broadcast tree rooted at v one can clearly distribute the information by sending it along M_t in round t.

We prove the theorem by proving

$$\Pr(\exists \text{ broadcast tree rooted at vertex } 1) = 1 - o(n^{-1})$$

We decompose $G_{n,p}$ as the union of independent copies of G_{n,p_1}, G_{n,p_2}, G_{n,p_3}, where $p_2 = p_3 = (4.5 \ln n)/n$ and $1 - p = (1 - p_1)(1 - p_2)(1 - p_3)$. Note that this yields $p_1 \geq (9 \ln n)/n$.

We (try to) construct our tree in three phases, where in Phase i, we use the edges of $G_{n,p_i}, i = 1, 2, 3$.

Phase 1

Here we use a simple greedy approach to construct $W_1, W_2, \ldots, W_{\nu-2}$.

In the following algorithm when a vertex $v \in W_t$ needs to find a vertex w to be matched to in M_t it searches for the *next* vertex in order that (i) is not in W_t, and (ii) is in $N(v)$. The pointer s_v keeps track of where we are in v's list.

GREEDY SEARCH

begin

$s_v := 0$ for all $v \in [n]$;

$W_0 := \{1\}$;

for $t = 0$ **to** $\nu - 3$ **do**

begin

$W_{t+1} := W_t$;

for $v \in W_t$ **do**

begin

A: $s_v = s_v + 1;$

if $s_v > n$ **then** FAIL;

B: **if** $s_v \in W_{t+1}$ **then goto** A;

C: **if** $(v, S_v) \notin G_{n,p_1}$ **then goto** A;

$W_{t+1} := W_t \cup \{s_v\}$

end

end

end

Phase 2

Find a matching $M_{\nu-2}$ of $W_{\nu-2}$ into $[n] - W_{\nu-2}$ using the edges of G_{n,p_2}. $W_{\nu-1}$ is equal to the set of vertices covered by $M_{\nu-2}$.

Phase 3

Find a matching $M_{\nu-1}$ of $[n] - W_{\nu-2}$ into $W_{\nu-1}$ using the edges of G_{n,p_3}.

Probability of Failure

If Phase 1 fails then s_v reaches $n+1$ for some $v \in [n]$. Now $|W_{\nu-2}| < n/2$ and so for this v, Statement B has caused a jump to A less than $n/2$ times. So we must have executed Statement C at least $n/2$ times and there have been at most $\nu - 3$ cases where an edge of G_{n,p_1} was found. Now when C is executed, the edge (v, s_v) has not been previously examined, and so occurs with probability p_1 independently of the history of the process so far. Thus if $B(\cdot,\cdot)$ denotes a binomial random variable then

$$
\begin{aligned}
\mathbf{Pr}(\text{Phase 1 fails}) &\leq n\mathbf{Pr}(B(n/2, p_1) \leq \nu - 3) \\
&= \mathrm{o}(n^{-1})
\end{aligned}
$$

on using the Chernoff bound $\Pr(\mathrm{B}(m, q) \leq (1-\epsilon)mq) \leq \mathrm{e}^{-\epsilon^2 mq/2}$.

The failure probabilities for Phases 2 and 3 can be estimated as in Erdős and Rényi [2]. For both Phases we must match $\leq n/2$ vertices into $\geq n/2$ vertices. Thus our failure probability is dominated by that for no perfect matching in a random bipartite graph with $n/2 + n/2$ vertices and edge probability p_2. This is $\mathrm{o}(n^{-1})$ as required, completing the proof of our theorem. □

Of course we do not believe that 18 is the correct constant. One can easily reduce it by being a little more careful with estimates. It does seem however that our method will not give us the least constant and we leave it at 18 for readability.

References

[1] P.Dolan. *Spanning Trees in Random Graphs*, Random Graphs Volume 2., A.M.Frieze and T.Łuczak (Editors), John Wiley and Sons (1992), 47-58.

[2] P.Erdős and A.Rényi, *On random matrices*, Publ. Math. Inst. Hungar. Acad. Sci. **8** (1964) 455-461.

[3] A.Gerbessiotis. *Broadcasting in Random Graphs.* Discrete Mathematics, to appear.

[4] E.Scheinerman and J.Wierman. *Optimal and Near-Optimal Broadcasting in Random Graphs.* Discrete Applied Math **25** (1989), 289-297.