Rainbow powers of a Hamilton cycle in $G_{n,p}$

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Abstract

We show that the threshold for having a rainbow copy of a power of a Hamilton cycle in a randomly edge colored copy of $G_{n,p}$ is within a constant factor of the uncolored threshold. Our proof requires $(1+\varepsilon)$ times the minimum number of colors.

Key words: Rainbow colorings. random graphs.

1 Introduction

There has recently been great progress in our understanding of thresholds for monotone properties in the random graph $G_{n,p}$. Inspired by the work of Alweiss, Lovett, Wu and Zhang [1] on the Sunflower Conjecture, Frankston, Kahn, Narayanan and Park [4] showed that under fairly general conditions, the threshold for the existence of combinatorial objects is within a factor $O(\log n)$ of the point where the expected number of such objects begins to take off. Great though these results are, this is not the end of the story. In a paper remarkable for the strength of its result and for the simplicity of its proof, Park and Pham[10] proved the so-called Kahn-Kalai conjecture [8] which implies the result of [4].

Kahn, Narayanan and Park [7] tightened their analysis for the case of the square of a Hamilton cycle, removing the $O(\log n)$ factor and solving the existence problem up to a constant factor; a remarkable achievement, given the complexity of the proofs of earlier weaker results. Their result was generalized by Espuny Díaz and Person [3] and Spiro [12], both of whom defined more generalized conditions under which the $O(\log n)$ factor can be removed. Espuny Díaz and Person asked whether a rainbow generalization of their result could be proven [3]. Our main theorem here proves a rainbow version in a setting that is more general than the Kahn–Narayanan–Park result but less general than Espuny Díaz–Person and Spiro results. It is likely that our result could be extended to the full generality of the Espuny Díaz–Person and Spiro results with some additional effort.

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Some notation Given a set X and $0 \leq p \leq 1$, we let X_p denote a subset of X where each $x \in X$ is placed independently into X_p with probability p. Similarly, X_m is a random m-subset of X for $1 \leq m \leq |X|$.

Let $\mathcal{H} = \{A_1, A_2, \ldots, A_M\}$ be a hypergraph on vertex set X. A key notion in this analysis is that of *spread*. For a set $S \subseteq X$ we let $\langle S \rangle = \{T : S \subseteq T \subseteq X\}$ denote the subsets of X that contain S. We say that $\mathcal{H}$ is κ-spread if

$$|\mathcal{H} \cap \langle S \rangle| \leq \frac{|\mathcal{H}|}{\kappa^{|S|}}, \quad \forall S \subseteq X. \tag{1}$$

$\mathcal{H}$ is called r-bounded if $|A| \leq r$ for all $A \in \mathcal{H}$ and r-uniform if $|A| = r$ for all $A \in \mathcal{H}$. The following theorem was proved in [4]:

Theorem 1. *Let $\mathcal{H}$ be an r-bounded, κ-spread hypergraph and let $X = V(\mathcal{H})$. There is an absolute constant $K > 0$ such that if*

$$p \geq \frac{K \log r}{\kappa} \text{ or respectively } m \geq \frac{(K \log r)|X|}{\kappa} \tag{2}$$

then w.h.p. X_p or X_m respectively contains an edge of $\mathcal{H}$. More precisely, $\mathbb{P}(X_p$ contains an edge of $\mathcal{H}$ $) \geq 1 - \varepsilon_r$ where $\varepsilon_r \to 0$ as $r \to \infty$.

To apply this theorem to, say, Hamilton cycles, we let $X = \binom{[n]}{2}$ and we let $A_i, i = 1, 2, \ldots, \frac{1}{2}(n-1)!$ be the edge sets of the Hamilton cycles of K_n.

In the special case of $\mathcal{H}$ corresponding in this way to the squares of Hamilton cycles, [7] removed the $\log r$-factor from the bounds in (2).

We now turn to the main topic of this note. We suppose that each $x \in X$ is uniformly and independently given a random color from a set Q. Given a set $A \subseteq X$ we refer to A^* as the set after its elements have been colored. We say that A^* is *rainbow* colored if each $a \in A$ has a different color. Bell, Frieze and Marbach [2] attempted to extend the results of [4] to rainbow colorings. They proved

Theorem 2. *Let $\mathcal{H}$ be an r-bounded, κ-spread hypergraph and let $X = V(\mathcal{H})$ be randomly colored from $Q = [q]$ where $q \geq r$. Suppose also that $\kappa = \Omega(r)$, that is, there exists a constant $C_0 > 0$ such that $\kappa \geq C_0 r$ for all valid r. Then given $\varepsilon > 0$ there is a constant C_ε such that if r is sufficiently large and*

$$m \geq \frac{(C_\varepsilon \log_2 r)|X|}{\kappa} \tag{3}$$

then X_m contains a rainbow colored edge of $\mathcal{H}$ with probability at least $1 - \varepsilon$.

The constraint $\kappa = \Omega(r)$ rules out the square of a Hamilton cycle as there we have $r = 2n$ and $\kappa = O(n^{1/2})$. The aim of this note is to tackle this case while also removing the extra $\log r$-factor. Unfortunately, we have to increase the number of colors slightly, by a factor $(1 + \varepsilon_1)$ for arbitrary positive ε_1. Chapter 15 of [6] extracts a property used in [7] to make the following extra assumption about the hypergraph $\mathcal{H}$. For $A \in \mathcal{H}$ we let

$$f_{t,A} = |\{B \in \mathcal{H} : |B \cap A| = t\}|.$$

The assumption now is that there exist constants $0 < \alpha < 1, K_0$ independent of r such that

$$f_{t,A} \leq \left(\frac{K_0}{\kappa}\right)^t |\mathcal{H}| \text{ for all } A \in \mathcal{H} \text{ and } 1 \leq t \leq \alpha r. \tag{4}$$

As $\mathcal{H}$ remains κ-spread, it follows from (1) that

$$f_{t,A} \leq \frac{2^r}{\kappa^t}|\mathcal{H}| \text{ for all } A \in \mathcal{H} \text{ and } t > \alpha r \tag{5}$$

Let a hypergraph $\mathcal{H}$ be *edge transitive* if for every pair of edges A_i, A_j there exists a permutation $\pi : X \to X$ such that $\pi(A_i) = A_j$ and such that $\pi(A) \in \mathcal{H}$ for all $A \in \mathcal{H}$. The induced map $\pi : \mathcal{H} \to \mathcal{H}$ is a bijection. (When $\mathcal{H}$ is defined by the edges of K_n, all we usually require is a permutation of the vertices.)

We will prove the following:

Theorem 3. *Let $\varepsilon, \varepsilon_1 > 0$ be arbitrary positive constants. Suppose that $\mathcal{H}$ is a κ-spread, r-uniform and edge transitive hypergraph on which (4) holds. Let $X = V(\mathcal{H})$ be randomly colored from $Q = [q]$ where $q \geq (1+\varepsilon_1)r$. Then there exists $C = C(\varepsilon, \varepsilon_1)$ such that for sufficiently large r, κ,*

$$m \geq \frac{C|X|}{\kappa} \text{ implies that } \mathbb{P}(X_m^* \text{ contains a rainbow colored edge of } \mathcal{H}) \geq 1 - \varepsilon. \tag{6}$$

We will show in Section 3 that hypergraphs corresponding to powers of Hamilton cycles fit the premise of Theorem 3. ([7] verified (4) for squares of Hamilton cycles and for completeness, we verify (4) for all powers.) We prove Theorem 3 in the next section. We note that our proof is in some part inspired by a proof by Huy Pham [11] of the main result of [7].

2 Proof of Theorem 3

The proof will proceed in three stages. First, we will color all elements of X independently and uniformly at random from $[q]$, and will remove all sets in $\mathcal{H}$ that are not rainbow. We show that the number of remaining sets is with high probability close to its expectation.

Then, let $N = |X|$ and $m = \frac{CN}{\kappa}$ for sufficiently large $C = C(\varepsilon, \varepsilon_1)$. Let W_0 be chosen randomly from $\binom{X}{m}$. Let $p_1 = \frac{m}{N}$ and let W_1 be obtained from $X \setminus W_0$ by including each element with probability p_1. Proving Theorem 3 on $W_0 \cup W_1$ suffices to prove it for $X_{O(m)}$ by standard concentration bounds. The second stage (succeeding with high probability) will deal with W_0 while the third stage (succeeding with probability $1 - \varepsilon$) will deal with W_1.

We will use the notation $A \lesssim B$ to indicate that $A \leq (1 + o(1))B$ as $r \to \infty$. We will also assume that $q = (1 + \varepsilon_1)r$. This assumption comes without loss of generality because $C(\varepsilon, \varepsilon_1)$ will be strictly decreasing in ε_1, so if $q > (1 + \varepsilon_1)r$, we could set ε_2 such that $q = (1 + \varepsilon_2)r$ and use ε_2 in the proof instead.

2.1 The size of $\mathcal{H}^*$

Let $\mathcal{H} = \{A_1, A_2, \ldots, A_M\}$ and let $\mathcal{H}^*$ denote the rainbow edges of $\mathcal{H}$ after a uniform and independent random coloring. Similarly, let X^* denote X after it has been randomly colored. Let $(a)_b = a(a-1)\cdots(a-b+1)$ for positive integers a, b.

We use the Chebyshev inequality to prove concentration of $Z = |\mathcal{H}^*|$ around its mean. We have

$$\mathbb{E}(Z) = \frac{|\mathcal{H}|(q)_r}{q^r} \to \infty.$$

as $r \to \infty$, because spread (with $S \in \mathcal{H}$ in (1)) implies that $|\mathcal{H}| \geq \kappa^r$ and we have assumed that κ is sufficiently large.

Using the edge transitivity of $\mathcal{H}$ to obtain (7),

$$\mathbb{E}(Z^2) = \sum_{t=0}^{r} \sum_{A_i,A_j:|A_i\cap A_j|=t} \frac{(q)_t((q-t)_{r-t})^2}{q^{2r-t}}$$

$$\leq \mathbb{E}(Z)\left(1+\mathbb{E}(Z)+\sum_{t=1}^{r-1}\sum_{A_i:|A_i\cap A_1|=t}\frac{(q-t)_{r-t}}{q^{r-t}}\right) \tag{7}$$

$$\leq \mathbb{E}(Z)\left(1+\mathbb{E}(Z)+|\mathcal{H}|\left(\sum_{t=1}^{\alpha r}\left(\frac{K_0}{\kappa}\right)^t\frac{(q-t)_{r-t}}{q^{r-t}}+\sum_{t=\alpha r+1}^{r-1}\frac{\binom{r}{t}}{\kappa^t}\frac{(q-t)_{r-t}}{q^{r-t}}\right)\right). \tag{8}$$

Explanation for (8)**:** For the first sum we use (4) on A_i and for the second sum we use spread by summing over all $\binom{r}{t}$ t-subsets of A_i.

So,

$$\frac{\mathbb{E}(Z^2)}{\mathbb{E}(Z)^2} \leq \frac{1}{\mathbb{E}(Z)}+1+\sum_{t=1}^{\alpha r}\left(\frac{K_0}{\kappa}\right)^t\frac{(q-t)_{r-t}q^r}{q^{r-t}(q)_r}+\sum_{t=\alpha r+1}^{r-1}\frac{2^r}{\kappa^t}\frac{(q-t)_{r-t}q^r}{q^{r-t}(q)_r}=1+o(1)$$

as long as $\kappa, r\to\infty$. It follows that w.h.p.

$$|\mathcal{H}^*| \sim \frac{|\mathcal{H}|(q)_r}{q^r}. \tag{9}$$

Thus, for the rest of the proof we will assume $|\mathcal{H}^*| \geq (1-o(1))\frac{|\mathcal{H}|(q)_r}{q^r}$.

2.2 Random sample from X

Given a set $A^* \in \mathcal{H}^*$, we define

$$f^*_{t,A^*} = |\{B^* \in \mathcal{H}^* : |B^*\cap A^*| = t\}|$$

so that for $1 \leq t \leq \alpha r$, we have by (4) that

$$\mathbb{E}(f^*_{t,A^*}) = \frac{(q-t)_{r-t}}{q^{r-t}}f_{t,A} \leq \frac{(q-t)_{r-t}}{q^{r-t}}\left(\frac{K_0}{\kappa}\right)^t|\mathcal{H}|. \tag{10}$$

and for $t > \alpha r$, we have by (5) that

$$\mathbb{E}(f^*_{t,A^*}) = \frac{(q-t)_{r-t}}{q^{r-t}}f_{t,A} \leq \frac{(q-t)_{r-t}}{q^{r-t}}\frac{2^r}{\kappa^t}|\mathcal{H}| \tag{11}$$

For $A^* \in \mathcal{H}^*$ and $W_0^* \subseteq X^*$ with $|W_0^*| = m$, let $T^* = T^*(A^*, W_0^*)$ be $B^*\setminus W_0^*$ for some $B^* \in \mathcal{H}^*, B^* \subseteq A^*\cup W_0^*$ that minimizes $|B^*\setminus W_0^*|$.

Let $\omega\to\infty, \omega = o(r^{1/2})$. For $A^* \in \mathcal{H}^*$ we say that (A^*, W_0^*) is *bad* if $|T^*(A^*, W_0^*)| \geq \omega$. Otherwise (A^*, W_0^*) is *good*. Let W_0^* be a *success* if $|\{A^* \in \mathcal{H}^* : (A^*, W_0^*) \text{ is bad}\}| \leq |\mathcal{H}^*|/2$, that is, if the majority of sets in $\mathcal{H}^*$ have a relatively small T^*.

Lemma 4. $\mathbb{P}$(*success*) $\geq 1 - c_0^\omega$ *for some constant* $0 < c_0 < 1$.

Proof. Let ν_{bad} denote the number of bad pairs (A^*, W_0^*). Fix a function $\phi : 2^{X^*} \to \mathcal{H}^*$, where $\phi(S^*) \subseteq S^*$ whenever S^* contains a set in $\mathcal{H}^*$. We claim that

$$\nu_{bad} \le \sum_{t \ge \omega} \sum_{|Z^*| = m+t} \sum_{t' \ge t} 2^{t'} f^*_{t', \phi(Z^*)}. \tag{12}$$

Explanation for (12)**:** This equation follows from the key observation of recent threshold papers [7, 10]. We count the number of (A^*, W_0^*) with $|T^*(A^*, W_0^*)| = t$ for a given $t \ge \omega$. We first fix $Z^* = T^* \cup W_0^*$, which as these are disjoint has size $m + t$. Then, we let $t' = |\phi(Z^*) \cap A^*|$, noting that $\phi(Z^*) \subseteq Z^*$ as Z^* does contain a set in $\mathcal{H}^*$. Since $T^* \subseteq Z^*$ is chosen to minimize $|B^* \setminus W_0^*|, B^* \in \mathcal{H}^*, B^* \subseteq A^* \cup W_0^*$, and $\phi(Z^*)$ is a valid choice of B^*, we must have $T^* \subseteq \phi(Z^*) \cap A^*$, and so $t' = |\phi(Z^*) \cap A^*| \ge t$. Given t', we can specify one of the at most $f^*_{t', \phi(Z^*)}$ possibilities for A^* as a superset of $\phi(Z^*) \cap A^*$. We then specify $T^* \subseteq \phi(Z^*) \cap A^*$ in at most $2^{t'}$ ways, which uniquely gives $W_0^* = Z^* \setminus T^*$.

By linearity of expectation and Equations (10), (11), and (12), we get

$$\begin{aligned}
\mathbb{E}(\nu_{bad}) &\le \sum_{t \ge \omega} \sum_{|Z^*| = m+t} \left(\sum_{t'=t}^{\alpha r} \frac{(q-t')_{r-t'}}{q^{r-t'}} \left(\frac{2K_0}{\kappa} \right)^{t'} |\mathcal{H}| + \sum_{t' > \alpha r} \frac{(q-t')_{r-t'}}{q^{r-t'}} \frac{2^{r+t'}}{\kappa^{t'}} |\mathcal{H}| \right) &(13) \\
&\le (1+o(1)) \sum_{t \ge \omega} \sum_{|Z^*| = m+t} \left(\sum_{t'=t}^{\alpha r} \frac{(q-t')_{r-t'}}{q^{r-t'}} \left(\frac{2K_0}{\kappa} \right)^{t'} \frac{q^r |\mathcal{H}^*|}{(q)_r} + \sum_{t' > \alpha r} \frac{(q-t')_{r-t'}}{q^{r-t'}} \frac{2^{r+t'}}{\kappa^{t'}} \frac{q^r |\mathcal{H}^*|}{(q)_r} \right) \\
&\le (1+o(1)) \sum_{t \ge \omega} \sum_{|Z^*| = m+t} \left(\sum_{t'=t}^{\alpha r} \left(\frac{2eK_0}{\kappa} \right)^{t'} |\mathcal{H}^*| + \sum_{t' > \alpha r} \frac{e^{t'} 2^{r+t'}}{\kappa^{t'}} |\mathcal{H}^*| \right). &(14)
\end{aligned}$$

Continuing, and using (14),

$$\begin{aligned}
(1 - o(1))\ \mathbb{E}(\nu_{bad}) &\le \sum_{t \ge \omega} \binom{N}{m+t} \left(\sum_{t'=t}^{\alpha r} \left(\frac{2eK_0}{\kappa} \right)^{t'} |\mathcal{H}^*| + \sum_{t' > \alpha r} \frac{2^{r+t'} e^{t'}}{\kappa^{t'}} |\mathcal{H}^*| \right) \\
&\le \binom{N}{m} |\mathcal{H}^*| \sum_{t \ge \omega} \left(\frac{\kappa}{C} \right)^t \left(\sum_{t'=t}^{\alpha r} \left(\frac{2eK_0}{\kappa} \right)^{t'} + \sum_{t' > \alpha r} \frac{2^{r+t'} e^{t'}}{\kappa^{t'}} \right) \\
&\le \binom{N}{m} |\mathcal{H}^*| c^\omega \text{ for some } 0 < c < 1.
\end{aligned}$$

Now, let $w_{bad} = |\{W_0^* : \text{there are at least } |\mathcal{H}^*|/2 \text{ bad } (A^*, W_0^*)\}|$. Then the above equation gives that

$$(1 - o(1))\ \mathbb{E}(w_{bad}) \le 2 \binom{N}{m} c^\omega$$

and thus

$$\mathbb{P}(failure) = \frac{\mathbb{E}(w_{bad})}{\binom{N}{m}} \le (1 + o(1))\ 2c^\omega.$$

□

By taking $\omega \to \infty$ as $r \to \infty$, this means that success will happen with high probability.

2.3 Finishing the proof

Suppose now that W_0^* is a success and then let $\mathcal{R}^*$ denote the multi-hypergraph

$$\{T^*(A^*, W_0^*) : A^* \in \mathcal{H}^*, (A^*, W_0^*) \text{ is good}\}$$

where each good (A^*, W_0^*) contributes one element. Then let

$$\mathcal{R}_\ell^* = \{R^* \in \mathcal{R}^* : |R^*| = \ell\} \text{ for } 0 \le \ell < \omega.$$

We can assume that $\mathcal{R}_0^* = \emptyset$, as otherwise W_0^* contains an edge of $\mathcal{H}^*$ and we have already succeeded. Now, generate $W^* = W_0^* \cup W_1^*$ where W_1^* is distributed as $(X^* \setminus W_0^*)_{p_1}$. If $R^* \subseteq W_1^*$ for some $R^* \in \mathcal{R}^*$, then the $B^* \in \mathcal{H}^*$ for which $R^* = B^* \setminus W_0^*$ satisfies Theorem 3. Thus, we just need to show that with probability at least $1 - \varepsilon$ there exists such an $R^* \subseteq W_1^*$.

To aid in the calculations below, for each $R^* \in \mathcal{R}_\ell^*$ with $R^* \subseteq W_1^*$, take $A(R^*)$ to be an independent random variable with distribution Bernoulli$((\varepsilon_1 p_1)^{\omega-\ell})$. $R^* \in \mathcal{R}^*$ is *accepted* if $R^* \subseteq W_1^*$ and $A(R^*) = 1$. Let ν_R denote the number of accepted sets. It suffices to show $\mathbb{P}(\nu_R = 0) \le \varepsilon$, which we will do by Chebyshev's inequality. Then

$$\mathbb{E}(\nu_R) = \sum_{\ell=1}^{\omega} |\mathcal{R}_\ell^*| \frac{p_1^\ell (q - r + \ell)_\ell}{q^\ell} (\varepsilon_1 p_1)^{\omega-\ell} \sim |\mathcal{R}^*| (\varepsilon_1 p_1)^\omega \to \infty. \tag{15}$$

The claims in (15) follow from the fact that

$$\omega = o(r^{1/2}) \text{ and the fact that } |\mathcal{R}^*| \ge \tfrac{1}{2}|\mathcal{H}^*| \gtrsim \tfrac{1}{2} e^{-r} |\mathcal{H}| \ge \tfrac{1}{2}(\kappa/e)^r. \tag{16}$$

Now

$$\begin{aligned}
\mathbb{V}ar(\nu_R) &\le \sum_{t=1}^{\omega} \sum_{\ell_1,\ell_2=1}^{\omega} \mathbb{E}\left(\left| \left\{(R^*, S^*) : R^* \in \mathcal{R}_{\ell_1}^*, S^* \in \mathcal{S}_{\ell_2}^*, |R^* \cap S^*| = t\right\} \right|\right) \times \\
&\qquad \frac{p_1^{\ell_1}(q - r + \ell_1)_{\ell_1}}{q^{\ell_1}} \frac{p_1^{\ell_2 - t}(q - r + \ell_2 - t)_{\ell_2 - t}}{q^{\ell_2 - t}} \cdot (\varepsilon_1 p_1)^{2\omega - \ell_1 - \ell_2} \\
&\sim \sum_{t=1}^{\omega} \mathbb{E}\left(\left|\{R^*, S^* \in \mathcal{R}^* : |R^* \cap S^*| = t\}\right|\right) (\varepsilon_1 p_1)^{2\omega - t}.
\end{aligned} \tag{17}$$

(The same assumptions (16) suffice to obtain (17).)

Fix $R^* \in \mathcal{R}^*$ and then for $1 \le t \le \omega$,

$$\begin{aligned}
\mathbb{E}(|\{S^* \in \mathcal{R}^* : |R^* \cap S^*| = t\}|) &\le \sum_{s=t}^{r} \left(\frac{K_0}{\kappa}\right)^s \frac{(q)_{r-s}}{q^{r-s}} |\mathcal{H}| \\
&\le \left(\frac{K_0}{\kappa}\right)^t \frac{(q)_{r-t}}{q^{r-t}} |\mathcal{H}| \sum_{s=t}^{r} \left(\frac{(1+\varepsilon_1)K_0}{\varepsilon_1 \kappa}\right)^{s-t} \\
&\le 2\left(\frac{K_0}{\kappa}\right)^t \frac{(q)_{r-t}}{q^{r-t}} |\mathcal{H}| \lesssim 2\left(\frac{K_0}{\kappa}\right)^t \frac{(q)_{r-t}}{q^{r-t}} \frac{q^r}{(q)_r} |\mathcal{H}^*| \le 2\left(\frac{eK_0}{\kappa}\right)^t |\mathcal{H}^*|.
\end{aligned} \tag{18}$$

Explanation for (18)**:** R^* appears several times in $\mathcal{R}^*$ as $A^* \setminus W_0^*$ for some $A^* \in \mathcal{H}^*$. For each such A^* we count the number of sets $B^* \in \mathcal{H}^*$ for which $s = |B^* \cap A^*| \ge t$. This will bound the number of choices for S^* in the LHS of (18). For the sum we use (11) which is only valid for $t \le \alpha r$. For larger t, we proceed as in (8) and K_0^t by $\binom{r}{t} \le (e/\alpha)^t$ and assume that $K_0 \ge e/\alpha$.

So

$$\mathbb{V}ar(\nu_R) \lesssim 2|\mathcal{H}^*||\mathcal{R}^*| \sum_{t=1}^{\omega} \left(\frac{eK_0}{\kappa}\right)^t (\varepsilon_1 p_1)^{2\omega - t}$$

$$\leq 4|\mathcal{R}^*|^2 (\varepsilon_1 p_1)^{2\omega} \sum_{t=1}^{\omega} \left(\frac{eK_0}{\varepsilon_1 \kappa p_1}\right)^t \leq 4|\mathcal{R}^*|^2 (\varepsilon_1 p_1)^{2\omega} \sum_{t=1}^{\omega} \left(\frac{eK_0}{\varepsilon_1 C}\right)^t \leq \frac{12K_0}{\varepsilon_1 C} \mathbb{E}(\nu_R)^2.$$

(We have used $\kappa p_1 = \kappa m/N = C$ and $C \gg K_0$ to get the third inequality.)

The Chebyshev inequality implies that

$$\mathbb{P}(\nu_R = 0) \leq \frac{\mathbb{V}ar(\nu_R)}{\mathbb{E}(\nu_R)^2} \lesssim \frac{12K_0}{\varepsilon_1 C}.$$

Taking $C(\varepsilon, \varepsilon_1) \geq \frac{13K_0}{\varepsilon\varepsilon_1}$ then verifies (6). (We use $\mathbb{E}(\nu_R) \to \infty$ to justify the final conclusion.)

3 Powers of Hamilton cycles

We verify (4) for the hypergraph $\mathcal{H}$ whose edges correspond to the kth power of a Hamilton cycle. As in [7] we split this into two propositions and modify their proof for $k = 2$.

Proposition 1. *For $T \subseteq \binom{[n]}{2}$, with $t \leq n/3k$ edges, inducing c components,*

$$|\mathcal{H} \cap \langle T \rangle| \leq (2k)^{2t} \left(n - \left\lceil \frac{t + (2k-1)c}{k} \right\rceil + c - 1\right)!.$$

Proof. Let $T_1, \ldots, T_c$ be the components of the subgraph induced by the edges T and let $v = |V(T)|$ where $(V(A), E(A))$ is the set of (vertices, edges) used by a subgraph A. The upper bound on t implies that no T_j can "wrap around," and so $|E(T_j)| \leq k|V(T_j)| - (2k-1)$ for each j and so

$$t \leq kv - (2k-1)c. \tag{19}$$

We designate a root vertex v_j for each T_j and order $V(T_j)$ by some order $\prec_j$ that begins with v_j and in which each $v \neq v_j$ appears later than at least one of its neighbors. We may then bound $|\mathcal{H} \cap \langle T \rangle|$ as follows. To specify an $S \in \mathcal{H}$ containing T, we first specify a cyclic permutation of $\{v_1, \ldots, v_c\} \bigcup([n] \setminus V(T))$. By (19), the number of ways to do this (namely, $(n - v + c - 1)!$) is at most $\left(n - \left\lceil \frac{t+(2k-1)c}{k} \right\rceil + c - 1\right)!$. We then extend to a full cyclic ordering of $[n]$ (thus determining T) by inserting, for $j = 1, \ldots, c$, the vertices of $V(T_j) \setminus \{v_j\}$ in the order $\prec_j$. This allows at most $2k$ places to insert each vertex (since one of its neighbours has been inserted before it and the edge joining them must belong to T), so the number of possibilities here is at most $(2k)^v \leq (2k)^{2t}$, and the proposition follows. □

Proposition 2. *For $T \subseteq S \in \mathcal{H}, |T| = t \leq n/3k$, the number of subgraphs of T with c components is at most $(4ke)^t \binom{2t}{c}$.*

Proof. To specify a subgraph T of S we proceed as follows. We first choose root vertices $v_1, \ldots, v_c$ for the components, say $T_1, \ldots, T_c$, of T, the number of possibilities for this being at most $\binom{2t}{c}$. We then choose the sizes, say $t_1, \ldots, t_c$, of $T_1, \ldots, T_c$; here the number of possibilities is at most $\sum_{u=c}^{t} \binom{u-1}{c-1}$. (For $u < t$, the summand is the number of positive integer solutions to $x_1 + \cdots + x_c = u$.) Finally, we specify for each i a connected S_i of size t_i rooted at v_i in at most $\prod_{i=1}^{c}(2ke)^{t_i}$ ways. This comes for the fact that there are at most $(\Delta e)^{t-1}$ rooted subtrees of the infinite Δ-regular tree, see Knuth [9], p396, Ex11. Combining these estimates (with $\sum_{u=c}^{t}\binom{u-1}{c-1} = \binom{t}{c} < 2^t$) yields the proposition. $\square$

It follows from these two propositions that if $S \in \mathcal{H}$ and $1 \leq t \leq n/3k$ then

$$\begin{aligned}
\frac{f_{t,S}}{|\mathcal{H}|} &\leq \sum_{c=1}^{t}(2k)^{2t}\left(n - \left\lceil\frac{t+(2k-1)c}{k}\right\rceil + c - 1\right)! \times (4ke)^t\binom{2t}{c} \times \frac{1}{(n-1)!} \\
&\leq 2\sum_{c=1}^{t}(16k^3e)^t\binom{2t}{c}\left(\frac{e}{n-1}\right)^{\left\lceil\frac{t+(2k-1)c}{k}\right\rceil - c}\left(\frac{n - \left\lceil\frac{t+(2k-1)c}{k}\right\rceil + c - 1}{n-1}\right)^{n-\left\lceil\frac{t+(2k-1)c}{k}\right\rceil + c - 1} \\
&\leq e^{O(t)}\sum_{c=1}^{t} n^{-\left\lceil\frac{t+(2k-1)c}{k}\right\rceil + c} \\
&= O\left(\frac{O(1)}{n^{1/k}}\right)^t.
\end{aligned}$$

So the kth power of a Hamiltonian cycle satisfies the conditions with $r = kn$, $\kappa = O(n^{1/k})$, $\alpha = 1/3k$.

4 Final thoughts

Theorem 3 could possibly be improved in at least two ways. First, we could try to replace ε by $o(1)$. For specific examples such as the square of a Hamilton cycle, this can probably be done using the ideas of Friedgut [5], as suggested in [7]. Also, we can try to replace ε_1 by zero, which would require an improvement to the proof in Section 2.3 that we do not have at the moment.

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