The cover times of random walks on random uniform hypergraphs.

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Abstract

Random walks in graphs have been applied to various network exploration and network maintenance problems. In some applications, however, it may be more natural, and more accurate, to model the underlying network not as a graph but as a hypergraph, and solutions based on random walks require a notion of random walks in hypergraphs. At each step, a random walk on a hypergraph moves from its current position v to a random vertex in a randomly selected hyperedge containing v. We consider two definitions of cover time of a hypergraph H. If the walk sees only the vertices it moves between, then the usual definition of cover time, $C(H)$, applies. If the walk sees the complete edge during the transition, then an alternative definition of cover time, the inform time $I(H)$ is used. The notion of inform time models passive listening which fits the following types of situations. The particle is a rumor passing between friends, which is overheard by other friends present in the group at the same time. The particle is a message transmitted randomly from location to location by a directional transmission in an ad-hoc network, but all receivers within the transmission range can hear.

In this paper we give an expression for $C(H)$ which is tractable for many classes of hypergraphs, and calculate $C(H)$ and $I(H)$ exactly for random r-regular, s-uniform hypergraphs. We find that for such hypergraphs, **whp**, $C(H)/I(H) \sim s(r-1)/r$, if $rs = O((\log\log n)^{1-\epsilon})$. For random r-regular, s-uniform multi-hypergraphs, constant $r \geq 2$, and $3 \leq s \leq O(n^{\epsilon})$, we also prove that, **whp**, $I(H) = O((n/s)\log n)$, i.e. the inform time decreases directly with the edge size s.

Keywords: Random walks, Hypergraphs, Cover time, Random graphs

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1. Introduction

The idea of a random walk on a hypergraph is a natural one. The particle making the walk picks a random edge incident with the current vertex. The particle enters the edge, and exits via a random endpoint, other than the vertex of entry. Two alternative definitions of cover time are possible for this walk. Either the particle sees only the vertices it visits, or it inspects all vertices of the hyperedge during the transition across the edge.

A random walk on a hypergraph models the following process. The vertices of a network are associated into groups, and these groups define the edges of the network. In the simplest case, the network is a graph so the groups are exactly the edges of the graph. In general, the groups may be larger, and represent friends, a family, a local computer network, or all receivers within transmission range of a directed transmission in an ad-hoc network. In this case the network is modeled as a hypergraph, the hyperedges being the group relationships. An individual vertex can be in many groups, and two vertices are neighbours if they share a common hyperedge. Within the network a particle (message, rumor, infection, etc.) is moving randomly from vertex to neighbouring vertex. When this transition occurs all vertices in a given group are somehow affected (infected, informed) by the passage of the particle *within the group*. Examples of this type of process include the following. The particle is an infection passed from person to person and other family members also become infected with some probability. The particle is a virus traveling on a network connection in an intra-net. The particle is a message transmitted randomly from location to location by a directional transmission in an ad-hoc network, and all receivers within the transmission range can hear. The particle is a rumor passing between friends, which may be overheard by other friends present in the group at the same time.

Let $H = (V(H), E(H))$ be a hypergraph. For $v \in V = V(H)$ let $d(v)$ be the degree of v, i.e. the number of edges $e \in E = E(H)$ incident with v, and let $d(H) = \sum_{v\in V} d(v)$ be the total degree of H. For $e \in E$, let $|e|$ be the size of hyperedge e, i.e. the number of vertices $v \in e$, respecting multiplicity. Let $N(v)$ be the neighbour set of v, $N(v) = \{w \in V : \exists e \in E, e \supseteq \{v, w\}\}$. We regard $N(v)$ as a multi-set in which each $w \in N(v)$ has a multiplicity equal to the number of edges e containing both v and w. A hypergraph is r regular if each vertex is in r edges, and is s-uniform if every edge is of size s. A hypergraph is simple if no edge contains a repeated vertex, and no two edges are identical. We assume a particle or message originated at some vertex u and, at step t, is moving randomly from a vertex v to a vertex w in $N(v)$. We model the problem conceptually as a random walk $\mathcal{W}_u = (W_u(0), W_u(1), \ldots, W_u(t), \ldots)$ on the vertex set of hypergraph H, where $W_u(0) = u$, $W_u(t) = v$ and $W_u(t+1) = w \in N(v)$.

Several models arise for reversible random walks on hypergraphs. Assume that the walk $\mathcal{W}$ is at vertex v, and consider the transition from that vertex. In the first model (Model 1), an edge e incident with v is chosen proportional to $|e| - 1$. The walk then moves to a random endpoint of that edge, other than

v. This is equivalent to v choosing a neighbour w u.a.r. (uniformly at random) from $N(v)$, where vertex w is chosen according to its multiplicity in $N(v)$. The stationary distribution of v in Model 1 is given by

$$\pi_v = \frac{\sum_{e:v\in e}(|e|-1)}{\sum_{e\in E(H)}|e|(|e|-1)}.$$

In the case of graphs this reduces to $\pi_v = d(v)/2m$, where m is the number of edges in the graph. Alternatively (Model 2) when $\mathcal{W}$ is at v, edge e is chosen u.a.r. from the hyperedges incident with v, and *then* w is chosen u.a.r. from the vertices $w \in e, w \neq v$. The stationary distribution of v in Model 2 is given by

$$\pi_v = \frac{d(v)}{\sum_{u\in V(H)} d(u)},$$

which corresponds to the familiar formula for graphs. If the hypergraph is uniform (all edges have the same size) then the models are equivalent.

Random walks on graphs are a well studied topic, for an overview see e.g. [1, 11]. Random walks on hypergraphs were used in [5] to cluster together electronic components which are near in graph distance for physical layout in circuit design. For that application, edges were chosen inversely proportional to their size, and then a random vertex within the edge was selected. A random walk model is also used for generalized clustering in [13]. As before, the aim is to partition the vertex set, and this is done via the Laplacian of the transition matrix. This technique has applications in data mining (see [10]) and clustering images from the WWW (see [15] and references therein). The paper [3] directly considers notions of cover time for random walks on hypergraphs, using Model 2. A further discussion of [3] is given below.

For a hypergraph H, we define the (vertex) cover time $C(H)$, the edge cover time $C_E(H)$, and the inform time $I(H)$. The (vertex) cover time $C(H) = \max_u C_u(H)$, where $C_u(H)$ is the expected time for the walk $\mathcal{W}_u$ to visit all vertices of H. Similarly, the edge cover time $C_E(H) = \max_u C_{u,E}(H)$, where $C_{u,E}(H)$ is the expected time to visit all hyperedges starting at vertex u.

Suppose that the walk $\mathcal{W}_u$ is at vertex v. Using e.g. Model 2, the walk first selects an edge e incident with v and then makes a transition to $w \in e$. The vertices of e are said to be *informed* by this move. The *inform time* $I(H)$, introduced in [3] as the *radio cover time*, is the maximum over start vertices u, of the expected time at which all vertices of the graph are informed. More formally, let $\mathcal{W}_u(t) = (W_u(0), W_u(1), ..., W_u(t))$ be the trajectory of the walk. Let $e(j)$ be the edge used for the transition from $W(j)$ to $W(j+1)$ at step j. Let $\mathcal{S}_u(t) = \cup_{j=0}^{t-1} e(j)$ be the set of vertices spanned by the edges of $\mathcal{W}_u(t)$. Let $\boldsymbol{I}_u$ be the step t at which $\mathcal{S}_u(t) = V$ for the first time, and let $I(H) = \max_u \mathbf{E}(\boldsymbol{I}_u)$. We use the name "inform time" rather than "radio cover time" in [3] to indicate the relevance of this term beyond the radio networks.

Several upper bounds on the cover time $C(H)$ are readily obtainable, for example an analogue of the $O(nm)$ bound for graphs [2] based on a twice round the spanning tree argument. For Model 1, replace each edge e by a clique of size

$\binom{|e|}{2}$ to obtain an upper bound of $O(nm\overline{s^2})$ for connected hypergraphs. Here $\overline{s^2}$ is the expected squared edge size $(\sum_{e \in E(H)} |e|^2)/m$. Thus $C(H) = O(n^3 m)$. A better bound of $O(nm\overline{s}) = O(n^2 m)$ was shown in [3] for Model 2.

Similarly, a Matthews type bound of $O(\log n \cdot \max_{u,v} \mathbf{E}(\boldsymbol{H}_{u,v}))$ on the cover time exists, where $\mathbf{E}(\boldsymbol{H}_{u,v})$ is the expected hitting time of v starting from u. We contribute a bound on the cover time of a hypergraph given in Theorem 1, which allows us to calculate $C(H)$ for many classes of hypergraphs. To prove this bound, we first observe that $\mathbf{E}(\boldsymbol{H}_{u,v}) = O(T + \mathbf{E}_\pi(\boldsymbol{H}_v))$, where T is a suitable *mixing time* (defined in the statement of the theorem) and $\mathbf{E}_\pi(\boldsymbol{H}_v)$ is the expected hitting time of vertex v from stationarity. Then we bound $\mathbf{E}_\pi(\boldsymbol{H}_v)$ and apply Matthews' bound [12].

Theorem 1. *Let H be a connected hypergraph. Let P denote the transition matrix of an aperiodic random walk on H with stationary distribution π_v, $v \in V$. Let T be a mixing time such that $|P_u^{(t)}(v) - \pi_v| \leq \delta \pi_v$, for all $u, v \in V$ and $t \geq T$. Assume further that $\max_v T\pi_v = o(1)$, and that $T\delta = o(1)$. For a walk starting from v, let $R_v(T)$ be the expected number of returns to v during T steps. Then*

$$C(H) = \log n \cdot O\left(T + \max_v \frac{R_v(T)}{\pi_v}\right). \tag{1}$$

The bound (1) for $C(H)$ can be evaluated directly for many classes of random hypergraphs. For example, for random r-regular, s-uniform (simple) hypergraphs $\mathcal{G}(n, r, s)$, and random s-uniform hypergraphs $G_{n,p,s}$ where each edge occurs independently with probability p. Let $r \geq 2$ and $s \geq 3$ in $\mathcal{G}(n, r, s)$, and let $p \geq C \log n / \binom{n-1}{s-1}$ in $G_{n,p,s}$, where $C > 1$. Then, **whp**, there is a mixing time $T = O(\log^k n)$ for some constant k which satisfies the above conditions, and where moreover $\pi_v = \Theta(1/n)$, and $R_v(T) = 1 + O(1)$. In this case Theorem 1 implies that **whp** $C(H) = O(n \log n)$.

The calculation of inform time $I(H)$ seems more challenging. Avin *et al.* [3] show that Matthews' bound extends to $I(H)$: for any n-vertex hypergraph H, $\max_{u,v} \mathbf{E}(\tilde{\boldsymbol{H}}_{u,v})) \leq I(H) \leq O(\log n \cdot \max_{u,v} \mathbf{E}(\tilde{\boldsymbol{H}}_{u,v})$, where $\mathbf{E}(\tilde{\boldsymbol{H}}_{u,v})$ is the expected time when vertex v is informed starting from vertex u ($\mathbf{E}(\tilde{\boldsymbol{H}}_{u,v})$ is called the radio hitting time in [3]). For a random walk on an s-uniform hypergraph, in each period of $2 \cdot \max_x \mathbf{E}(\tilde{\boldsymbol{H}}_{x,v})$ steps, the walk traverses an edge containing v with probability at least $1/2$, so visits vertex v with probability at least $1/(2s)$. Hence $\mathbf{E}(\boldsymbol{H}_{u,v}) = O(s \cdot \max_x \mathbf{E}(\tilde{\boldsymbol{H}}_{x,v}))$, implying that $C(H) = O(s \log n \cdot I(H))$. Thus the speed-up of the inform time over the cover time is at most $O(s \log n)$.

Avin *et al.* [3] consider a special type of *directed hypergraphs*, called *radio hypergraphs*, and analyse $I(H)$ on *one-* and *two-dimensional mesh radio hypergraphs*, which are induced by a cycle and a square grid on a torus, respectively. Their result for the two-dimensional mesh can be stated in the following way. For a random walk on a $\sqrt{n} \times \sqrt{n}$ grid such that in each step all vertices within distance k from the current vertex are informed and the walk moves to a random vertex in this k-neighbourhood, the inform time is $I(H) = O((n/k^2)\log(n/k^2)\log n)$.

In this paper we calculate precisely $C(H)$, $I(H)$ and $C_E(H)$ for the case of simple random r-regular, s-uniform hypergraphs H. As far as we know, the first analysis of cover time and inform time for random walks on classes of general (undirected) hypergraphs. The proof of the following theorem is the main technical contribution of this paper. Throughout the paper "log" stands for the natural logarithm, and from now on $\mathcal{G}(n, r, s)$ denotes either the family of all simple n-vertex r-regular s-uniform hypergraphs, or the uniform distribution on such hypergraphs, depending on the context. The term "with high probability," abbreviated to **whp**, means with probability $1 - o(1)$, that is, with probability approaching 1 when n (in our case, the number of vertices) tends to infinity.

Theorem 2. *Suppose that $r \geq 2$ and $s \geq 3$ are constants and H is chosen u.a.r. from the set $\mathcal{G}(n, r, s)$ of all simple r-regular, s-uniform hypergraphs with n vertices. Then* **whp** *as $n \to \infty$,*

$$C(H) \sim \left(1 + \frac{1}{(r-1)(s-1)-1}\right) n \log n,$$

$$I(H) \sim \left(1 + \frac{s-1}{(r-1)(s-1)-1}\right) \frac{n}{s-1} \log n,$$

$$C_E(H) \sim \left(1 + \frac{s-1}{(r-1)(s-1)-1}\right) \frac{rn}{s} \log n.$$

In the case of graphs, $I(H) = C(H)$, and $C_E(H) \geq C(H)$. For hypergraphs, clearly $I(H) \leq C(H)$. However there is the possibility that $C_E(H) \leq C(H)$, as every edge can be visited without visiting every vertex. We must have $I(H) \leq C_E(H)$ as a vertex is informed whenever the walk covers an edge containing that vertex. Indeed, intuitively we should have $C_E(H)$ about r times $I(H)$, if every vertex has degree r. We note that our theorem gives $C_E(H) \sim r((s-1)/s)I(H)$.

Our proof of Theorem 2 also applies when s and/or r grow (slowly) with n. More specifically, checking all elements of the proof reveals that all assumptions on the values of r and s are satisfied, if $rs = O((\log\log n)^{1-\epsilon})$, for a constant $\epsilon > 0$. Therefore we have the following corollary.

Corollary 3. *If $r \geq 2$, $s \to \infty$, $rs = O((\log\log n)^{1-\epsilon})$ and $\epsilon > 0$ is a constant, then*

$$C(H) \sim n \log n \quad \text{and} \quad I(H) \sim \frac{r}{r-1}\frac{n}{s}\log n.$$

Thus in this case, seeing s vertices at each step of the walk leads to an $\Theta(s)$ speed up in cover time.

Corollary 3 gives the asymptotic value of inform time $I(H)$ for the case when $s \to \infty$, but allows s to increase only very slowly with n. For random *multi-hypergraphs*, and for r constant, we can give an order of magnitude result for $I(H)$ for much faster growing s, requiring only that $s = O(n^\delta)$, for some suitably small constant $\delta > 0$. A multi-hypergraph allows loops (the same vertex appearing two or more times in the same hyperedge) and parallel hyperedges. Obviously, as s increases it is difficult to avoid this condition. A standard way to generate random multi-hypergraphs is to use the configuration (pairing) model, described in Section 3.2. For random multi-hypergraphs $\mathcal{M}(n, r, s)$ generated using the configuration model we have the following result, which is proved in Section 9.

Theorem 4. *Let $r \geq 2$ and $3 \leq s = O(n^\delta)$, for a suitably small constant $\delta > 0$. Then* **whp** *for $H \in \mathcal{M}(n, r, s)$ the inform time $I(H)$ satisfies*

$$I(H) = O\left(\frac{n}{s} \log n\right).$$

2. Proof of Theorem 1

To prove Theorem 1, we use the bound $C(H) = O(\log n \cdot \max_{u,v} \mathbf{E}(\boldsymbol{H}_{u,v}))$, the observation that $\mathbf{E}(\boldsymbol{H}_{u,v}) = O(T + \mathbf{E}_\pi(\boldsymbol{H}_v))$, and the bound on $\mathbf{E}_\pi(\boldsymbol{H}_v)$ in Lemma 5 below. The quantity $\mathbf{E}_\pi(\boldsymbol{H}_v)$ is the expected hitting time of a vertex v from the stationary distribution π. The value of $\mathbf{E}_\pi(\boldsymbol{H}_v)$ is given by

$$\mathbf{E}_\pi(\boldsymbol{H}_v) = Z_{vv}/\pi_v, \tag{2}$$

where

$$Z_{vv} = \sum_{t=0}^{\infty} (P_v^{(t)}(v) - \pi_v), \tag{3}$$

and $P_v^{(t)}(v)$ is the probability that the random walk starting from vertex v is back at v at step t. A proof of this can be found in e.g. Lemma 11 in [1, Chapter 2]. For a walk $\mathcal{W}_v$ starting from v define

$$R_v(T) \;=\; \sum_{t=0}^{T-1} P_v^{(t)}(v). \tag{4}$$

Thus $R_v(T)$ is the expected number of returns made by $\mathcal{W}_v$ to v during T steps, in the hypergraph H. We note that $R_v(T) \geq 1$, as $P_v^{(0)}(v) = 1$.

Lemma 5. *Let T be a mixing time of a random walk $\mathcal{W}_u$ on H satisfying $|P_u^{(t)}(v) - \pi_v| \leq \delta\pi_v$ for all $u, v \in V$, and $t \geq T$. Assuming that $T\delta = o(1)$, and $T\pi_v = o(1)$, then*

$$\mathbf{E}_\pi(\boldsymbol{H}_v) \leq (1 + o(1))\frac{R_v(T)}{\pi_v}. \tag{5}$$

Proof. Let $D(t) = \frac{1}{2}\max_u \sum_{x\in V} |P_u^{(t)}(x) - \pi_x|$. From [1] Chapter 2 Lemma 20, we have that $D(s+t) \leq 2D(s)D(t)$. Hence, as $D(T) \leq \delta$, then $D(kT) \leq (2\delta)^k$.

$$\begin{aligned} Z_{vv} &= \sum_{t=0}^{\infty}(P_v^{(t)}(v) - \pi_v) \leq \sum_{t<T}(P_v^{(t)}(v) - \pi_v) + T\sum_{k\geq 1}(2\delta)^k \\ &\leq R_v(T) + T\pi_v + O(T\delta) = (1+o(1))R_v(T). \end{aligned}$$

□

3. Proof of Theorem 2: preliminaries

We explain the proof of the value of $C(H)$ of Theorem 2; the proofs of $I(H)$ and $C_E(H)$ are similar. We reduce the walk $\mathcal{W}_{H,u}$ on the hypergraph H to an equivalent walk $\mathcal{W}_{G,u}$ on an associated graph $G(H) = (V, F)$, as explained in Section 3.3, and then apply some techniques developed for random walks on graphs.

In Section 3.1 we state Lemma 6 on which the proof of Theorem 2 is based. This lemma, proven in [7], gives a precise estimate of the probability that a random walk $\mathcal{W}_u$ on a graph G does not visit a given vertex v within t steps after a suitably defined mixing time T. The general idea is to apply Lemma 6 to the associated graph $G(H)$, but to do so, we have to derive a bound on the mixing time of $G(H)$ and calculate the parameter p_v of the random walk in $G(H)$ defined in (12). We can calculate p_v, if v is a *tree-like vertex* in hypergraph H, which means, informally, that there are no short cycles nearby v. In Section 3.2 we define formally this property of being a tree-like vertex, and show that it holds for most vertices of a random $H \in \mathcal{G}(n, r, s)$ **whp**.

In Section 3.4, we establish the conductance of the graph $G(H)$ to obtain a bound on the mixing time (via the relation between the conductance and the mixing time given in Section 3.1). We also prove that the conditions of Lemma 6 hold for the tree-like vertices in $G(H)$, and derive the parameter p_v for such vertices. Using the tools presented in Section 3, we prove then in Section 4 the formula for $C(H)$ stated in Theorem 2, establishing matching upper and lower bounds on $C(H)$ in Sections 4.1 and 4.2, respectively. In Section 5 we sketch how the calculations of $C(H)$ can be adapted to derive the formulas for $I(H)$ and $C_E(H)$. The analysis of $I(H)$ and $C_E(H)$ follows the analysis of $C(H)$, but with the following added difficulty. Instead of dealing with graph $G(H)$, we need to defined a special contraction Γ of $G(H)$ and to consider the random walk in Γ.

3.1. Random walk background

Let $G = (V, E)$ denote a fixed connected graph with n vertices and m edges. Let P be the matrix of transition probabilities of the random walk. We consider the random walk $\mathcal{W}_u$ which starts from a vertex u, and let $P_u^{(t)}(v) = \mathbf{Pr}(W_u(t) =$

v). We assume the random walk $\mathcal{W}_u$ is ergodic, so it has stationary distribution π, where $\pi_v = d(v)/(2m)$. Let Φ_G be the conductance of G, defined as

$$\Phi_G = \min_{S \subseteq V, \pi_S \leq 1/2} \Phi_G(S),$$
$$\Phi_G(S) = \frac{\sum_{x \in S} \sum_{y \in \bar{S}} \pi_x P_x(y)}{\pi_S}. \tag{6}$$

Then, (see e.g. [14]) with $\Phi = \Phi_G$,

$$|P_u^{(t)}(x) - \pi_x| \leq (\pi_x/\pi_u)^{1/2} e^{-\Phi^2 t/2}. \tag{7}$$

From now on, T will stand for a mixing time $t \geq T$ which satisfies the following condition for all $t \geq T$:

$$\max_{u,x \in V} |P_u^{(t)}(x) - \pi_x| \leq n^{-3}. \tag{8}$$

If (8) holds, we say the distribution of the walk is in *near stationarity*. In our analysis we need low, logarithmic mixing times for some auxiliary graphs which we will derive from the random hypergraph H. We show such mixing times by bounding the conductance of those graphs and using (7). In particular, if $\Phi_G = \Omega(1)$ and $\min_{v \in V} \pi_v \geq n^{-2}$, then there exists $T = O(\log n)$ which satisfies (8). The graphs which we consider have $O(rsn)$ edges, so $\pi_v = \Omega((rsn)^{-1}) \geq n^{-2}$ (we consider only simple, unweighted random walks).

We consider the returns to vertex v made by a walk $\mathcal{W}_v$, starting at v. Define

$$R_v(T, z) = \sum_{t=0}^{T-1} P_v^{(t)}(v) z^t, \tag{9}$$

where z is a complex variable. Thus $R_v(T, 1)$ is equal to $R_v(T)$ in (4): the expected number of returns to v in steps $0, 1, \ldots, T-1$.

Lemma 6, given below and proven in [7], is the main tool in our analysis. Let $v \in V$. We list the conditions required by Lemma 6.
(o) T is such that (8) holds for each $t \geq T$.
(i) For some (small) constant $\theta > 0$ and some (large) constant $K > 0$ we have:

$$\min_{|z| \leq 1 + 1/KT} |R_v(T, z)| \geq \theta, \tag{10}$$

(ii) $T\pi_v = o(1)$ and $T\pi_v = \Omega(n^{-2})$.

Lemma 6. [7] *Assume conditions (o), (i) and (ii) above hold for a graph G and a vertex v in G. Let $\mathcal{A}_v(t)$ be the event that the random walk $\mathcal{W}_u$ on graph G does not visit vertex v at steps $T, T+1, \ldots, t$. Then,*

$$\mathbf{Pr}(\mathcal{A}_v(t)) = \frac{(1 + O(T\pi_v))}{(1 + p_v)^t} + O(T^2 \pi_v e^{-t/KT}), \tag{11}$$

where p_v is given by the following formula, with $R_v = R_v(T)$:

$$p_v = \frac{\pi_v}{R_v(1 + O(T\pi_v))}. \tag{12}$$

3.2. Configuration model and tree-like vertices in random hypergraphs

To prove required structural properties of random hypergraphs, we need a workable model of an r-regular s-uniform hypergraph. We use a hypergraph version of the *configuration model* of Bollobás [4]. A configuration $C = C(n, r, s)$ consists of a partition of rn labeled points $\{a_{1,1}, ..., a_{1,r}, ..., a_{n,1}, ..., a_{n,r}\}$ into unordered sets $E_i,\ i = 1, ..., rn/s$ of size s. We assume naturally that s divides rn. We refer to these sets as the hyperedges of the configuration, and to the sets $v_i = \{a_{i,1}, ..., a_{i,r}\}$ as the vertices. By identifying the points of v_i, we obtain an r-regular, s-uniform multi-hypergraph $H(C)$: it may have parallel hyperedges (hyperedges containing exactly the same vertices) and loops (hyperedges with two or more points from the same vertex). In general, many configurations map to the same underlying hypergraph $H(C)$.

The set $\mathcal{C}(n, r, s)$ of all configurations $C(n, r, s)$ with the uniform measure defines the measure μ on r-regular s-uniform multi-hypergraphs, where $\mu(H)$ is the probability that $H = H(C)$ for a random configuration $C \in \mathcal{C}(n, r, s)$. We denote by $\mathcal{M}(n, r, s)$ the family of all n-vertex r-regular s-uniform multi-hypergraphs with the probability distribution μ. The measure $\mu(H)$ depends only on the number of parallel edges and loops at each vertex in H, and as an example all *simple* hypergraphs, i.e. those without parallel edges or loops, have equal measure μ. Thus $\mathcal{M}(n, r, s)$ restricted to simple hypergraphs is the family $\mathcal{G}(n, r, s)$ of r-regular s-uniform simple hypergraphs with uniform probability distribution.

The probability a u.a.r. sampled configuration gives a simple hypergraph is bounded below by a number dependent only on r and s. For the values of r, s considered in this paper, the probability that $H(C)$ is simple is $\Omega(e^{-(r-1)(s-1)/2})$ [6]. This and the fact that all simple hypergraphs have equal measure μ imply that if some property of multi-hypergraphs holds with probability at most $p(n, r, s)$ in the configuration model (i.e., for the probability space $\mathcal{M}(n, r, s)$), then this property holds with probability $O(p(n, r, s)e^{(r-1)(s-1)/2})$ for simple r-regular s-uniform random hypergraphs (i.e., for the probability space $\mathcal{G}(n, r, s)$).

To use Lemma 6, we need the parameter R_v for (12). To calculate R_v, the expected number of returns made by $\mathcal{W}_v$ to vertex v during T steps, we need to identify the local structure of a typical vertex of a random hypergraph H. A sequence $v_1, v_2, \ldots, v_k \in V$ is said to define a *path* of *length* $k-1$ if there are *distinct* edges $e_1, e_2, \ldots, e_{k-1} \in E$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ for $1 \le i \le k-1$. A sequence $v_1, v_2, \ldots, v_k \in V$, $k \ge 3$, is said to define a *cycle* of *length* k if there are *distinct* edges $e_1, e_2, \ldots, e_k \in E$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ for $1 \le i \le k$, with $v_{k+1} = v_1$. A path/cycle is short if its length is at most ω. A vertex $v \in V(H)$ is said to be *locally-tree-like to depth* k if there does not exist a path from v of length at most k to a cycle of length at most k. An edge $e \in E(H)$ is locally-tree-like to depth k, if it contains only vertices which are locally-tree-like to depth k. We introduce the threshold parameter

$$\omega = (\log \log n)^{1-\epsilon},$$

where ϵ is any constant such that $0 < \epsilon < 1$, and say that a vertex, or en edge, is *tree-like*, if it is locally-tree-like to depth ω. We argue that almost all vertices of H are tree-like.

Lemma 7. *If $rs \leq \omega$, then,* **whp***, there are at most $(rs)^{3\omega}$ vertices in a hypergraph $H \in \mathcal{G}(n,r,s)$ that are not tree-like.*

Proof. We work with the configuration model $H(C)$. The expected number of vertices on cycles of length $k \leq \omega$ can be bounded above by

$$\sum_{k=3}^{\omega} skn^k \left(\frac{rs}{n}\right)^k \leq O\left(s\omega(rs)^{\omega}\right).$$

The Markov inequality implies that the probability that there are more than $(rs)^{2\omega}$ vertices on short cycles is at most $(rs)^{-\omega/2}$. Thus the probability that there are more than $(rs)^{2\omega}$ vertices on short cycles in $H \in \mathcal{G}(n,r,s)$ is $O((rs)^{-\omega/2}e^{rs/2})$, which is $o(1)$ since $rs \leq \omega$. For each such vertex there are at most $(rs)^{\omega}$ vertices reachable by a walk of length ω. Therefore, **whp** there are at most $(rs)^{3\omega}$ vertices which are within short (at most ω) distance from a short cycle. □

Lemma 8. *If $rs \leq \omega$, then,* **whp***, there are no short paths joining distinct short cycles in a hypergraph $H \in \mathcal{G}(n,r,s)$.*

Proof. If such a structure exists then there exists a walk $v_1, v_2, \ldots, v_k$ of length at most 3ω and a pair $i, j \in [k]$ and edges $f_1, f_2 \in E$ such that $v_1, v_i \in f_1$ and $v_k, v_j \in f_2$. The probability of this is at most

$$\sum_{k=5}^{3\omega} s^2k^2n^k \left(\frac{rs}{n}\right)^{k+1} = O\left(\frac{s^3\omega^3(rs)^{3\omega}}{n}\right)$$

in the configuration model, so $o(1)$ in the $\mathcal{G}(n,r,s)$ probability distribution. □

We use the configuration model for hypergraphs also in Section 6 to bound the conductance of associated graphs.

3.3. Construction of contracted graph

To calculate $C(H), I(H)$ and $C_E(H)$, we replace the hypergraph H with a clique graph $G(H)$, and using $G(H)$, for a given tree-like vertex v or edge e of $G(H)$ we construct contraction graphs $\Gamma(v)$ and $\Gamma(e)$.

Clique graph $G(H)$ is obtained from H by replacing each hyperedge $e \in E(H)$ with a clique of size $|e|$ on the vertex set of e. Observe that $G(H)$ is actually a *multi-graph*. Formally, $G(H) = (V, F)$ where $F = \bigcup_{e \in E(H)} \binom{e}{2}$. We can think of the walk $\mathcal{W}_u$ on H as a walk on $G(H)$. Thus, the cover time of $G(H)$ is the cover time of H. Graph $G(H)$ is regular with degree $r(s-1)$, so $d(G(H)) = r(s-1)n$. To estimate the cover time $C(H)$, we use Lemma 6 applied to graph $G(H)$.

Inform-contraction graph $\Gamma(v)$ is used in the analysis of the inform time $I(H)$. Let S_v be the multi-set of edges $\{w,x\}$ in $G(H)$, not containing v, but which are contained in hyperedges incident with vertex v in H i.e.

$$S_v = \{\{w,x\} : \exists e \in E(H), v \in e, \text{ and } w, x \in e \setminus \{v\}\}.$$

Since H is r-regular and s-uniform, each S_v has size $r\binom{s-1}{2}$.

A vertex v is informed if either v is visited or S_v is visited by $\mathcal{W}_u$ (viewing $\mathcal{W}_u$ as a random walk on $G(H)$). To compute the probability that v or S_v are visited we subdivide each edge $f = \{w,x\}$ of S_v by introducing an artificial vertex a_f. Thus f is replaced by $\{w, a_f\}, \{a_f, x\}$. This transforms multi-graph $G(H)$ into multi-graph $G_v(H)$. Next let $D_v = \{v\} \cup \{a_f : f \in S_v\}$ and note that D_v is an independent set in $G_v(H)$. Now contract D_v to a single vertex $\gamma = \gamma(D_v)$, retaining all parallel edges. This transforms multi-graph $G_v(H)$ into multi-graph $\Gamma(v)$. It is easy to check that the degree of vertex γ in $\Gamma(v)$ is $d(\gamma) = r(s-1)^2$. Furthermore,

$$d(\Gamma(v)) = d(G_v(H)) = r(s-1)n + r(s-1)(s-2).$$

For a random walk in $\Gamma(v)$ the stationary distribution of γ is thus

$$\pi_\gamma^{\Gamma(v)} = \frac{(s-1)}{(n+s-2)}. \tag{13}$$

Suppose now that $\mathcal{W}_u$ is a random walk in $G(H)$ starting from a vertex $u \neq v$, and $\mathcal{X}_u$ is a random walk in $\Gamma(v)$ starting from the same vertex u (thus, in $\Gamma(v)$, $u \neq \gamma$). The paths in $G(H)$ which start from u, do not contain vertex v and do not contain any edge from S_v, are exactly the same as the paths in $\Gamma(v)$ which start from u and do not contain vertex γ. Moreover, for any such path P, the probability that $\mathcal{W}_u$ follows path P in $G(H)$ is exactly the same as the probability that $\mathcal{X}_u$ follows the same path P in $\Gamma(v)$, because the transition probabilities of the edges on P are the same in $G(H)$ and in $\Gamma(v)$. Thus, viewing $\mathcal{W}_u$ as a random walk on in both H and $G(H)$, we have

$$\begin{aligned}
&\mathbf{Pr}(\mathcal{W}_u \text{ doesn't inform } v \text{ in steps } 0,1,\ldots,t;\ H) \\
&= \mathbf{Pr}(\mathcal{W}_u \text{ doesn't visit } S_v \cup \{v\} \text{ in steps } 0,1,\ldots,t;\ G(H)) \\
&= \mathbf{Pr}(\mathcal{X}_u \text{ doesn't visit } \gamma \text{ in steps } 0,1,\ldots,t;\ \Gamma(v)) \\
&= \mathbf{Pr}(\mathcal{X}_u(\sigma) \neq \gamma, 0 \leq \sigma \leq t;\ \Gamma(v)).
\end{aligned} \tag{14}$$

To estimate the inform time $I(H)$, we use Lemma 6 applied to graph $\Gamma(v)$, and the following lemma.

Lemma 9. *Let $v \in V$, and consider a random walk $\mathcal{W}_x$ in H (viewing $\mathcal{W}_x$ also as a random walk on $G(H)$) and a random walk $\mathcal{X}_u$ in $\Gamma(v)$. Let T be a mixing time satisfying* (8) *in both $G(H)$ and $\Gamma(v)$. For $t \geq T$, let $\mathcal{B}_v(t)$ be the event that the random walk $\mathcal{W}_x$ does not inform vertex v at steps $T, T+1, \ldots, t$, and* *let $\mathcal{A}_\gamma(t)$ be the event that the random walk $\mathcal{X}_u$ in $\Gamma(v)$ does not visit vertex γ at steps $T, T+1, \ldots, t$. Then*

$$\mathbf{Pr}(\mathcal{B}_v(t); G(H)) \;=\; \mathbf{Pr}(\mathcal{A}_\gamma(t); \Gamma(v)) \left(1 + O\left(\frac{s}{n}\right)\right), \tag{15}$$

where the probabilities are those derived from the walk in the given graph.

Proof. Using (8) and (14), we have

$$\begin{aligned}
&\mathbf{Pr}(\mathcal{A}_\gamma(t); \Gamma(v)) \;= \\
&\quad= \sum_{y \neq \gamma} P_u^{(T)}(y; \Gamma(v))\, \mathbf{Pr}(\mathcal{X}_y(\sigma) \neq \gamma,\ 0 \leq \sigma \leq t - T; \Gamma(v)) \\
&\quad= \sum_{y \neq \gamma} \pi_y^{\Gamma(v)}\, (1 + O(n^{-2}))\, \mathbf{Pr}(\mathcal{X}_y(\sigma) \neq \gamma,\ 0 \leq \sigma \leq t - T; \Gamma(v)) \\
&\quad= (1 + O(n^{-2})) \frac{d(G(H))}{d(\Gamma(v))} \\
&\qquad \times \sum_{y \neq \gamma} \pi_y^{G(H)} \mathbf{Pr}(\mathcal{W}_y \notin S_v \cup \{v\} \text{ in steps } 0, 1, \ldots, t - T;\ G(H)).
\end{aligned} \tag{16}$$

Using (8) and (14), we also get

$$\begin{aligned}
&\mathbf{Pr}(\mathcal{B}_v(t); H) \;= \\
&\quad= \sum_{y \neq v} P_u^{(T)}(y; H)\, \mathbf{Pr}(\mathcal{W}_y \text{ doesn't inform } v \text{ in steps } 0, 1, \ldots, t - T;\ H) \\
&\quad= \sum_{y \neq v} P_u^{(T)}(y; G(H))\, \mathbf{Pr}(\mathcal{W}_y \notin S_v \cup \{v\} \text{ in steps } 0, 1, \ldots, t - T;\ G(H)) \\
&\quad= (1 + O(n^{-2})) \\
&\qquad \times \sum_{y \neq \gamma} \pi_y^{G(H)} \mathbf{Pr}(\mathcal{W}_y \notin S_v \cup v \text{ in steps } 0, 1, \ldots, t - T;\ G(H)).
\end{aligned} \tag{17}$$

Comparing (16) and (17), and checking that $d(\Gamma(v))/d(G(H) =$ $(1 + O(s/n))$, we get (15). □

Edge-Contraction graph $\Gamma(e)$ is used in the analysis of the edge cover time $C_E(H)$. For a given hyperedge $e \in E(H)$, first transform multi-graph $G(H)$ into a multi-graph $G_e(H)$ as follows. For each of the edges $f = \{u, v\}$ in the clique e in $G(H)$ (the clique corresponding to the hyperedge e), subdivide f using a new vertex a_f. Thus each such f is replaced by edges $\{u, a_f\}$ and $\{a_f, v\}$, and the resulting graph is referred to as $G_e(H)$. Let $D_e = \{a_f : f \subseteq e \in E(H)\}$. Contract D_e to a vertex γ to transform multi-graph $G_e(H)$ into a multi-graph $\Gamma(e)$, similarly to $\Gamma(v)$. The degree of γ in $\Gamma(e)$ is $d(\gamma) = s(s-1)$. Furthermore,

$$d(\Gamma(e)) = d(G_e(H)) = r(s-1)n + s(s-1).$$

For a random walk in $\Gamma(e)$ the stationary distribution of γ is thus

$$\pi_\gamma^{\Gamma(e)} = \frac{s}{rn + s}. \tag{18}$$

Suppose now that $\mathcal{W}_u$ is a random walk in $G(H)$ starting from a vertex $u \in V(H)$, and $\mathcal{Y}_u$ is a random walk in $\Gamma(e)$ starting from the same vertex u (thus, in $\Gamma(e)$, $u \neq \gamma$). The paths in $G(H)$ which start from u and do not contain any edge from the clique e, are exactly the same as the paths in $\Gamma(e)$ which start from u and do not contain vertex γ. Moreover, for any such path P, the probability that $\mathcal{W}_u$ follows path P in $G(H)$ is exactly the same as the probability that $\mathcal{Y}_u$ follows the same path P in $\Gamma(e)$ (the transition probabilities of the edges on P are the same in $G(H)$ and in $\Gamma(e)$). Thus, viewing $\mathcal{W}_u$ as a random walk also on H, we have

$$\begin{aligned}
&\mathbf{Pr}(\mathcal{W}_u \text{ doesn't traverse hyperedge } e \text{ in steps } 0,1,\ldots,t;\ H) \\
&\quad = \mathbf{Pr}(\mathcal{W}_u \text{ doesn't traverse an edge of clique } e \text{ in steps } 0,1,\ldots,t;\ G(H)) \\
&\quad = \mathbf{Pr}(\mathcal{Y}_u(\sigma) \neq \gamma, 0 \leq \sigma \leq t;\ \Gamma(e)) \qquad (19)
\end{aligned}$$

To estimate the edge cover time $C_E(H)$, we use Lemma 6 applied to graph $\Gamma(e)$, and the following lemma, which can be proven in an analogous way as Lemma 9.

Lemma 10. *Let $e \in E(H)$, and consider a random walk $\mathcal{W}_x$ in H (viewing $\mathcal{W}_x$ also as a random walk on $G(H)$) and a random walk in $\mathcal{Y}_u$ in $\Gamma(e)$. Let T be a mixing time satisfying* (8) *in both $G(H)$ and $\Gamma(e)$. For $t \geq T$, let $\mathcal{B}_e(t)$ be the event that the random walk $\mathcal{W}_x$ does not visit hyperedge e at steps $T, T+1, \ldots, t$, and let $\mathcal{A}_\gamma(t)$ be the event that the random walk $\mathcal{Y}_u$ in $\Gamma(e)$ does not visit vertex γ at steps $T, T+1, \ldots, t$. Then*

$$\mathbf{Pr}(\mathcal{B}_e(t); H) \;=\; \mathbf{Pr}(\mathcal{A}_\gamma(t); \Gamma(e)) \left(1 + O\left(\frac{s}{n}\right)\right). \qquad (20)$$

3.4. Conditions and parameters for Lemma 6

Our proof of Theorem 2 is based on applying Lemma 6 to graphs $G(H)$, $\Gamma(v)$ and $\Gamma(e)$. To apply Lemma 6, we need suitable upper bounds on the mixing times in these graphs. We obtain such bounds from the lower bounds on conductance given in the following lemma, which is proven in Section 6.

Lemma 11. *Let $r \geq 2$, $s \geq 3$, and $rs = o(\log n)$. For a random hypergraph $H \in \mathcal{G}(n,r,s)$, the conductance of each of the graphs $G(H)$, $\Gamma(v)$, and $\Gamma(e)$ is $\Omega(1)$* **whp**.

We use Lemma 11 and Inequality (7) in a straightforward verification of the following lemma.

Lemma 12. *Let $r \geq 2$, $s \geq 3$, and $rs = o(\log n)$. There is $T = O(\log n)$ which* **whp** *satisfies the mixing time condition* (8) *in each of the graphs $G(H)$, $\Gamma(v)$ and $\Gamma(e)$, if H is a random hypergraph $H \in \mathcal{G}(n,r,s)$.*

The next steps towards applying Lemma 6 to graphs $G(H)$, $\Gamma(v)$ and $\Gamma(e)$ are to obtain precise estimates of the parameters R_v for the values p_v in (12), and to check that the technical condition (i) of this lemma holds. We summarize these parts of the analysis in the two lemmas below, which are proven in Section 7. We note that the condition (ii) of Lemma 6 clearly holds since $T = O(\log n)$.

Lemma 13.

(i) *Let v be tree-like in H, then in $G(H)$ the value of p_v is given by*

$$p_v = (1+o(1))\frac{1}{n}\frac{(r-1)(s-1)-1}{(r-1)(s-1)}. \tag{21}$$

(ii) *Let v be tree-like in H, then in $\Gamma(v)$ the value of $p_{\gamma(v)}$ is given by*

$$p_{\gamma(v)} = (1+o(1))\frac{s-1}{n}\frac{(r-1)(s-1)-1}{r(s-1)-1}. \tag{22}$$

(iii) *Let e be tree-like in H, then in $\Gamma(e)$ the value of $p_{\gamma(e)}$ is given by*

$$p_{\gamma(e)} = (1+o(1))\frac{s}{rn}\frac{(r-1)(s-1)-1}{r(s-1)-1}. \tag{23}$$

Lemma 14. *Let v (resp. e) be a tree like vertex (resp. edge) in H. Then v (resp. $\gamma(v)$, $\gamma(e)$) satisfies the condition (i) of Lemma 6 in $G(H)$ (resp. $\Gamma(v)$, $\Gamma(e)$).*

4. Proof of Theorem 2: estimate the cover time $C(H)$

4.1. Upper bound on the cover time $C(H)$

We are assuming from now on that the hypergraph H satisfies the conditions stated in Lemmas 7 and 8, and that the mixing time $T = O(\log n)$ satisfies (8) (see Lemma 12). We view the random walk $\mathcal{W}_u$ in H as a random walk in the clique graph $G = G(H)$.

Let

$$t_0 = (1+o(1))\frac{(r-1)(s-1)}{(r-1)(s-1)-1} n\log n,$$

where the $o(1)$ term is large enough so that all inequalities below are satisfied. Let $T_G(u)$ be the time taken to visit every vertex of G by the random walk $\mathcal{W}_u$. Let U_t be the number of vertices of G which are not visited by $\mathcal{W}_u$ in the interval $[T, t]$. We note the following:

$$C_u = C_u(H) = C_u(G) = \mathbf{E}T_G(u) = \sum_{t\geq 0}\mathbf{Pr}(T_G(u) > t), \tag{24}$$

$$\mathbf{Pr}(T_G(u) > t) = \mathbf{Pr}(U_t \geq 1) \leq \mathbf{E}U_t. \tag{25}$$

It follows from (24) and (25) that for all $t \geq T$

$$C_u \leq t + \sum_{\sigma\geq t}\mathbf{E}U_\sigma = t + \sum_{v\in V}\sum_{\sigma\geq t}\mathbf{Pr}(\mathcal{A}_v(\sigma)), \tag{26}$$

where $\mathcal{A}_v(\sigma)$ is the event that vertex v is not visited in the interval $[T, \sigma]$ (as defined in the statement of Lemma 6). Let V_1 be the set of tree-like vertices and let $V_2 = V - V_1$. We apply Lemma 6. For $v \in V_1$, from (21) we have $t_0 p_v = (1+o(1))\log n$ and $(p_v)^{-1} \le (2+o(1))n$. Hence,

$$\begin{aligned}\sum_{\sigma\ge t_0} \mathbf{Pr}(\mathcal{A}_v(\sigma)) &\le (1+o(1))\sum_{\sigma\ge t_0}\left((1+p_v)^{-\sigma} + O\left(e^{\sigma/(KT)}\right)\right)\\ &\le (1+o(1))(1+p_v)^{-t_0}\sum_{\varrho\ge 0}(1+p_v)^{-\varrho} + O\left(e^{t_0/(2KT)}\right)\\ &\le (1+o(1))n^{-1}p_v^{-1} \;\le\; 3.\end{aligned}$$

Furthermore, also from Lemma 6,

$$\mathbf{Pr}(\mathcal{A}_v(3n)) \le (1+o(1))(1+p_v)^{-3n} \le e^{-1}. \tag{27}$$

Suppose next that $v \in V_2$. It follows from Lemmas 7 and 8 that we can find $w \in V_1$ such that $dist(v,w) \le \omega$. Hence from (27), with $\nu = 3n+\omega$, we have

$$\mathbf{Pr}(\mathcal{A}_v(\nu)) \le 1 - (1-e^{-1})(rs)^{-\omega}, \tag{28}$$

since if our walk visits w, it will with probability at least $(rs)^{-\omega}$ visit v within the next ω steps. Thus if $\zeta = (1-e^{-1})(rs)^{-\omega}$,

$$\begin{aligned}\sum_{\sigma\ge t_0}\mathbf{Pr}(\mathcal{A}_v(\sigma)) &\le \sum_{\sigma\ge t_0}(1-\zeta)^{\lfloor\sigma/\nu\rfloor} \;\le\; \sum_{\sigma\ge t_0}(1-\zeta)^{\sigma/(2\nu)}\\ &= \frac{(1-\zeta)^{t_0/(2\nu)}}{1-(1-\zeta)^{1/(2\nu)}} \;\le\; 3\nu\zeta^{-1}.\end{aligned} \tag{29}$$

Thus for all $u \in V$, recalling that $|V_2| \le (rs)^{3\omega}$ (lemma 7) and assuming that $(rs)^{4\omega} = o(\log n)$,

$$C_u \;\le\; t_0 + 3|V_1| + 3|V_2|\nu\zeta^{-1} \;=\; t_0 + O((rs)^{4\omega}n) \;=\; t_0 + o(t_0). \tag{30}$$

We conclude that

$$C(H) \;=\; \max_{v\in V} C_u \;\le\; (1+o(1))t_0.$$

4.2. Lower bound on the cover time $C(H)$

For any vertex u, we can find a set of vertices S, such that at time $t_1 = t_0(1-o(1))$, the probability the set S is covered by the walk $\mathcal{W}_u$ tends to zero. Hence $T_G(u) > t_1$ **whp** which implies that $C(H) \ge (1-o(1))t_0$. We construct S as follows. Let $S \subseteq V_1$ be some maximal set of locally tree-like vertices all of which are at least distance $2\omega+1$ apart. Thus $|S| \ge (n-(rs)^{3\omega})(rs)^{-(2\omega+1)}$.

Let $S(t)$ denote the subset of S which has not been visited by $\mathcal{W}_u$ in the interval $[T,t]$. Now, using Lemma 6,

$$\mathbf{E}|S(t)| \;=\; (1-o(1))\sum_{v\in S}\left(\frac{1+o(1)}{(1+p_v)^t} + o(n^{-2})\right).$$

Setting $t_1 = (1-\epsilon)t_0$ where $\epsilon = 2\omega^{-1}$, we have

$$\mathbf{E}|S(t_1)| \;=\; (1+o(1))|S|e^{-(1-\epsilon)t_0 p_v} \;\geq\; (1+o(1))\frac{n^{2/\omega}}{(rs)^{2\omega+1}} \;\geq\; n^{1/\omega}. \qquad (31)$$

Let $Y_{v,t}$ be the indicator for the event that $\mathcal{W}_u$ has not visited vertex v at time t. Thus $\sum_{v\in S} Y_{v,t} = |S(t)|$. Let $Z = \{v, w\} \subset S$. It can be shown, by merging v and w into a single node Z and using Lemma 6, that

$$\mathbf{E}(Y_{v,t_1}Y_{w,t_1}) = \frac{1+o(1)}{(1+p_Z)^{t_1+2}} + o(n^{-2}), \qquad (32)$$

where $p_Z \sim p_v + p_w$. Thus

$$\mathbf{E}(Y_{v,t_1}Y_{w,t_1}) = (1+o(1))\mathbf{E}(Y_{v,t_1})\mathbf{E}(Y_{w,t_1}). \qquad (33)$$

Using (31) and (33), it can be shown that

$$\mathbf{Pr}(|S(t_1)| > T) \;\geq\; \frac{(\mathbf{E}|S(t_1)| - T)^2}{\mathbf{E}((|S(t_1)| - T)^2)} = 1 - o(1).$$

Since at most T of $S(t_1)$ can be visited in the first T steps, the probability that not all vertices are covered at time t_1 is equal to $1 - o(1)$, so $C(H) \geq t_1$.

5. Proof of Theorem 2: estimate $I(H)$ and $C_E(H)$

The estimation of $I(H)$ and $C_E(H)$ is done very similarly as in Sections 4.1 and 4.2. We briefly outline only the upper bound proof for $I(H)$. Let $I_u(H)$ be the expected time for $\mathcal{W}_u$ to inform all vertices. Then for $t \geq T$, similarly to (26),

$$I_u(H) \leq t + \sum_{v\in V}\sum_{\sigma\geq t} \mathbf{Pr}(\mathcal{B}_v(\sigma))$$

where $\mathcal{B}_v(\sigma)$ is the event that vertex v is not informed in the interval $[T, \sigma]$ (as defined in the statement of Lemma 9). Let

$$t_0 = (1+o(1))\left(1 + \frac{s-1}{(r-1)(s-1)-1}\right)\frac{n}{s-1}\log n.$$

For tree-like vertices v we use Lemma 6, applied to graph $\Gamma(v)$ and vertex $\gamma(v)$ with $p_{\gamma(v)}$ from (22), and Lemma 9. For non-tree-like vertices we use the argument as in (27)-(29) and obtain $I_u(H) \leq t_0 + o(t_0)$.

The estimation of $C_E(H)$ is based on Lemma 6, applied to graph $\Gamma(e)$ for a tree-like edge e with $p_{\gamma(e)}$ from (23), and Lemma 10.

6. Conductance of graphs $G(H)$, $\Gamma(v)$ and $\Gamma(e)$: proof of Lemma 11

In this section we estimate the conductance of graphs $G(H)$, $\Gamma(v)$ and $\Gamma(e)$. We first show in Lemma 15 a bound, in the configuration model, on the number of hyperedges which are almost fully contained within the same set of at most $n/2$ vertices. This lemma will imply an $\Omega(1)$ bound on the conductance of each of the graphs $G(H)$, $\Gamma(v)$ and $\Gamma(e)$, as stated in Lemma 11.

Lemma 15. *Suppose that $r \geq 2$, $s \geq 3$. There exist constants $\delta > 0$ and $\epsilon > 0$ such that if $s = O(n^{\delta})$ and a configuration C is sampled u.a.r. from $\mathcal{C}(n, r, s)$, then the probability that the following property holds is $o(n^{-1/8})$: there is a subset S of $t \leq n/2$ vertices and a subset F of $rt(1-\epsilon)/s$ hyperedges in the multi-hypergraph $H(C)$ such that each hyperedge in F has at least $(1-\epsilon)s$ vertices in S.*

Proof. Let $N(t, k, r, s)$ be the expected number of sets S of configuration vertices of size t such that the number of hyperedges containing at least $(1-\epsilon)s$ vertices from S is at least $k = rt(1-\epsilon)/s$. We will prove the lemma by showing that

$$\sum_{t=1}^{n/2} N(t, k, r, s) \; = \; o(n^{-1/8}). \tag{34}$$

For a given subset of ks points, the probability that these points form k hyperedges is equal to

$$\frac{F(ks)F(rn-ks)}{F(rn)},$$

where $F(a) = a!/((a/s)!(s!)^{(a/s)})$, for $s \mid a$, is the number of partitions of a elements into pairwise-disjoint unordered sets of size s.

Consider a fixed subset S of t vertices. To form k hyperedges which have large intersection with S, we first select $(1-\epsilon)ks$ points from the rt points corresponding to the vertices in S, and then we select the remaining ϵks points. Thus we have the following bound.

$$\begin{aligned} N(t,k,r,s) \leq & \binom{n}{t}\binom{rt}{(1-\epsilon)ks}\binom{rn-(1-\epsilon)ks}{\epsilon ks}\frac{F(ks)F(rn-ks)}{F(rn)} \\ \leq & \binom{n}{t}\binom{rt}{ks}\binom{ks}{\epsilon ks}\binom{rn}{\epsilon ks}\frac{F(ks)F(rn-ks)}{F(rn)}. \end{aligned} \tag{35}$$

Inequality (35) holds because for $p \geq q \geq j \geq 0$,

$$\binom{p}{j} \leq \binom{p}{q}\binom{q}{q-j}.$$

Note that if $s \mid a, b$ and $a > b$, then

$$\frac{F(b)F(a-b)}{F(a)} = \frac{\binom{a/s}{b/s}}{\binom{a}{b}} = \Theta(\sqrt{s})\left(\frac{b}{a}\right)^{b(s-1)/s}\left(1-\frac{b}{a}\right)^{(a-b)(s-1)/s}. \tag{36}$$

The second equality in (36) follows from the fact that Stirling's formula for the factorial gives the following approximation for the binomial coefficients, where $p \geq q \geq 1$ are integers:

$$\binom{p}{q} = \Theta(1)\sqrt{\frac{p}{q(p-q)}}\left(\frac{p}{q}\right)^{q}\left(1-\frac{q}{p}\right)^{q-p}. \tag{37}$$

We will use also the following bound on the binomial coefficients, which can be derived using Stirling's formula.

$$\binom{p}{q} \leq \left(\frac{pe}{q}\right)^{q}. \tag{38}$$

Using (36), (37) and (38) we get the following bounds on the factors of the right-hand side of (35):

$$\begin{aligned}
\frac{F(ks)F(rn-ks)}{F(rn)} &= \Theta(\sqrt{s})\left(\frac{ks}{rn}\right)^{k(s-1)}\left(1-\frac{ks}{rn}\right)^{(rn-ks)(s-1)/s} \\
\binom{n}{t} &= \Theta(1)\sqrt{\frac{n}{t(n-t)}}\left(\frac{n}{t}\right)^{t}\left(1-\frac{t}{n}\right)^{t-n} \\
\binom{rt}{ks} &= \Theta(1)\sqrt{\frac{rt}{ks(rt-ks)}}\left(\frac{rt}{ks}\right)^{ks}\left(1-\frac{ks}{rt}\right)^{ks-rt} \\
\binom{ks}{\epsilon ks} &\leq \left(\frac{e}{\epsilon}\right)^{\epsilon ks} \\
\binom{rn}{\epsilon ks} &\leq \left(\frac{rne}{\epsilon ks}\right)^{\epsilon ks}
\end{aligned}$$

Thus, assuming $t \leq n/2$ and that $ks = rt(1-\epsilon)$ where $\epsilon > 0$ constant,

$$\begin{aligned}
N(t,k,r,s) =& O(\sqrt{s})\left(\frac{rn}{ks}\right)^{k}\left(\frac{n}{t}\right)^{t-ks}\left(\frac{e}{\epsilon}\right)^{\epsilon ks}\left(\frac{rne}{\epsilon ks}\right)^{\epsilon ks} \\
&\times\left(1-\frac{ks}{rn}\right)^{(rn-ks)(s-1)/s}\left(1-\frac{ks}{rt}\right)^{ks-rt}\left(1-\frac{t}{n}\right)^{t-n} \\
=& O(\sqrt{s})\left(\frac{n}{(1-\epsilon)t}\right)^{rt(1-\epsilon)/s}\left(\frac{t}{n}\right)^{rt(1-\epsilon)-t}\left(\frac{e}{\epsilon}\right)^{2\epsilon tr}\left(\frac{1}{1-\epsilon}\right)^{\epsilon tr}\left(\frac{n}{t}\right)^{\epsilon tr} \\
&\times\left(1-\frac{t(1-\epsilon)}{n}\right)^{r(n-t(1-\epsilon))(s-1)/s}\left(\frac{1}{\epsilon}\right)^{\epsilon rt}\left(1-\frac{t}{n}\right)^{t-n} \\
=& O(\sqrt{s})\left(\Psi_t\right)^{tr}\left(\frac{(1-t(1-\epsilon)/n)^{r(1-1/s)}}{1-t/n}\right)^{n-t},
\end{aligned} \tag{39}$$

where

$$\Psi_t = e^{2\epsilon}\left(\frac{1}{1-\epsilon}\right)^{(1-\epsilon)/s+\epsilon}\left(\frac{1}{\epsilon}\right)^{3\epsilon}\left(1-\frac{t(1-\epsilon)}{n}\right)^{\epsilon(1-1/s)}\left(\frac{t}{n}\right)^{(1-2\epsilon)(1-1/s)-1/r-\epsilon}. \tag{40}$$

To establish an upper bound, we first consider the third (last) factor in (39). We write

$$\frac{(1-t(1-\epsilon)/n)^{r(1-1/s)}}{1-t/n} = \left(1+\frac{\epsilon t}{n-t}\right)^{r(1-1/s)} \left(1-\frac{t}{n}\right)^{r(1-1/s)-1}$$
$$\le \exp\left\{\frac{t}{n}\left(1-r(1-1/s)(1-2\epsilon)\right)\right\}.$$

Now $r(1-1/s) \ge 4/3$ and so if $\epsilon < 1/8$ we find that the contribution of the third factor in (39) is less than one. Hence

$$N(t,k,r,s) = O(\sqrt{s})\,(\Psi_t)^{tr}. \tag{41}$$

Considering the factor $(\Psi_t)^{tr}$ in (39), we have

$$\Psi_t \le e^{2\epsilon}\left(\frac{1}{1-\epsilon}\right)^{(1-\epsilon)/s+\epsilon}\left(\frac{1}{\epsilon}\right)^{3\epsilon}\left(\frac{t}{n}\right)^{(1-\epsilon)(1-1/s)-1/r-\epsilon} \tag{42}$$
$$\le e^{2\epsilon}\left(\frac{1}{1-\epsilon}\right)^{(1-\epsilon)/3+\epsilon}\left(\frac{1}{\epsilon}\right)^{3\epsilon}\left(\frac{t}{n}\right)^{(1-\epsilon)(2/3)-1/2-\epsilon} \tag{43}$$
$$\le e^{2\epsilon}\left(\frac{1}{1-\epsilon}\right)^{(1-\epsilon)/3+\epsilon}\left(\frac{1}{\epsilon}\right)^{3\epsilon}\left(\frac{1}{2}\right)^{(1-\epsilon)(2/3)-1/2-\epsilon}. \tag{44}$$

Inequality (42) follows from (40). For Inequality (43), we use $r \ge 2$ and $s \ge 3$, and assume that ϵ is small enough so that $(1-\epsilon)(2/3) - 1/2 - \epsilon > 0$. Inequality (44) holds because $t \le n/2$. The expression in (44) tends to $(1/2)^{1/6} < 1$ as ϵ tends to 0, so for some constant $c < 1$ and sufficiently small ϵ, $\Psi_t \le c$. Thus for $t_0 = \log n/\log(1/c)$,

$$\sum_{t=t_0}^{n/2} N(t,k,r,s) \le O(\sqrt{s}) \sum_{t=t_0}^{n/2} \Psi_t^{tr} = O\left(\sqrt{s}\,c^{rt_0}\right) = O\left(\frac{\sqrt{s}}{n^2}\right) = O(n^{-1}). \tag{45}$$

For $t = O(\log n)$, Inequality (43) implies

$$\Psi_t \;\le\; O(1)\left(\frac{1}{n}\right)^{(1-o(1))((1-\epsilon)(2/3)-1/2-\epsilon)} \le o(n^{-1/7}),$$

with the last inequality holding for sufficiently small constant $\epsilon > 0$. Thus, if $s = O(n^\delta)$ for a suitably small constant $\delta > 0$,

$$\sum_{t=1}^{t_0} N(t,k,r,s) = o\left(\sqrt{s}\,n^{-1/7}\log n\right) = o(n^{-1/8}). \tag{46}$$

The bounds (45) and (46) give the bound (34). □

Lemma 15 and the relation between the configuration model and random simple hypergraphs discussed in Section 3.2 give the following corollary.

Corollary 16. *For $r \geq 2$, $s \geq 3$, and $rs = o(\log n)$, there exists $\epsilon > 0$ such that* **whp** *a random simple r-regular s-uniform hypergraph does not have a subset S of $t \leq n/2$ vertices and a subset F of $(1-\epsilon)rt/s$ hyperedges such that each hyperedge in F has at least $(1-\epsilon)s$ vertices from S.*

. Proof of Lemma 11. Going back to (6) we see that if G is a d-regular graph then $\Phi_G(S) = \frac{e(S:\overline{S})}{d|S|}$, were $e(S:\overline{S})$ denotes the number of edges with one endpoint in S and the other in $\overline{S} = V \setminus S$. Note that in this case $\pi(S) \leq 1/2$, if $|S| \leq n/2$.

Let $H \in \mathcal{G}(n,r,s)$ be a random hypergraph. For a subset S of vertices of H, let F_S be the set of all hyperedges which contain at least $(1-\epsilon)s$ vertices from S, where the constant $\epsilon > 0$ is from Corollary 16. Corollary 16 implies that **whp** for each S of size $t \leq n/2$, F_S contains at most $(1-\epsilon)rt/s$ hyperedges. If F_S contains at most $(1-\epsilon)rt/s$ hyperedges, then there are at most $(1-\epsilon)rt$ pairs (v,e) such that $v \in S$, $v \in e \in E(H)$, and $|e \cap S| \geq (1-\epsilon)s$. This implies that there are at least ϵrt pairs (v,e) such that $v \in S$, $v \in e \in E(H)$, and $|e \cap \overline{S}| \geq \epsilon s$. Since each such pair (v,e) contributes at least ϵs to $e(S:\overline{S})$ in $G(H)$, we have

$$\Phi_G(S) = \frac{e(S:\overline{S})}{d|S|} \geq \frac{\epsilon^2 rts}{r(s-1)t} = \Omega(1).$$

Thus **whp** the conductance of graph $G(H)$ is $\Omega(1)$.

Considering the contracted graph $\Gamma = \Gamma(v), \Gamma(e)$, note first that contracting vertices cannot reduce conductance. This is because we minimise the same $\Phi(S)$ value over a smaller collection of sets S. It is a simple matter to see that subdividing at most $r\binom{s}{2}$ edges within S increases the degree of S by at most $rs(s-1)$ and thus $\Phi_\Gamma = \Omega(1)$ **whp**. □

The above proof of Lemma 11 repeated for a random multi-hypergraph $H \in \mathcal{M}(n,r,s)$, using directly Lemma 15 instead of Corollary 16, gives the constant conductance and logarithmic mixing time for $\mathcal{M}(n,r,s)$.

Lemma 17. *Let $r \geq 2$ and $3 \leq s = O(n^\delta)$, for a suitably small constant $\delta > 0$. For a random multi-hypergraph $H \in \mathcal{M}(n,r,s)$,* **whp** *the conductance of each of the graphs $G(H)$, $\Gamma(v)$ and $\Gamma(e)$ is $\Omega(1)$, and their mixing time satisfying condition* (8) *is $T = O(\log n)$.*

7. Returns to a tree-like vertex (proof of Lemma 13)

7.1. Returns in $G(H)$

We consider the random walk in $G(H)$. For a vertex v and integer $k \geq 1$, let $N_k(v)$ denote the set of vertices w for which there is a path of length at most k from v to w. The following construction models an infinite extension of the neighbourhood of v in $G = G(H)$, for a tree-like vertex v. Let $\mathcal{T}_G^*$ be an infinite graph (with a tree-like structure) defined recursively as a root h joined to each vertex of $r-1$ disjoint cliques $C_1, C_2, \ldots, C_{r-1}$ of size $s-1$. Each vertex in

$C_1 \cup \cdots \cup C_{r-1}$ is the root of a further disjoint copy of $\mathcal{T}_G^*$. For $\mathcal{T}_G$ we take a root vertex h and join it to each vertex of r disjoint cliques $C_1, C_2, \ldots, C_r$ of size $s-1$. Each vertex in $C_1 \cup \cdots \cup C_r$ is the root of a disjoint copy of $\mathcal{T}_G^*$. If v is tree-like in H, then provided $k \le \omega$, the subgraph of $G(H)$ induced by $N_k(v)$ is isomorphic to the first k levels of $\mathcal{T}_G$ (that is, the root plus the k levels).

We first compute the expected number of returns R_G to the root for a random walk on $\mathcal{T}_G$. We can then argue as in the proof of Lemma 8 of [7] that $R_v = R_G + o(1)$ for a tree-like vertex v of $G(H)$. We can project a walk on $\mathcal{T}_G$ onto the non-negative integers by mapping a vertex v of $\mathcal{T}_G$ to its distance Δ_v from the root. Each vertex $v \in \mathcal{T}_G$ has degree $(s-1)r$, and if $v \ne h$ and the walk is at v, then it moves to a neighbour w where

$$\Delta_w = \begin{cases} \Delta_v + 1 & \text{probability } \frac{r-1}{r} \\ \Delta_v & \text{probability } \frac{s-2}{r(s-1)} \\ \Delta_v - 1 & \text{probability } \frac{1}{r(s-1)} \end{cases} \tag{47}$$

Now R_G is the expected number of returns to the origin of a random walk on the non-negative integers, with probabilities defined as in (47).

We note the following result (see e.g. [9]), for a random walk on the non-negative integers $\{0, 1, \ldots\}$ with transition probabilities at $k > 0$ of $q < p$ for moves left and right respectively. Starting at vertex 1, the probability of ultimate return to the origin 0 is

$$\rho = \frac{q}{p}. \tag{48}$$

Since the walk always moves to 1 from the origin, then the expected number of returns R to the origin is given by

$$R_G = \frac{1}{1-\rho}. \tag{49}$$

For the random walk on $\mathcal{T}_G$, (47) gives $p = (r-1)/r$ and $q = 1/(r(s-1))$, so

$$\rho = \frac{1}{(r-1)(s-1)}. \tag{50}$$

For a tree-like vertex v and each $t = 0, 1, \ldots, \omega$, the probability that the random walk on $G(H)$ starting from v is at v at step t is the same as the probability that the random walk on the non-negative integers starting from 0 is at 0 at step t. For the random walk on the non-negative integers, it can be shown that the expected number of returns to 0 after step ω is $o(1)$. For the random walk on $G(H)$, the number of returns to v in steps $\omega, \omega+1, \ldots, T-1$ is also $o(1)$ (this can be shown using (7) and the assumption that $T\pi_v = o(1)$). Thus we have

$$R_v = R_G + o(1) = \frac{(r-1)(s-1)}{(r-1)(s-1)-1} + o(1). \tag{51}$$

Finally, for tree-like vertices v we have that the value of p_v in (12) is given by

$$p_v = (1 + o(1))\frac{1}{n}\frac{(r-1)(s-1)-1}{(r-1)(s-1)}. \tag{52}$$

7.2. Returns in $\Gamma(v)$

The following construction models an infinite extension of the neighbourhood of γ in $\Gamma = \Gamma(v)$, for a tree-like vertex v. Let $\mathcal{T}_\Gamma$ be an infinite multi-graph consisting of a root h (corresponding to γ) joined to $r(s-1)$ distinct vertices $w_{i,j}, i = 1, 2, \dots, r,\ j = 1, 2, \dots s-1$ (corresponding to the vertices in cliques with v, which are neighbours of γ in Γ) and with $s-1$ parallel edges between h and each $w_{i,j}$. Each vertex $w = w_{i,j}$ is the root of a disjoint copy of the infinite tree $\mathcal{T}_G^*$ defined in Section 7.1.

Consider the random walk on $\mathcal{T}_\Gamma$ starting from the root h. For the first return to h to happen, the walk first moves from h to its neighbour w, then it leaves vertex w and eventually comes back to w (not visiting h), repeating such "looping" for $k \geq 0$ times, and finally, on the k-th return to w, the walk goes back to h for the first time. Thus the probability P_γ of a return to h of a walk on $\mathcal{T}_\Gamma$ starting at h is given by

$$P_\gamma = \sum_{k=0}^{\infty} \rho^k (1-\hat{\rho})^k \hat{\rho} = \frac{\hat{\rho}}{1-\rho(1-\hat{\rho})} \tag{53}$$

where ρ is the probability of a return to the root w of a $\mathcal{T}_G^*$ given in (50), and

$$\hat{\rho} = \frac{s-1}{s-1+(r-1)(s-1)} = \frac{1}{r} \tag{54}$$

is the probability of moving from a $w_{i,j}$ to the root h in a single step. Plugging the values (50) and (54) into (53) gives

$$P_\gamma = \frac{s-1}{r(s-1)-1}.$$

Therefore, the number of returns to the root of $\mathcal{T}_\Gamma$ is $R_\Gamma = 1/(1-P_\gamma)$, and using arguments as in Section 7.1, we get

$$R_\gamma = R_\Gamma + o(1) = \frac{r(s-1)-1}{(r-1)(s-1)-1} + o(1). \tag{55}$$

Thus for a tree-like vertex v, using (13), the value of $p_{\gamma(v)}$ in (12) is given by

$$p_{\gamma(v)} = (1+o(1))\frac{s-1}{n}\frac{(r-1)(s-1)-1}{r(s-1)-1}. \tag{56}$$

7.3. Returns in $\Gamma(e)$

Let e be a tree-like hyperedge in H. Let $\mathcal{T}'_\Gamma$ be an infinite multi-graph consisting of a root h (corresponding to $\gamma(e)$ in $\Gamma(e)$) joined to s distinct vertices $w_i, i = 1, 2, \dots s$ (corresponding to the vertices in the clique of edge e) and with $s-1$ parallel edges between h and each w_i. Each vertex $w = w_i$ is the root of a disjoint copy of the infinite tree $\mathcal{T}^*_G$ defined in Section 7.1.

The corresponding random walks on non-negative integers are exactly the same for $\mathcal{T}'_\Gamma$ and for $\mathcal{T}_\Gamma$. Hence, for a tree-like hyperedge e, the expected number R_γ of returns to γ in $\Gamma(e)$ is as in (55). Therefore, using (18), the value of $p_{\gamma(e)}$ in (12) is given by

$$p_{\gamma(e)} = (1+o(1))\frac{s}{rn}\frac{(r-1)(s-1)-1}{r(s-1)-1}. \tag{57}$$

8. Technical condition (10) of Lemma 6

In our analysis of the cover and inform times of a random hypergraph $H \in \mathcal{G}(n, r, s)$, we apply Lemma 6 to graph $G(H)$ and to a tree-like vertex v, to graph $\Gamma(v)$ for a tree-like vertex v and to vertex $\gamma(v)$, and to graph $\Gamma(e)$ for a tree-like hyperedge e and to vertex $\gamma(e)$. We show in this section that the condition (10) of Lemma 6 holds in each of these three cases. (Thus this section contains the proof of Lemma 14.)

8.1. Graph $G(H)$

We consider a tree-like vertex v in graph $G(H)$. Let $r_t = P_v^{(t)}(v)$, the probability that the random walk which starts from vertex v is back at v at step t. Thus $R_v(T,z) = \sum_{t=0}^{T-1} r_t z^t$ and $R_v = R_v(T,1) = \sum_{t=0}^{T-1} r_t$. Observe first that the condition (10) holds whenever $R_v \le 2-\epsilon$ for some constant $\epsilon > 0$. Indeed, for $|z| \le 1+\lambda$, where $\lambda = 1/KT$,

$$\begin{aligned} |R_v(T,z)| &\ge r_0 - (1+\lambda)^T \sum_{t=1}^{T-1} r_t \\ &= 1 - (1+\lambda)^T (R_v - 1) \ \ge\ 1 - (1+\lambda)^T(1-\epsilon) \ >\ \epsilon/2. \end{aligned} \tag{58}$$

The last inequality above holds because K is a suitably large constant.

We consider graph $G(H)$ and a tree-like vertex v. We write (51) as

$$R_v = 1 + \frac{1}{(r-1)(s-1)-1} + o(1)$$

and see that if $r \ge 3$ or $s \ge 4$, then $R_v < 5/3$, so we only need to consider the case $r = 2$ and $s = 3$. For any z,

$$|R_v(T,z)| \geq |R_v| - |R_v - R_v(T,z)| = R_v - \left|\sum_{j=1}^{T-1} r_j(z^j - 1)\right|$$
$$\geq R_v - \sum_{j=1}^{T-1} r_j|z^j - 1|. \quad (59)$$

Now $R_v \sim 2$ in our case, so to show that $|R_v(T,z)| \geq \epsilon$ for some constant $\epsilon > 0$ we only need to show that the sum $\sum_{j=1}^{T-1} r_j|z^j - 1|$ is at most $2 - \epsilon$. Take z such that $|z| \leq 1 + \lambda$, write $z = |z|\tilde{z}$, where $\tilde{z} = e^{i\theta}$, and obtain

$$\sum_{j=1}^{T-1} r_j|z^j - 1| \leq \sum_{j=1}^{T-1} r_j(|z^j - \tilde{z}^j| + |\tilde{z}^j - 1|) = \sum_{j=1}^{T-1} r_j(|z|^j - 1) + \sum_{j=1}^{T-1} r_j|\tilde{z}^j - 1|$$
$$\leq ((1+\lambda)^T - 1)(R_v - 1) + \sum_{j=1}^{T-1} |\tilde{z}^j - 1|$$
$$\leq \frac{2}{K} + \sum_{j=1}^{T-1} |\tilde{z}^j - 1|. \quad (60)$$

We also have
$$|\tilde{z}^j - 1| = (2(1 - \cos j\theta))^{1/2} = 2|\sin(j\theta/2)|,$$
so
$$\sum_{j=1}^{T-1} r_j|\tilde{z}^j - 1| = 2\sum_{j=1}^{T-1} r_j|\sin(j\theta/2)|. \quad (61)$$

Note that $r_1 = 0$, and use (47) to calculate that $r_2 = \frac{1}{4}$ and $r_3 = \frac{1}{16}$. Suppose first that $\theta \notin I = [\frac{3\pi}{8}, \frac{5\pi}{8}] \cup [\frac{11\pi}{8}, \frac{13\pi}{8}]$. Then $|\sin\theta| \leq \sin\frac{3\pi}{8}$ and so

$$\sum_{j=1}^{T-1} r_j|\sin(j\theta/2)| \leq \sum_{j=1}^{T-1} r_j - r_2\left(1 - \sin\frac{3\pi}{8}\right) \leq R_v - 1 - 2\epsilon \leq 1 - \epsilon, \quad (62)$$

for some constant $\epsilon > 0$. On the other hand, if $\theta \in I$ then $|\sin(3\theta/2)| \leq \sin\frac{7\pi}{16}$ and then

$$\sum_{j=1}^{T-1} r_j|\sin(j\theta/2)| \leq \sum_{j=1}^{T-1} r_j - r_3\left(1 - \sin\frac{7\pi}{16}\right) \leq 1 - \epsilon. \quad (63)$$

Inequality (60) with $K = 2/\epsilon$, and (61), (62) and (63) imply that the sum in (59) is at most $2 - \epsilon$.

8.2. Graphs $\Gamma(v)$ *and* $\Gamma(e)$

We now consider the condition (10) of Lemma 6 for vertex γ in graph $\Gamma(v)$, where v is a tree-like vertex in H, and in graph $\Gamma(e)$, where e is a tree-like hyperedge in H. In both cases R_γ is the same (up to an additive term $o(1)$) and is given in (55). We write this R_γ as

$$R_\gamma = 1 + \frac{s-1}{(r-1)(s-1)-1} + o(1),$$

and see that we only need to consider the case when $r = 2$, since $R_v < 7/4$ for $r \geq 3$. Unfortunately, for $r = 2$, R_v is now strictly greater than 2 and we do not see how the straightforward approach which we used for graph $G(H)$ in Section 8.1 could work in this case. Instead, one could follow a more general approach developed in [8] (see the proof of Lemma 14 in [8]). However, rather than adapting the analysis from [8] to the random walks considered here, we introduce in the following lemma a new result, which can be viewed as simplification and further generalisation of the approach from [8]. This lemma immediately implies that the condition (10) holds for vertex γ in graphs $\Gamma(v)$ and $\Gamma(e)$.

Lemma 18. *Consider an arbitrary n-vertex graph G and a vertex v in G. Let T be a mixing time satisfying (8). If $T = o(n^3)$, $T\pi_v = o(1)$ and R_v is bounded from above by a constant, then the condition* (10) *holds for $\theta = 1/4$ and any constant $K \geq 3R_v$.*

PROOF. For a random walk $\mathcal{W}_v$ on G starting from v, let β_t be the probability that the first return to vertex v is at time t. We consider functions

$$\begin{aligned}\beta(z) &= \sum_{t=1}^{T-1} \beta_t z^t, \\ \alpha(z) &= \sum_{t=0}^{\infty} \alpha_t z^t = 1 + \beta(z) + \beta^2(z) + \beta^3(z) + \cdots = \frac{1}{1-\beta(z)}.\end{aligned}$$

We have $\alpha_0 = 1$ and for $t \geq 1$,

$$\alpha_t = \sum_{k\geq 1} \sum_{\substack{1 \leq t_1, t_2, \ldots, t_k \leq T-1 \\ t_1 + t_2 + \cdot + t_k = t}} \beta_{t_1}\beta_{t_2}\cdots\beta_{t_k}.$$

Thus α_t is the probability of the event Υ_t that the random walk which started at vertex v is at v at time t and has never left v for $T-1$ steps. This means that $\alpha_t = r_t$ for $t \leq T-1$, so

$$R_v(T, z) = \alpha(z) - \sum_{t=T}^{\infty} \alpha_t z^t.$$

We show that the condition (10) holds for $\theta = 1/4$ and any constant $K \geq 3R_v$ by showing that for every z such that $|z| \leq 1 + 1/(KT)$, we have $|\alpha(z)| \geq 1/3$ and $|\sum_{t=T}^{\infty} \alpha_t z^t| = o(1)$.

For integer $\tau \geq 1$, we denote $R_\tau = R_v(\tau, 1)$; in particular, $R_T = R_v$. Let $B_T = \beta(1) = \sum_{t=1}^{T-1} \beta_t$ be the probability that the walk returns to v within T steps; $0 < B_T < 1$. We use the fact that $R_T(1 - B_T) \sim 1$, which can be shown in the following way. Since

$$\begin{aligned} R_T &= 1 + \sum_{t=1}^{T-1} \beta_t R_{T-1-t} \;\leq\; 1 + B_T R_T, \\ R_{2T} &= 1 + \sum_{t=1}^{2T-1} \beta_t R_{2T-1-t} \;\geq\; \sum_{t=1}^{T-1} \beta_t R_T \;=\; 1 + B_T R_T, \\ R_{2T} - R_T &= \sum_{t=T}^{2T-1} r_t \leq T(\pi_v + n^{-3}) = o(1), \end{aligned}$$

we have

$$R_T + o(1) = 1 + B_T R_T.$$

This implies that

$$B_T = 1 - \frac{1}{(1 + o(1))R_T}. \tag{64}$$

Denoting $\lambda = 1/(KT)$, for each z such that $|z| \leq 1 + \lambda$, we have

$$\begin{aligned} |\alpha(z)| &= \left| \frac{1}{1 - \beta(z)} \right| \;\geq\; \frac{1}{1 + |\beta(z)|} \;\geq\; \frac{1}{1 + \beta(|z|)} \\ &\geq \frac{1}{1 + B_T(1+\lambda)^T} \;\geq\; \frac{1}{1 + (1+\lambda)^T} \;\geq\; \frac{1}{2 + 1/K + o(1)} \;\geq\; \frac{1}{3}. \end{aligned}$$

Next we show that in the circle $|z| \leq 1 + \lambda$, the sum $\sum_{t=T}^{\infty} \alpha_t z^t$, which is the difference between $R_v(T, z)$ and $\alpha(z)$, is $o(1)$. Let $0 = t_0 < t_1 < t_2 < \cdots$ be the steps when the random walk is at vertex v, and let $t \geq T$. The event Υ_t is the union $\bigcup_{j \geq 1} \Upsilon_{t,j}$ of pairwise disjoint events

$$\Upsilon_{t,j} \;=\; \{t_j = t, \text{ and } t_i - t_{i-1} < T \text{ for } i = 1, 2, \ldots, j\}. \tag{65}$$

For $j \geq 1$, we define pairwise disjoint events

$$\Psi_{t,j} \;=\; \{t_j = t, \text{ and } t_i - t_{i-1} < T \text{ for } i = 1, 2, \ldots, \lfloor t/T \rfloor - 1\}. \tag{66}$$

That is, $\Psi_{t,j}$ is the event that each of the first $\lfloor t/T \rfloor - 1$ returns takes fewer than T steps, and the j-th return occurs at step t. We have $\Upsilon_{t,j} = \Psi_{t,j} = \emptyset$, for $j < \lfloor t/T \rfloor - 1$ (because for such j, $t_j < t$), and comparing the definitions (65) and (66), we see that $\Upsilon_{t,j} \subseteq \Psi_{t,j}$, for $j \geq \lfloor t/T \rfloor - 1$. Therefore, with $q = \lfloor t/T \rfloor - 1$,

$$\alpha_t \;=\; \mathbf{Pr}\left[\Upsilon_t\right] \;=\; \mathbf{Pr}\left[\bigcup_{j \geq q} \Upsilon_{t,j}\right] \;\leq\; \mathbf{Pr}\left[\bigcup_{j \geq q} \Psi_{t,j}\right] \;\equiv\; \sigma_t. \tag{67}$$

We now bound σ_t. We have

$$\Psi_t \equiv \bigcup_{j \geq q} \Psi_{t,j} = \{\mathcal{W}_v(t) = v\} \cap \{t_i - t_{i-1} < T, \text{ for } i = 1, 2, \ldots, q\}.$$

We denote

$$\widehat{\Psi}_t = \{t_i - t_{i-1} < T, \text{ for } i = 1, 2, \ldots, q\}.$$

We assume first that $t \geq 2T$, so $q \geq 1$. The step t_q when the q-th return to vertex v occurs is such that $q \leq t_q < qT \leq t - T$. Considering all possible values of t_q, we express $\widehat{\Psi}_t$ as the union of pairwise disjoint events:

$$\begin{aligned}
\widehat{\Psi}_t &= \textstyle\bigcup_{\tau=q}^{t-T} \widehat{\Psi}_{t,\tau}, \\
\widehat{\Psi}_{t,\tau} &= \{t_i - t_{i-1} < T, \text{ for } i = 1, 2, \ldots, q, \text{ and } t_q = \tau\}.
\end{aligned}$$

Using the above notation, we have

$$\begin{aligned}
\sigma_t &= \mathbf{Pr}[\Psi_t] \;=\; \mathbf{Pr}\left[(\mathcal{W}_v(t) = v) \cap \widehat{\Psi}_t\right] \\
&= \mathbf{Pr}\left[\widehat{\Psi}_t\right] \times \mathbf{Pr}\left[\mathcal{W}_v(t) = v \mid \widehat{\Psi}_t\right] \\
&\leq (B_T)^q \times \max_{q \leq \tau \leq t-T} \left\{\mathbf{Pr}\left[\mathcal{W}_v(t) = v \mid \widehat{\Psi}_{t,\tau}\right]\right\} && (68) \\
&= (B_T)^q \times \max_{q \leq \tau \leq t-T} \{\mathbf{Pr}\left[\mathcal{W}_v(t) = v \mid \mathcal{W}_v(\tau) = v\right]\} && (69) \\
&= (B_T)^q \times \max_{q \leq \tau \leq t-T} \{\mathbf{Pr}\left[\mathcal{W}_v(t-\tau) = v\right]\} \\
&\leq (B_T)^q (\pi_v + n^{-3}). && (70)
\end{aligned}$$

Inequality (68) holds because $\widehat{\Psi}_t$ is the event that each of the first q returns to vertex v takes fewer than T steps. The differences $t_i - t_{i-1}$ are independent of each other and, by definition, $\mathbf{Pr}[t_i - t_{i-1} < T] = B_T$, so $\mathbf{Pr}\left[\widehat{\Psi}_t\right] = (B_T)^q$. Equation (69) holds because the random walk is a Markov chain, so the only relevant property of the condition $\widehat{\Psi}_{t,\tau}$ is that the walk is at vertex v at step τ. Finally, Inequality (70) follows from (8), because $t - \tau \geq T$.

If $T \leq t < 2T$, then $q = 0$ and we simply have $\Psi_t = \{\mathcal{W}_v(t) = v\}$, so in this case the bound (70) holds as well.

Using (67) and (70), if $|z| \leq 1 + \lambda$, then

$$\begin{aligned}
\left|\sum_{t=T}^{\infty} \alpha_t z^t\right| &\leq \sum_{t=T}^{\infty} \alpha_t |z|^t \;\leq\; \sum_{t=T}^{\infty} \sigma_t (1+\lambda)^t \;\leq\; \sum_{q \geq 0} \sum_{t=(q+1)T}^{(q+2)T-1} \sigma_t (1+\lambda)^t \\
&\leq \sum_{q \geq 0} T (B_T)^q (\pi_v + n^{-3})(1+\lambda)^{(q+2)T} \\
&\leq T(\pi_v + n^{-3})(1+\lambda)^{2T} \sum_{q \geq 0} \left(B_T (1+\lambda)^T\right)^q \\
&\leq o(1) \sum_{q \geq 0} \left(B_T e^{1/K}\right)^q && (71) \\
&= o(1) \frac{1}{1 - B_T e^{1/K}} \\
&\leq o(1) \frac{1}{1 - e^{1/K}(1 - 1/(2R_T))} = o(1). && (72)
\end{aligned}$$

Inequality (71) follows from the assumption that $T\pi_v = o(1)$ and $T = o(n^3)$. The inequality on line (72) follows from (64). The final $o(1)$ bound on line (72) holds for any constant $K \geq 3R_T$. $\square$

9. Proof of Theorem 4

Let $\mathbf{E}_\pi(H_v)$ denote the expected hitting time of a vertex v from the stationary distribution π. The quantity $\mathbf{E}_\pi(\boldsymbol{H}_v)$ can be expressed as $\mathbf{E}_\pi(H_v) = Z_{vv}/\pi_v$, where the value of $Z_{vv} > 1$ is given by $Z_{vv} = \sum_{t=0}^{\infty}(P_v^{(t)}(v) - \pi_v)$. These relationships were stated in (2) and (3), but we repeat them here for convenience.

Let $\mathcal{A}_t(v) = \mathcal{A}_{t,u}(v)$ denote the event that a random walk $\mathcal{W}_u$ does not visit vertex v in steps $0, ..., t$ (as before, but now looking at all steps, including the initial T steps). Our proof of Theorem 4 uses the following crude bound for $\mathbf{Pr}(\mathcal{A}_t(v))$ in terms of $\mathbf{E}_\pi(\boldsymbol{H}_v)$.

Lemma 19. *Let T be a mixing time of a random walk $\mathcal{W}_u$ on G satisfying* (8). *Then*

$$\mathbf{Pr}(\mathcal{A}_t(v)) \leq \exp\left(-\lfloor t/(T + 3\mathbf{E}_\pi(\boldsymbol{H}_v))\rfloor\right).$$

PROOF. Let $\rho \equiv P_u^{(T)}$ be the distribution of $\mathcal{W}_u$ on G after T steps, and let $\mathbf{E}_\rho(\boldsymbol{H}_v)$ be the expected time to hit v starting from ρ. As $\pi_x = \Omega(1/n^2)$ for any connected graph, it follows from (8) that

$$\mathbf{E}_\rho(\boldsymbol{H}_v) = (1 + o(1))\mathbf{E}_\pi(\boldsymbol{H}_v). \tag{73}$$

Let $H_v(\rho)$ be the time to hit v starting from ρ, and let $\tau = T + 3\mathbf{E}_\pi(\boldsymbol{H}_v)$. Then

$$\mathbf{Pr}(\mathcal{A}_t(v)) \leq \mathbf{Pr}[\boldsymbol{H}_v(\rho) \geq 3\mathbf{E}_\pi(\boldsymbol{H}_v)] \leq \frac{1}{e}.$$

By restarting the process $\mathcal{W}_u$ at $\mathcal{W}(0) = u, \mathcal{W}(\tau), \mathcal{W}(2\tau), \ldots, \mathcal{W}(\lfloor t/\tau\rfloor\tau)$ we obtain

$$\mathbf{Pr}(\mathcal{A}_\tau(v)) \leq e^{-\lfloor t/\tau\rfloor}.$$

$\square$

Lemma 20. *Let T be a mixing time of a random walk $\mathcal{W}_u$ on a graph G with conductance Φ_G, and satisfying* (8). *Let vertex $v \in V$ be such that $T \cdot \pi_v = o(1)$, then*

$$\mathbf{E}_\pi(\boldsymbol{H}_v) = O\left(\frac{1}{\Phi_G^2 \pi_v}\right). \tag{74}$$

PROOF. Using (7) with $x = u = v$, then

$$|P_v^t(v) - \pi_v| \leq e^{-\Phi_G^2 t/2},$$

and

$$Z_{vv} = \sum_{t\geq 0}(P_v^t(v) - \pi_v) \leq \sum_{t\geq 0} e^{-\Phi_G^2 t/2} = O\left(\frac{1}{\Phi_G^2}\right). \tag{75}$$

The bound (74) follows from (75) and (2). $\square$

We are now completing the proof of Theorem 4. Let H be a random multi-hypergraph from $\mathcal{M}(n, r, s)$. As usual we replace the random walk on H with a walk on the multi-graph $G(H)$. To estimate $I(H)$ we consider the inform contraction graph $\Gamma(v)$, with contracted vertex γ.

From Lemma 17 have that **whp** $\Phi_{G(H)}$, $\Phi_{\Gamma(v)} \geq c$, for some constant $c > 0$, and the mixing time of a random walk on H, $G(H)$, and $\Gamma(v)$ is $T = O(\log n)$. Assume that H has these properties. In H and $G(H)$, $\pi_u = 1/n$ for all u, and in $\Gamma(v)$, $\pi_\gamma = (s-1)/(n+s-2)$. Thus for any x, and $s \leq n$,

$$|P_u^{(t)}(x) - \pi_x| \leq n^{1/2} e^{-\Phi^2 t/2}.$$

From Lemma 20, as $\Phi_{\Gamma(v)} > c$ constant we have, for some constant $B > 0$,

$$\mathbf{E}_\pi(\boldsymbol{H}_\gamma) \;\leq\; \frac{Bn}{s}, \tag{76}$$

From Lemma 19 we have, for a random walk $\mathcal{W}_u$ on $\Gamma(v)$ that

$$\mathbf{Pr}(\mathcal{A}_t(\gamma)) \leq \exp(-\lfloor t/(T + 3\mathbf{E}_\pi(\boldsymbol{H}_\gamma))\rfloor). \tag{77}$$

Let $t^* = 3(T + 3Bn/s)\log n = O((n/s)\log n)$; recall that $s = O(n^\delta)$ for a positive constant $\delta < 1$. Then (76) and (77) imply

$$\mathbf{Pr}(\mathcal{A}_{t^*}(\gamma)) \leq e^{-\lfloor 3\log n\rfloor} \leq n^{-2},$$

and

$$\mathbf{Pr}(\text{there exists a vertex } v \text{ not informed at } t^*) \leq n^{-1}.$$

Finally, for each vertex u in H, the inform time $I_u(H)$ when walks start from u is

$$\begin{aligned} I_u(H) &= \sum_{t\geq 0} \mathbf{Pr}(\text{there exists a vertex } v \text{ not informed at } t) \\ &\leq t^* + t^* \sum_{k\geq 1} \mathbf{Pr}(\text{there exists a vertex } v \text{ not informed at } kt^*) \\ &\leq t^* + t^* \sum_{k\geq 1} n^{-k} \;=\; (1+o(1))t^*. \end{aligned}$$

10. Conclusions

We have considered random walks on hypergraphs and the following scenario: when the walk passes in the current step from a vertex v to a vertex u inside a hyperedge e, then all vertices in e are aware that something is happening, say, they all become "informed." This scenario leads to a natural question of calculating, or estimating, for a given hypergraph H its *inform time* $I(H)$, that is, the expected number of steps needed to inform all vertices. In particular, we would like to calculate the speed-up of the inform time over the traditional cover time, when vertices become aware of the activity only when the walk passes directly through them.

We have derived precise estimations of the cover time and inform time (and the edge cover time) for random r-regular s-uniform hypergraphs, where $r \geq 2$ and $s \geq 3$ are constants. Our formulas show that for such hypergraphs the speed-up of the inform time is roughly $s(1-1/r)$, for a large s. It seems natural to hope for a $\Omega(s)$ speed-up for s-uniform hypergraphs, since the "informing process" affects up to $s-1$ new vertices in each step, while the traditional "covering process" affects at most one new vertex in each step. Our precise estimations of the cover and inform times, and the speed-up, for random hypergraphs extend to the case of slowly growing r and s, but requiring that rs is $O((\log \log n)^{1-\epsilon})$, for an arbitrary constant $\epsilon > 0$. For random multi-hypergraphs we have been able to consider the case of a faster growing s, and have shown an $\Omega(s)$ speed-up of the inform time, if $s = O(n^{\delta})$ for a small positive constant δ.

An immediate open question is whether our analysis of the inform and cover times of random hypergraphs can be extended to larger values of s. More generally, are there other classes of hypergraphs for which formulas for the inform and cover times can be derived? It would be especially interesting to be able to say something about geometric hypergraphs, where hyperedges are defined somehow by geometric proximity of vertices. However, to proceed in this direction, first a good model of such hypergraphs would need to be developed.

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