A note on the chromatic number of the square of a sparse random graph

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Abstract

We show that w.h.p the chromatic number χ of the square of $G_{n,p}, p = c/n$ is asymptotically equal to the maximum degree $\Delta(G_{n,p})$. This improves an earlier result of Garapaty et al [5] who proved that $\chi(G_{n,p}^2) \leq 6 \cdot \Delta(G_{n,p})$ w.h.p.

1 Introduction

Let $p = c/n$ where $c > 0$ is a constant. The chromatic number of $G_{n,p}$ is well-understood, at least for sufficiently large c. Łuczak [6] proved that if $G = G_{n,p}$ then $\chi(G) \sim \frac{c}{2\log c}$. This was refined by Achlioptas and Naor [1] and further refined later by Coja-Oghlan and Vilenchik [2].

The square of a graph G is obtained from G by adding edges for all pairs of vertices at distance two or less from each other. Atkinson and Frieze [3] showed that w.h.p. the independence number of $G_2 = G_{n,p}^2$ is asymptotically equal to $\frac{4n\log c}{c^2}$, for large c. Garapaty, Lokshtanov, Maji and Pothen [5] studied the chromatic number of powers of $G_{n,p}$. Let $\Delta = \Delta(G_{n,p}) \sim \frac{\log n}{\log\log n}$ be the maximum degree in $G = G_{n,p}$ (for a proof of this known claim about the maximum degree, see for example [4], Theorem 3.4). Garapaty et al proved, in the case of the square G_2 of $G_{n,p}, p = c/n$ that $\chi(G_2) \leq 6 \cdot \frac{\log n}{\log\log n}$ w.h.p. We strengthen this and prove

Theorem 1. *Let $p = c/n$, $c > 0$ constant. Let G_2 denote the square of $G_{n,p}$. Then, w.h.p. $\chi(G_2) \sim \Delta(G_{n,p}) \sim \frac{\log n}{\log\log n}$.*

We will show that w.h.p. we can properly color G_2 with $q = \Delta(1 + 3\theta^{1/3})$ colors, where $\theta = o(1)$ is given in (1). Note that the neighbors of a vertex form a clique in G_2 and so the lower bound in the theorem is trivial.

Remark 1. *The value of c does not contribute to the main term in the claim of Theorem 1. Thus we would expect that we could replace $p = c/n$ by $p \leq \omega/n$ for some slowly growing function $\omega = \omega(n) \to \infty$. Indeed, a careful examination of the proof below verifies this so long as $c = o(\log\log n)$.*

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2 Proof of Theorem 1

2.1 Structural properties

We can use the following high probability bounds for Δ taken from [4], Theorem 3.4:

$$\frac{\log n}{\log\log n}\left(1-\frac{3\log\log\log n}{\log\log n}\right)\leq\Delta\leq\frac{\log n}{\log\log n}\left(1+\frac{3\log\log\log n}{\log\log n}\right)$$

This implies that w.h.p.

$$n^{1-\theta}\leq\Delta^{\Delta}\leq n^{1+\theta}\text{ where }\theta=\frac{4\log\log\log n}{\log\log n}. \tag{1}$$

Let $d(v)$ denote the degree of v in $G_{n,p}$. For $0<\alpha\leq 1$, let $V_\alpha=\{v:d(v)\geq\alpha\Delta\}$ and let W_α denote the closed neighborhood of V_α i.e V_α plus the neighbors of V_α.

Fix

$$\varepsilon=\theta^{1/2}.$$

The next few lemmas are needed to analyse the coloring of vertices in W_ε.

Lemma 2. *W.h.p., $v,w\in V_{2/3}$ implies that $dist(v,w)\geq 10$. (Here $dist(.,.)$ is graph distance in $G_{n,p}$.)*

Define

$$L_m=\left\{(\ell_1,\ell_2,\ldots,\ell_m)\in\{\varepsilon\Delta,\varepsilon\Delta+1,\ldots,\Delta\}^m\ :\sum_{i=1}^m\ell_i\geq(1+\theta^{1/3})\Delta\right\}$$

Lemma 3. *Suppose that $m\leq 2/\varepsilon$. Then w.h.p. there does not exist a connected subset $S\subseteq[n]$ of $G_{n,p}$ with at most $3m$ vertices containing vertices $w_i\in i=1,2,\ldots,m$ such that $(d(w_i),i=1,2,\ldots,m)\in L_m$.*

Corollary 4. *A vertex $v\notin V_\varepsilon$ has at most $\Delta_1=\left(1+2\theta^{1/3}\right)\Delta$ G_2-neighbors in W_ε, w.h.p.*

Proof. Suppose v has more than Δ_1 G_2-neighbors in W_ε. Let T be the tree obtained by Breadth-First-Search to depth three from v in $G_1=G_{n,p}$. Let this tree have levels $L_0=\{v\},L_1,L_2,L_3$. Let the G_1 neighbors of v be $\{u_1,u_2,\ldots,u_k\}$. Let $F_{i,t},t=2,3$ denote the vertices in L_t separated from v in T by u_i.

We now define a subtree T_1 of T that will take the place of S in Lemma 3. To obtain T_1 we do the following: suppose that $u_1,u_2,\ldots,u_p$ are the neighbors of v in V_ε. Delete the neighbors of $u_i,i\in[1,p]$, except for v. Suppose that $X_i=F_{i,2}\cap V_\varepsilon\neq\emptyset$ for $i\in[p+1,q]$ and that $F_{i,2}\cap V_\varepsilon=\emptyset$ for $i\in[q+1,k]$. Choose a vertex $x_i\in X_i$ for each $i\in[p+1,q]$ and delete $X_i\setminus\{x_i\}$ from T. Suppose also that $Y_i=F_{i,3}\cap V_\varepsilon\neq\emptyset$ for $i\in[q+1,r]$. Choose a vertex $x_i\in X_i$with a neighbor y_i in Y_i for each $i\in[q+1,r]$ and delete $X_i\setminus\{x_i\},Y_i\setminus\{y_i\}$ from T. T_1 is the tree that survives these deletions. For $i\in[r+1,k]$, we delete u_i and the vertices $F_{i,2}\cup F_{i,3}$ from T.

Vertex v has at most

$$D=\sum_{i=1}^r d(u_i)+(k-r)\leq\sum_{i=1}^p d(u_i)+(r-p)\varepsilon\Delta+(k-r)\leq\sum_{i=1}^p d(u_i)+(r-p+1)\varepsilon\Delta.$$

G_2 neighbors in W_ε. Our assumption is that $D>\Delta_1$.

Next let $M = \sum_{w \in V(T_1) \cap V_\varepsilon} d(w)$. Then,

$$M \geq \sum_{i=1}^{p} d(u_i) + (r-p)\varepsilon\Delta \geq D - \varepsilon\Delta > (1 + 2\theta^{1/3} - \theta^{1/2})\Delta > (1 + \theta^{1/3})\Delta.$$

We get a contribution of at least $\varepsilon\Delta$ from a member of $F_{i,2}$ for $i \in [p+1, q]$ and a contribution of at least $\varepsilon\Delta$ from the surviving member of $F_{i,3}$ for $i \in [q+1, r]$.

The number of vertices N in the tree T_1 satisfies

$$N \leq 1 + p + 2(q-p) + 3(r-q) \leq 3|V(T_1) \cap V_\varepsilon|.$$

Putting $m = |V(T_1) \cap V_\varepsilon|$, we see that this contradicts Lemma 3, provided $m \leq 2/\varepsilon$. Assume then that $m > 2/\varepsilon$. It follows from Lemma 2 that either T consists of $u_1 \in V_{2/3}$ and the neighbors of u_1 and then the corollary holds trivially. Otherwise, $M > m\varepsilon\Delta > 2\Delta$ and one can delete a vertex of degree less than $2\Delta/3$ and reduce m by one keep $M > (1 + \theta^{1/3})\Delta$, eventually leading to a contradiction. □

A similar argument gives

Corollary 5. *A vertex has at most Δ_1 G_2-neighbors in V_ε.*

Proof. Suppose v has more than Δ_1 G_2-neighbors in V_ε. Let T be the tree obtained by Breadth-First-Seach to depth two from v in $G_1 = G_{n,p}$. Remove all leaves from T that are not in V_ε and repeat. We are left with a set of G_1-neighbors W_0 of v in V_ε and set of G_1-neighbors $u_1, u_2, \ldots, u_k$ of v that are not in V_ε. In addition we have sets $W_1, W_2, \ldots, W_k \subseteq V_\varepsilon$ such that u_i is a G_1-neighbor of all vertices in $W_i, i = 1, 2, \ldots, k$. The G_2-degree of v is given by $D = \sum_{w \in W_0} d(w) + k + \sum_{i=1}^{k} |W_i| > \Delta_1$. The number of G_2-neighbors in V_ε is $m_1 = \sum_{i=0}^{k} |W_i|$ and the tree T contains $1 + k + m_1 \leq 2m_1 + 1$ vertices. Let $W = \bigcup_{i=0}^{k} W_i$ and add v to W if $v \in V_\varepsilon$. Then let $W = \{w_1, w_2, \ldots, w_m\}$ where $m = m_1 + 1_{v \in V_\varepsilon}$. If $m \leq 2/\varepsilon$ then we contradict Lemma 3. Otherwise, $M = \sum_{i=1}^{m} d(w_i) \geq m d_{\min}$ where $d_{min} = \min \{d(w) : w \in W\}$. But $d_{min} \geq \varepsilon\Delta$ and so $M > 2\Delta$. It follows from Lemma 2 that $d_{min} < 2\Delta/3$ and so we can reduce m by one and keep $M > 4\Delta/3$. Continuing in this way, we eventually reduce m to below $2/\varepsilon$ and keep $M > (1 + \theta^{1/3})\Delta$. But now we contradict Lemma 3. □

The next part of our strategy is to bound the number of G_2-edges contained in any set S that is disjoint from W_ε. We prove a high probability bound of $(5.5 + 2c)\varepsilon\Delta|S|$.

Remark 2. *This will imply that the vertices of $[n] \setminus W_\varepsilon$ can be list-colored using at most $(5.5 + 2c)\varepsilon\Delta + 1$ colors.*

This remark follows from Lemmas 6, 7 and 8.

For large sets we can use the following:

Lemma 6. *The total number of edges in G_2 is less than $c(c+1)n$ w.h.p.*

For $2 \leq s \leq n$ let ν_s be the maximum number of G_2-edges in a set of size s. Then we have that

$$\text{if } k_0 = 2/\varepsilon^2 \text{ then } \nu_s \leq 10k_0c^3 s \text{ for } n/(10ck_0) \leq s \leq n. \tag{2}$$

If $S \cap W_\varepsilon = \emptyset$ then a vertex outside S has at most $\varepsilon\Delta$ neighbors in S. For a fixed set S let $a_{k,S}$ denote the number of (vertex, set) pairs v, T where $v \notin S$ and $|T| = k$ and $T = N(v) \cap S$.

Lemma 7. *The following holds w.h.p. Let* $A_1(S) = \sum_{k \leq \varepsilon\Delta} a_{k,S} k^2$ *bound the number of* G_2*-edges due to the* $a_{k,S}$*. Then* $A_1(S) \leq 5\varepsilon\Delta|S|$ *for all* $|S| \leq n/(10ck_0)$*.*

We now have to deal with the number of edges uv and the number of paths of length two uvw where $\{u, v, w\} \subseteq S$. Denote this by $A_2(S)$.

Lemma 8. *W.h.p.,* $A_2(S) \leq (2c + 1/2)\varepsilon\Delta|S|$ *for all* $|S| \leq n/(10ck_0)$*.*

Thus,

$$A_1(S) + A_2(S) \leq (5.5 + 2c)\varepsilon\Delta|S|. \tag{3}$$

This, together with (2), verifies what we claimed in Remark 2.

3 Coloring G_2

Given the above we color G_2 as follows:

(a) We color V_ε with $q = \Delta(1 + 3\theta^{1/3})$ colors. We do this greedily i.e we arbitrarily order the vertices in V_ε and then in this order, we color a vertex with the lowest index available color. Corollary 5 implies that any vertex v has at most $\Delta\left(1 + 2\theta^{1/3}\right)$ G_2-neighbors in V_ε and so there will be an unused color.

(b) We color $\widehat{W_\varepsilon} = W_\varepsilon \setminus V_\varepsilon$ with $q = \Delta(1 + 3\theta^{1/3})$ colors. We do this greedily i.e we arbitrarily order the vertices in $\widehat{W_\varepsilon}$ and then in this order, we color a vertex with the lowest index available color. Corollary 4 implies that any vertex $v \notin V_\varepsilon$ has at most $\Delta\left(1 + 2\theta^{1/3}\right)$ G_2-neighbors in $\widehat{W_\varepsilon}$ and so there will be an unused color.

(c) We then color $[n] \setminus W_\varepsilon$ with at most $\Delta\left(1 + 2\theta^{1/3} + (5.5 + 2c)\theta^{2/3}\right)$ colors. This follows from (3) and Corollary 4.

3.1 Proof of Lemma 2

Let $\ell_0 = 2\Delta/3 - 10$. We have

$$\begin{aligned}\mathbb{P}(\exists v, w \in V_{2/3} : dist(v, w) < 10) &\leq \sum_{k=1}^{9} \binom{n}{k} k! p^{k-1} \left(\sum_{\ell=\ell_0}^{n-1} \binom{n}{\ell} p^\ell (1-p)^{n-10-\ell} \right)^2 \\ &\leq \sum_{k=1}^{9} n c^{k-1} n^{-4/3+o(1)} = o(1).\end{aligned}$$

3.2 Proof of Lemma 3

Then,

$$\mathbb{P}(\exists S) \leq \sum_{m=2}^{2/\varepsilon} \sum_{s=m}^{3m} \binom{n}{s} s^{s-2} p^{s-1} \binom{s}{m} \sum_{D \geq (1+\theta^{1/3})\Delta} \sum_{\ell_1 + \cdots + \cdots \ell_m = D} \prod_{i=1}^{m} \left(\sum_{k=\ell_i}^{n-s} \binom{n-s}{k} p^k (1-p)^{n-s-k} \right)$$

$$\leq \sum_{m=2}^{2/\varepsilon}\sum_{s=m}^{3m}\binom{n}{s}s^{s-2}p^{s-1}2^s \sum_{D\geq(1+\theta^{1/3})\Delta}\ \sum_{\ell_1+\cdots+\cdots\ell_m=D}\ \prod_{i=1}^m n^{-\ell_i/\Delta+O(\theta)}$$
$$\leq \frac{2n}{c}\sum_{m=2}^{2/\varepsilon}\left(\frac{2ec}{3m}\right)^{3m}\sum_{D\geq(1+\theta^{1/3})\Delta}\binom{D-1}{m-1}n^{-D+O(\theta m)}$$
$$\leq \frac{2n}{c}\sum_{m=2}^{2/\varepsilon}\left(\frac{2ec}{3m}\right)^{3m}\left(\frac{(1+\theta^{1/2})\Delta}{m}\right)^{m}n^{-(1+\theta^{1/3}-O(\theta^{1/2}))}$$
$$\leq n^{1+o(\theta^{1/2})-(1+\theta^{1/3}-O(\theta^{1/2}))} = o(1).$$

Explanation: There are $\binom{n}{s}$ choices for S. Then there are at most s^{s-2} choices for a spanning tree of S. Then we choose the vertices of large degree in $\binom{s}{m}$ ways. D is the total degree of the large degree vertices. The product bounds the probability that the selected vertices have large degree.

3.3 Proof of Lemma 6

Let $d(i)$ denote the degree of vertex i in $G_{n,p}$. The expected number of edges in G_2 is

$$\mathbb{E}\left(\sum_{i=1}^n \frac{d(i)(d(i)+1)}{2}\right) = \frac{n}{2}\sum_{j=1}^{n-1} j(j+1)\binom{n-1}{j}p^j(1-p)^{n-1-j} = \frac{c^2(n-1)(n-2)+cn^2}{2n}.$$

To show concentration round the mean, we use the following theorem from Warnke [7]:

Theorem 9. *Let $X=(X_1,X_2,\ldots,X_N)$ be a family of independent random variables with X_k taking values in a set Λ_k. Let $\Omega=\prod_{k\in[N]}\Lambda_k$ and suppose that $\Gamma\subseteq\Omega$ and $f:\Omega\to\mathbf{R}$ are given. Suppose also that whenever $\mathbf{x},\mathbf{x}'\in\Omega$ differ only in the kth coordinate*

$$|f(\mathbf{x})-f(\mathbf{x}')| \leq \begin{cases} c_k & if\ \mathbf{x}\in\Gamma. \\ d_k & otherwise. \end{cases}$$

If $W=f(X)$, then for all reals $\gamma_k>0$,

$$\mathbb{P}(W\geq \mathbb{E}(W)+t) \leq \exp\left\{-\frac{t^2}{2\sum_{k\in[N]}((c_k+\gamma_k(d_k-c_k)^2))}\right\} + \mathbb{P}(X\notin\Gamma)\sum_{k\in[N]}\gamma_k^{-1}.$$

We use Theorem 9 with $N=n$, $W=\sum_{i=1}^n \frac{d(i)(d(i)+1)}{2}$, $X_i=\{j<i:\{j,i\}\text{ is an edge of }G_{n,p}\}, i=1,2,\ldots,n$ and $\Gamma=\{\Delta(G_{n,p})\leq\log n\}$. In which case we can take $c_k=\log^2 n$, $d_k=n^2$ and $\mathbb{P}(X\notin\Gamma)\leq(\log n)^{-\frac{1}{2}\log n}$. Then we can take $\gamma_k=n^{-4}$ for $k\in[n]$ and $t=n^{3/2}$ to complete the proof of Lemma 6.

3.4 Proof of Lemma 7

Let $|S|\leq n/(10ck_0)$. We will prove:

(a) For $k_0 < k_1 < k_2 \le \varepsilon\Delta$, for all sets $S \subseteq [n] \setminus V_\varepsilon$,

$$\sum_{k=k_1}^{k_2} a_{k,S} \le \frac{(1+\varepsilon)|S|}{k_1}.$$

(b) $a_{k,S} \le (10c)^{k_0}|S|$ for $k \le k_0$.

We have, where $M_{u,k_1,k_2} = \left\{(x_2,\dots,x_{\varepsilon\Delta}) : \sum_{k=k_1}^{k_2} x_k = u\right\}$.

$$\begin{aligned}
&\mathbb{P}\left(\exists S, |S| = s \le n/(10ck_0) : \sum_{k=k_1}^{k_2} a_{k,S} \ge t\right) \\
&\le \sum_{s=k_1^{1/2}}^{n/(10ck_0)} \binom{n}{s} \sum_{u\ge t} \sum_{\mathbf{x}\in M_{u,k_1,k_2}} \binom{n}{x_{k_1},\dots,x_{k_2},n-u} \prod_{k=k_1}^{k_2} \left(\binom{s}{k}\left(\frac{c}{n}\right)^k\right)^{x_k} \\
&\le \sum_{s=k_1^{1/2}}^{n/(10ck_0)} \left(\frac{ne}{s}\right)^s \sum_{u\ge t} \sum_{\mathbf{x}\in M_{u,k_1,k_2}} \binom{n}{x_{k_1},\dots,x_{k_2},n-u} \prod_{k=k_1}^{k_2} \left(\frac{sec}{k_1 n}\right)^{k_1 x_k} \\
&= \sum_{s=k_1^{1/2}}^{n/(10ck_0)} \left(\frac{ne}{s}\right)^s \sum_{u\ge t} \left(\frac{sec}{k_1 n}\right)^{k_1 u} \sum_{\mathbf{x}\in M_{u,k_1,k_2}} \binom{n}{x_{k_1},\dots,x_{k_2},n-u} \\
&\le \sum_{s=k_1^{1/2}}^{n/(10ck_0)} \left(\frac{ne}{s}\right)^s \sum_{u\ge t} \left(\frac{sec}{k_1 n}\right)^{k_1 u} \binom{n}{u}.
\end{aligned}$$

Putting $t = (1+\varepsilon)s/k_1$, we have, for large k_1 i.e. for $k_1 > 2/\varepsilon$,

$$\begin{aligned}
&\mathbb{P}\left(\exists S, |S| = s \le n/(10ck_0) : \sum_{k=k_1}^{k_2} a_{k,S} \ge \frac{(1+\varepsilon)s}{k_1}\right) \\
&\le \sum_{s=k_1^{1/2}}^{n/(10ck_0)} \left(\frac{ne}{s}\right)^s \sum_{u\ge t} \left(\frac{sec}{k_1 n}\right)^{k_1 u} \binom{n}{u} \\
&\le 2\sum_{s=k_1^{1/2}}^{n/(10ck_0)} \left(\frac{ne}{s}\right)^s \left(\frac{sec}{k_1 n}\right)^{(1+\varepsilon)s} \binom{n}{(1+\varepsilon)s/k_1} \\
&\le 2\sum_{s=k_1^{1/2}}^{n/(10ck_0)} \left(\frac{ne}{s}\right)^s \left(\frac{sec}{k_1 n}\right)^{(1+\varepsilon)s} \left(\frac{nek_1}{(1+\varepsilon)s}\right)^{(1+\varepsilon)s/k_1} \\
&= 2\sum_{s=k_1^{1/2}}^{n/(10ck_0)} \left(\left(\frac{s}{n}\right)^{\varepsilon-(1+\varepsilon)/k_1} \frac{e^{2+\varepsilon+(1+\varepsilon)/k_1}c^{1+\varepsilon}}{k_1^{(1+\varepsilon)(k_1-1)/k_1}}\right)^s \\
&= o(n^{-2}).
\end{aligned}$$

We also have

$$\mathbb{P}(\exists S, |S| = s \le n/(10ck_0) : a_{k,S} \ge t) \le \binom{n}{s}\binom{n}{t}\left(\binom{s}{k}\left(\frac{c}{n}\right)^k\right)^t$$

$$\leq \left(\frac{ne}{s}\right)^s \left(\frac{ne}{t}\right)^t \cdot \left(\frac{se}{k} \cdot \frac{c}{n}\right)^{kt}. \tag{4}$$

We put $t = (10c)^{k_0}s$ for $2 \leq k < k_0$. Then we have, where $L = (10c)^{k_0}$,

$$\begin{aligned}
\mathbb{P}(\exists S, |S| = s \leq n/(10ck_0) : 2 \leq k \leq k_0, a_{k,S} \geq Ls) &\leq \left(\frac{ne}{s}\right)^s \cdot \left(\frac{ne}{Ls}\right)^{Ls} \cdot \left(\frac{sec}{kn}\right)^{Lks} \\
&= \left(\left(\frac{s}{n}\right)^{L(k-1)-1} \cdot e \cdot \left(\frac{e}{L}\right)^L \cdot \left(\frac{ec}{k}\right)^{Lk}\right)^s \\
&\leq \left(\left(\frac{s}{n}\right)^{L(k-1)-1} \cdot e^{cL+2}\right)^s. \qquad (5)
\end{aligned}$$

If $k \geq 3$ then the bracketed term σ_s in (5) is at most $\left(\frac{se^c}{n}\right)^L = o(1)$ and so $\sum_{s\geq 1} \sigma_s^s = o(1)$. If $k = 2$ we write $\sigma_s = \left(\frac{se^c}{n}\right)^{L-1} \cdot e^{c+2} = o(1)$, as well.

So, by dividing $[1, \varepsilon\Delta]$ into intervals of size $\varepsilon\Delta/2^i, i \geq 1$, we get that for all $|S| \leq n/(10ck_0)$

$$\begin{aligned}
A_1(S) &\leq \sum_{k=2}^{k_0} (10c)^{k_0} s + \sum_{i\geq 1} \frac{(1+\varepsilon)s}{\varepsilon\Delta/2^i} \cdot \frac{(\varepsilon\Delta)^2}{2^{2i-2}} \\
&\leq 2\varepsilon^{-1/2}(10c)^{2/\varepsilon}s + 4(1+\varepsilon)\varepsilon\Delta s \leq 5\varepsilon\Delta s. \qquad (6)
\end{aligned}$$

3.5 Proof of Lemma 8

The number of such paths is equal to $\sum_{v\in S} \frac{d_S(v)(d_S(v)+1)}{2}$ where $d_S(v)$ is the degree of v in S.

$$A_2(S) = \sum_{v\in S, d(v)\leq\varepsilon\Delta} \frac{d_S(v)(d_S(v)+1)}{2} \leq \varepsilon\Delta \sum_{v\in S} \frac{d_S(v)+1}{2} = \varepsilon\Delta(e(S) + |S|/2), \tag{7}$$

where $e(S)$ is the number of $G_{n,p}$ edges entirely contained in S. Now

$$\begin{aligned}
\mathbb{P}(\exists S, s = |S| \leq n/(10ck_0) : e(S) \geq 2cs) \leq \sum_{s=2}^{n/(10ck_0)} \binom{n}{s} \binom{\binom{s}{2}}{2cs} \left(\frac{c}{n}\right)^{2s} &\leq \sum_{s=2}^{n/(10ck_0)} \left(\frac{ne}{s} \cdot \left(\frac{se}{4n}\right)^2\right)^s \\
&= \sum_{s=2}^{n/(10ck_0)} \left(\frac{s}{n} \cdot \left(\frac{e}{4}\right)^2\right)^s = o(n^{-1}).
\end{aligned}$$

So,

$$A_2(S) \leq \varepsilon\Delta(2c + 1/2)|S| \text{ for all } |S| \leq n/(10ck_0), \ w.h.p.$$

4 Conclusions

While we have shown that $\chi(G_2) \sim \Delta$ w.h.p., it is possible that $\chi(G_2) = \Delta + 1$ w.h.p. This would be quite pleasing, but we are not confident enough to make this a conjecture. It is of course interesting to further consider $\chi(G_2)$ when $np \to \infty$. Note that when $np \gg n^{1/2}$, the diameter of $G_{n,p}$ is equal to 2 w.h.p. In which case $G_2 = K_n$. One can also consider higher powers of $G_{n,p}$ as was done in [3] and [5]. Such considerations are more technically challenging.

References

[1] D. Achlioptas and A. Naor, The two possible values of the chromatic number of a random graph, *Annals of Mathematics* 162 (2005) 1335-1351.

[2] A. Coja–Oghlan and D. Vilenchik, Chasing the k-colorability threshold, Proceedings of the 54th IEEE Annual Symposium on Foundations of Computer Science.

[3] A.M. Frieze and G. Atkinson, On the b-independence number of sparse random graphs, *Combinatorics, Probability and Computing* 13, 295-310.

[4] A.M. Frieze and M. Karoński, Introduction to Random Graphs, Cambridge University Press, 2015.

[5] K. Garapaty, D. Lokshtanov, H. Maji and A. Pothen, The Chromatic Number of Squares Of Random Graphs, *Journal of Combinatorics* 14 (2023) 507-537.

[6] T. Łuczak, The chromatic number of random graphs, *Combinatorica* 11 (1991) 45-54.

[7] L. Warnke, On the Method of Typical Bounded Differences, *Combinatorics, Probability and Computing* 25 (2016 269–299.