Random Graphs

Alan Frieze

Carnegie Mellon University

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$G_{n,p}$: Each edge e of the complete graph K_n is included independently with probability $p = p(n)$.

Whp $G_{n,p}$ has $\sim \binom{n}{2}p$ edges, provided $\binom{n}{2}p \to \infty$

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$G_{n,m}$: Vertex set $[n]$ and m random edges.

If $m \sim \binom{n}{2}p$ then $G_{n,p}$ and $G_{n,m}$ have "similar" properties.

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Erdős (1947): **Whp** the maximum size of a clique or independent set in $G_{n,1/2}$ is $\leq 2\log_2 n$.

Therefore

$$R(k,k) \geq 2^{k/2}.$$

Random graphs first used to prove existence of graphs with certain properties:

Mantel (1907): There exist triangle free graphs with arbitrarily large chromatic number.
Erdős (1959): There exist graphs of arbitrarily large girth and chromatic number.

$m = cn$, $c > 0$ is a large constant. **Whp** $G_{n,m}$ has $o(n)$ vertices on cycles of length $\leq \log\log n$ and no independent set of size more than $\frac{2\log c}{c}n$.

So removing the vertices on small cycles gives us a graph with girth $\geq \log\log n$ and chromatic number $\geq \frac{c+o(1)}{2\log c}$.

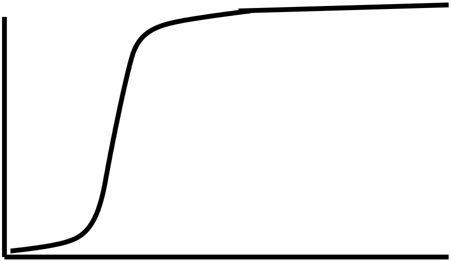
Erdős and Rényi began the study of random graphs in their own right.

On Random Graphs I (1959): $m = \frac{1}{2}n(\log n + c_n)$

$$\lim_{n\to\infty} \mathbf{Pr}(G_{n,m} \text{ is connected}) = \begin{cases} 0 & c_n \to -\infty \\ e^{-e^{-c}} & c_n \to c \\ 1 & c_n \to +\infty \end{cases}$$

$$= \lim_{n\to\infty} \mathbf{Pr}(\delta(G_{n,m}) \geq 1)$$

$\mathbf{Pr}(G_{n,m} \textit{ is connected})$



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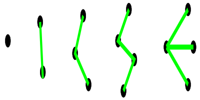
$n^{2/3}\log n$	Components are of the form 

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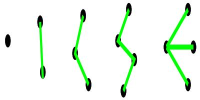
$n^{2/3}\log n$	Components are of the form 
$n^{\frac{k-1}{k}}\log n$	Components are trees of vertex size $1, 2, \ldots, k+1$. Each possible such tree appears.

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$\frac{1}{2}cn$, $c < 1$	Mainly trees. Some unicyclic components. Maximum component size $O(\log n)$
$\frac{1}{2}n$	Complicated. Maximum component size order $n^{2/3}$. Has subsequently been the subject of moreintensive study e.g. Janson, Knuth, Łuczak and Pittel (1993).
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Only very simple probabilistic tools needed. Mainly first and second moment method.

Connectivity threshold

$$p = (1+\epsilon)\frac{\log n}{n}$$

X_k = number of k-components, $1 \leq k \leq n/2$.
$X = X_1 + X_2 + \cdots + X_{n/2}$
$G_{n,p}$ is connected iff $X = 0$.

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$$\begin{aligned}
\mathbf{Pr}(X \neq 0) &\le \mathbf{E}(X) \\
&\le \sum_{k=1}^{n/2} \binom{n}{k} k^{k-2} p^{k-1} (1-p)^{k(n-k)} \\
&\le \frac{n}{\log n} \sum_{k=1}^{n/2} \left(\frac{e \log n}{n^{(1+\epsilon)(1-k/n)}} \right)^k \\
&\to 0.
\end{aligned}$$

Hitting Time: Consider $G_0, G_1, \ldots, G_m, \ldots,$ where G_{i+1} is G_i plus a random edge.

Let m_k denote the minimum m for which $\delta(G_m) \geq k$.

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  Ajtai, Komlós and Szemerédi (1985), Bollobás (1984).
- **Whp** At time m_2 there are $(\log n)^{n-o(n)}$ distinct Hamilton cycles.
  Cooper and Frieze (1989).

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  Ajtai, Komlós and Szemerédi (1985), Bollobás (1984).
- **Whp** m_k is the "time" when G_m first has $k/2$ edge disjoint Hamilton cycles.
  Bollobás and Frieze (1985).

Some Open Problems

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Is it true that **whp** G_m has $\delta(G_m)/2$ Hamilton cycles, for $m = 1, 2, \ldots, \binom{n}{2}$?

It is known to be true as long as $\delta(G_m) = o(\textit{average degree})$.

It is known that $G_{n,1/2}$ has $\sim n/4$ edge disjoint Hamilton cycles, Frieze and Krivelevich (2005).

Some Open Problems

Is it true that if we include the edges of the n-cube, Q^n with constant probability $p > 1/2$ then the resulting random subgraph is Hamiltonian **whp**?

It is known to have a perfect matching **whp** - Bollobás (1999).

Some Open Problems

If we randomly color the edges of $G_{n,Kn\log n}$ with Kn colors and K is sufficiently large, then **whp** there exists a Hamilton cycle with every edge a different color – Cooper and Frieze (2002).

If we only have $\sim \frac{1}{2}n\log n$ random edges, then how many colors do we need to get such a cycle **whp**?

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If we replace Hamilton Cycle by Spanning Tree then the problem is solved: The hitting time for a multi-colored spanning tree is the maximum of the hitting time for connectivity and the appearance of $n-1$ colors – Frieze and McKay (1994).

Some Open Problems

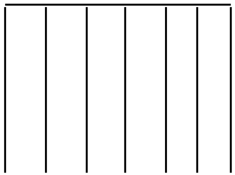
If we consider digraphs and ask for a multi-colored Hamilton cycle or spanning arborescence then nothing(?) is known.

Some Open Problems

Is it true that if T is a degree bounded tree with n vertices then **whp** $G_{n,Kn\log n}$ contains a **spanning** copy of T, for sufficiently large $K = K(T)$. Problem posed by Jeff Kahn.

True if T has a linear number of leaves.

The tree below seems to be a difficult one:



$n^{1/2}$ paths of length $n^{1/2}$

Small Subgraphs

Given a **fixed** graph H, one can ask when does $G_{n,p}$ contain a copy of H.

If X_H is the number of copies of H in $G_{n,p}$ then

$$\mathbf{E}(X_H) \sim C_H n^{v_H} p^{e_H}$$

where C_H is a constant, v_H, e_H are the number of vertices and edges in H.

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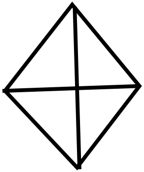
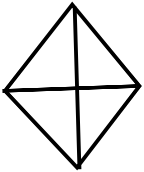
Does $\mathbf{E}(X_H) \to \infty$ imply that there is a copy of H **whp**?



If $p = o(n^{-2/3})$ then $\mathbf{E}(X_H) \to 0$.
If $p = \omega n^{-2/3}$ then $\mathbf{E}(X_H) \to \infty$ and a copy of H exists **whp**.



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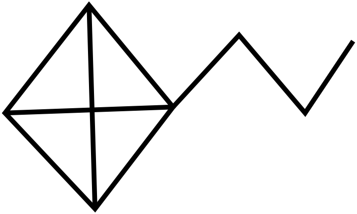

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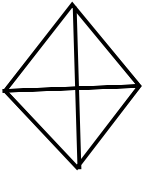

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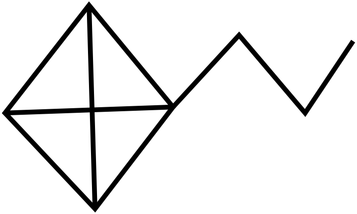

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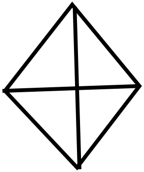

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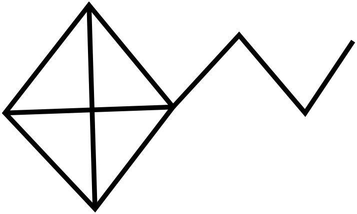

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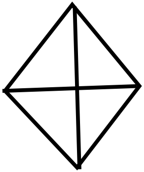

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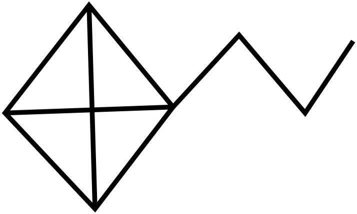

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Study of this problem has led to important probabilistic tools: Suen's inequality (1980), Janson's Inequality (1990) and the concentration inequality for multivariate polynomials by Kim and Vu (2004).

Graph Coloring

Graph Coloring

Matula (1970) showed using the second moment method that **whp** the maximum size $\alpha(G_{n,1/2})$ of an independent set is

$$2\log_2 n - 2\log_2\log_2 n + O(1).$$

Thus, **whp** $\chi(G_{n,1/2}) \geq\sim \frac{n}{2\log_2 n}$

Bollobás and Erdős (1976) and Grimmett and McDiarmid (1975) showed that **whp** a simple greedy algorithm uses $\sim \frac{n}{\log_2 n}$ colors.

Graph Coloring

A simple first moment calculation shows that **whp** $\alpha(G_{n,d/n})$ is

$$\leq 2\frac{\log d}{d}n$$

for d sufficiently large.

Thus, **whp**

$$\chi(G_{n,d/n}) \gtrsim \frac{d}{2\log d}$$

Shamir and Upfal (1984) showed that a slight modification of the greedy algorithm uses $\sim \frac{d}{\log d}$ colors.

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Martingale Tail Inequalities

Azuma/Hoeffding

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Let $Z = Z(X_1, \ldots, X_N)$ where $X_1, \ldots, X_N$ are independent. Suppose that changing one X_i only changes Z by ≤ 1. Then

$$\mathbf{Pr}(|Z - \mathbf{E}(Z)| \geq t) \leq e^{-t^2/(2n)}.$$

“Discovered” by Shamir and Spencer (1987) and by Rhee and Talagrand (1988).

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Let Z be the maximum number of independent sets in a collection $S_1, \dots, S_Z$ where each $|S_i| \sim 2\log_2 n$ and $|S_i \cap S_j| \le 1$.

$$\mathbf{E}(Z) = n^{2-o(1)} \textit{ and changing one edge changes Z by } \le 1$$

So,

$$\mathbf{Pr}\Big(\exists S \subseteq [n] : \ |S| \ge \frac{n}{(\log_2 n)^2} \textit{ and S doesn't contain a}$$
$$(2 - o(1))\log_2 n \textit{ independent set}\Big) \le 2^n e^{-n^{2-o(1)}} = o(1).$$

So, we color $G_{n,1/2}$ with color classes of size $\sim 2\log_2 n$ until there are $\le n/(\log_2 n)^2$ vertices uncolored and then give each remaining vertex a new color.

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Suppose $k \sim \frac{2\log d}{d} n$ and X_k is the number of independent k-sets in $G_{n,d/n}$

$$\mathbf{Pr}(X_k \neq 0) \geq \frac{\mathbf{E}(X_k)^2}{\mathbf{E}(X_k^2)} \geq e^{-a_1 n}.$$

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But Azuma-Hoeffding gives

$$\mathbf{Pr}(|\alpha(G_{n,d/n}) - \mathbf{E}(\alpha)| \geq \epsilon_1 n) \leq e^{-a_2 n}.$$

Here $a_2 > a_1$ and so $\mathbf{E}(\alpha) \geq \frac{(2-\epsilon_2)\log d}{d} n$ and ...

Taking a similar (but much more computationally challenging) approach Łuczak (1991) showed that

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Then Łuczak (1991) proved that **whp** there was a two point concentration for $\chi(G_{n,d/n})$ i.e. $\exists k_d$ such that **whp**

$$\chi(G_{n,d/n}) \in \{k_d, k_d + 1\}.$$

Achlioptas and Naor (2005) showed that k_d is the smallest integer ≥ 2 such that $d < d_k = 2k \log k$.

If $d > d_k$ and X_k is the number of k-colorings of $G_{n,d/n}$ then $\mathbf{E}(X_k) \to 0$.

If $d \leq d_{k-1}$ then
$\mathbf{Pr}(G_{n,d/n}\ is\ k-colorable) \geq \mathbf{E}(X_k)^2/\mathbf{E}(X_k^2) \geq \xi > 0$.

Using the results of Friedgut (1999) and Achlioptas and Friedgut (1999) we see that this implies $G_{n,d/n}\ is\ k-colorable$ **whp** for $d \leq d_{k-1}$.

Some Open Problems

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Is it the case that there exist $d_3 < d_4 < \cdots < d_k < \cdots$ such that $d_k < d < d_{k+1}$ implies that **whp** $\chi(G_{n,d/n}) = k$?

The results of Friedgut (1999) and Achlioptas and Friedgut (1999) suggests strongly that this is true.

Some Open Problems

What is the Chromatic number of a random r-regular graph $G_{n,r}$?

Achlioptas and Moore (2005) show that provided $r = O(1)$ the chromatic number is 3 point concentrated around the smallest integer k such that $r < 2k \log k$.

Shi and Wormald (2005) show that **whp** a random 4-regular graph has chromatic number 3 and a random 6-regular graph has chromatic number 4.

Cooper, Frieze, Reed and Riordan (2002) show that if $r \to \infty$ then **whp**

$$\chi(G_{n,r}) \sim \frac{r}{2 \log r}.$$

Some Open Problems

Is there a polynomial time algorithm that **whp** can color $G_{n,1/2}$ with $\frac{(1-\epsilon)n}{\log_2 n}$ colors?

Randomly generated k-colorable graphs, $k = O(1)$, with $O(n)$ edges can be colored quickly, Alon and Kahale (1994).

Some Open Problems

What is the game chromatic number χ_g of the random graph $G_{n,1/2}$?

There are two players: A and B who alternately *properly* color the vertices of G. A tries to color the whole graph and B tries to force a situation where some vertex cannot be colored. χ_g is the minimum number of colors which guarantees a win for A.

Bohman, Frieze and Sudakov (2005) show that **whp**

$$(1-\epsilon)\frac{n}{\log_2 n} \leq \chi_g(G_{n,1/2}) \leq (2+\epsilon)\frac{n}{\log_2 n}.$$

The diameter of random graphs

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Suppose $d \geq 2$ is a positive integer and $p^d n^{d-1} = \log(n^2/c)$ so that average degree is $\tilde{\Theta}(n^{1/d})$. Then

$$\lim_{n\to\infty} \mathbf{Pr}(\textit{diameter } G_{n,p} = d + \delta) = \begin{cases} e^{-c/2} & \delta = 0 \\ 1 - e^{-c/2} & \delta = 1 \end{cases}$$

Bollobás (1981).

Basically, there are $\tilde{\Theta}(n^{k/d})$ vertices at distance $\leq k$ from a fixed vertex v.

The diameter of random graphs

Diameter of the Giant Component of $G_{n,c/n}$: Fernholz and Ramachandran (2005).

One would expect this to be $\sim A(c) \log n$ **whp**. They show that

$$A(c) = \frac{2}{-\log W} + \frac{1}{\log c}$$

where W is the solution in $(0, 1)$ of $We^{-W} = ce^{-c}$.

Here $W \to 0$ as $c \to \infty$, so the diameter is "like" $\log_c n$ for large c, as one would expect.

Algorithms and Differential Equations

Karp and Sipser (1981) described a simple greedy matching algorithm for finding a large matching in the random graph $G_{n,c/n}$.

If there is a vertex v of degree one, choose a random degree one vertex and the edge incident to it; otherwise choose a random edge.

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If there is a vertex v of degree one, choose a random degree one vertex and the edge incident to it; otherwise choose a random edge.

They show that the algorithm is asymptotically optimal i.e. the matching it produces is within $1 - o(1)$ of optimal.

Aronson, Frieze and Pittel (1998) showed that **whp** this algorithm only makes $\tilde{\Theta}(n^{1/5})$ "mistakes".

The proof of the above results rests on the fact that the progress of the algorithm can **whp** be tracked by the solution of a differential equation.

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Karp and Sipser introduced this approach (via Kurtz theorem) to the “CS/Probabilistic Combinatorics” community and Wormald has “championed” its applications.

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Let $X_0(m)$ be the number of isolated vertices in G_m. Then

$$\mathbf{E}(X_0(m+1) - X_0(m) \mid G_m) = -2\frac{X_0(m)}{n}. \qquad (1)$$

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Let $x_0(t) = X_0(tn)/n$ for $t > 0$. Then (1) suggests the equation

$$x_0' = -2x_0$$

which has the solution

$$x_0 = e^{-2t}$$

or

$$X_0(m) \sim ne^{-2m/n}.$$

More typical example: From “Hamilton Cycles in 3-Out” – Bohman and Frieze (2006).

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$$
\begin{aligned}
\mathbf{E}(y'_{i,j,0} - y_{i,j,0}) &= -\frac{jy_{i,j,0}}{\mu} - \sum_{a,b} \frac{by_{a,b,1}}{\mu}\left((b-1)\frac{iy_{i,j,0}}{\mu-1} + \hat{a}\,\frac{jy_{i,j,0}}{\mu-1}\right) \\
&\quad + \sum_{a,b} \frac{by_{a,b,1}}{\mu}\left((b-1)\frac{(i+1)y_{i+1,j,0}}{\mu-1} + \hat{a}\,\frac{(j+1)y_{i,j+1,0}}{\mu-1}\right) + \tilde{O}(\mu^{-1}) \\
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\mathbf{E}(y'_{L,j,0} - y_{L,j,0}) &= -\frac{jy_{L,j,0}}{\mu} - \sum_{a,b} \frac{by_{a,b,1}}{\mu}\left((b-1)\frac{3y_{3,j,0}}{\mu-1} + \hat{a}\,\frac{jy_{L,j,0}}{\mu-1}\right) \\
&\quad + \sum_{a,b} \frac{by_{a,b,1}}{\mu} \cdot \hat{a}\,\frac{(j+1)y_{L,j+1,0}}{\mu-1} + \tilde{O}(\mu^{-1}). \\
&\stackrel{d}{=} \phi^{in}_{L,j,0}(y) + \tilde{O}(\mu^{-1}) \\
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Eigenvalues of Random Graphs

Let A be the adjacency matrix of $G_{n,p}$. Then **whp**

$$\lambda_1(A) = (1 + o(1)) \max\{\sqrt{\Delta}, np\}.$$

Krivelevich and Sudakov (2003)

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Now let A be the adjacency matrix of a random d-regular graph, $d \geq 3$. $\lambda_1(A) = d$ and **whp**, for any constant $\epsilon > 0$,

$$|\lambda_i(A)| \leq 2\sqrt{d-1} + \epsilon \qquad\qquad 2 \leq i \leq n$$

Friedman (2004)

Typical Graphs

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Unstructured, randomly generated(?) real world graphs like the **WWW** seem to have a different distribution to $G_{n,p}$, e.g. the number of vertices of degree k drops off like $k^{-\alpha}$ instead of $e^{-\alpha k}$.

Albert, Barabási and Jeong (1999), Faloutsos, Faloutsos and Faloutsos (1999), Broder, Kumar, Maghoul, Raghavan, Rajagopalan, Stata, Tomkins and Wiener (2002)

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Modelling Choices:
Fix a degree sequence and make each graph with this degree sequence equally likely: Bender and Canfield (1978), Bollobás (1980), Molloy and Reed (1995) and Cooper and Frieze(digraphs) (2004).

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Modelling Choices:
Fix a degree sequence $d_1, d_2, \ldots, d_n$ and make edge (i, j) occur independently with probability proportional to $d_i d_j$: Chung and Lu (2002), Mihail and Papadimitriou (2002)

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Modelling Choices:
Preferential Attachment Model: Vertex set $v_1, v_2, \ldots, v_n, \ldots$;
Vertex v_{n+1} chooses m random neighbours in $v_1, \ldots, v_n$ with probability proportional to their degree.

Introduced as a model of the web by Barabási and Albert (1999).

Properties of the Preferential Attachment Model PAM

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- Classifying special interest groups in web graphs: Cooper (2002)

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$$d_k\left(\frac{k}{2}+1\right) \sim d_{k-1}\frac{k-1}{2} + 1_{k=m}$$

$$d_k \sim \frac{2m(m+1)}{(k+2)(k+1)k}t \qquad \textit{for } k \geq m.$$

Some Open Problems

What is the second eigenvalue of the transition matrix of a random walk on PAM?

It should be $O(1/m)$.

Some Open Problems

What is the size of the smallest dominating set in PAM?

Some Open Problems

What is the expected time to for a random walk to get within distance d of every vertex?

$d = 0$ is Cover Time and is understood.

Should be $o(n)$ for $d \geq 2$.

Some Open Problems

Forest Fire Model Leskovec, Kleinberg and Faloutsos (2005).

v_{t+1} randomly chooses an ambassador node w from $v_1, v_2, \ldots, v_{t+1}$ and we get the edge (v, x). Then a random process constructs a tree rooted at w, all of whose nodes are joined to v_{t+1}.

The graph produced is difficult to analyse rigorously.

How many edges? What is the diameter? ...

Achlioptas Problem

Suppose that $e_1, f_1, e_2, f_2, \ldots,$ is a random sequence of pairs of edges e_i, f_i. You have to choose, on-line, one of e_i, f_i for $i = 1, 2, \ldots$. Can you avoid creating a giant component for significantly beyond $n/2$ choices?

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Bohman and Frieze (2001): If one of e_i, f_i is disjoint from $e_1, f_1, \ldots, e_{i-1}, f_{i-1}$ then choose this edge, otherwise just take e_i.

Whp one can choose $.544n$ edges before creating a giant.

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Suppose that $e_1, f_1, e_2, f_2, \ldots,$ is a random sequence of pairs of edges e_i, f_i. You have to choose, on-line, one of e_i, f_i for $i = 1, 2, \ldots$. Can you avoid creating a giant component for significantly beyond $n/2$ choices?

Subsequently several authors: Bohman and Kravitz (2005), Spencer and Wormald (2005) and Flaxman, Gamarnik and Sorkin (2004) studied algorithms for delaying and/or speeding up the emergence of a giant component.

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In particular, $.544n$ can been significantly improved. SW improve it to $.829n$ and it is know Bohman, Frieze and Wormald that $.983n$ is an upper bound for the delay.

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Related off-line problems were considered in Bohman, Frieze and Wormald, Bohman and Kim.

In particular, the BK and SW papers show that for a restricted class of algorithm, differential equations can be used to accurately predict the emergence of a giant, by tracking the parameter

$$Z = \frac{1}{n} \sum_i |C_i|^2.$$

Where $C_1, C_2, \ldots$ are the components of the graph induced by the edges selected so far.

The giant should appear when this parameter becomes unbounded.

Open Questions

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Analyze the algorithm that always chooses the edge which produces the smallest increase in Z. When does a giant component appear?

The differential equations method has problems here, because the natural system of equations is infinite.

Open Questions

Consider speeding up or delaying the occurrence of other graph properties e.g. avoid 3-colorability.

Game Version

Suppose there are two players, Creator and Destroyer. Creator plays on odd rounds and Destroyer plays on even rounds. Creator wants to construct a giant component as soon as possible and Destroyer wants to delay the occurrence for as long as possible.

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Beveridge, Bohman, Frieze and Pikhurko (2006) show that the best strategy for Creator is to try to maximize the increase in Z and the best strategy for Destroyer is to try to minimize the increase in Z.

If they both play optimally, then it takes roughly $n/2$ rounds to create a giant, since they tend to cancel each others advantage over just choosing randomly.

Random Geometric Graphs

Choose points $X_1, X_2, \ldots, X_n$ randomly from the unit square $[0,1]^2$ and then join X_i, X_j by an edge if $|X_i - X_j| \leq r$. Lets call the graph $X_{n,r}$.

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If $\pi r^2 n = (1+\epsilon)\log n$ then $X_{n,r}$ is Hamiltonian **whp**.
Díaz, Mitsche and Pérez (2006)

Open Question

Given $X_1, X_2, \ldots, X_n$ and an integer k, we can define the k-nearest neighbour graph, where each X_i is joined by an edge to its k nearest points.

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Given $X_1, X_2, \ldots, X_n$ and an integer k, we can define the k-nearest neighbour graph, where each X_i is joined by an edge to its k nearest points.

For what value of k does the graph have a giant component **whp**?

Teng and Yao show that $k > 1$ is necessary and $k \geq 212$ is sufficient.

Experiments "suggest" $k = 3$ is the right answer.

THANK YOU