On rainbow trees and cycles

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Both theorems use the (lop-sided) local lemma.

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There exists an absolute constant $c > 0$ *such that if an edge colouring of* K_n *is* cn*-bounded then there exist rainbow cycles of all sizes* $3 \leq k \leq n$.

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For smaller k we use the following: If $c > 0$ and an edge colouring of K_n is cn-bounded, then there exists a set $S \subseteq [n]$ such that $|S| = N = n/2$ and the induced colouring of the edges of S is $c'N$-bounded where $c' = c(1 + 1/(\ln n)^2)$.

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To prove this, we take a random $n/2$ set S.

To complete the theorem, we take c sufficiently small and we apply this $\sim \log_2 n$ times until we have shown the existence of rainbow cycles of length $N \leq k \leq n$ where $cN \leq 1$ and a set of N vertices for which the edge coloring is cN bounded.

Rainbow Trees

Theorem

Given a real constant $\epsilon > 0$ and a positive integer Δ, there exists a constant $c = c(\epsilon, \Delta)$ such that if $n \geq (1 - \epsilon)\Delta$ and an edge colouring of K_n is cn-bounded, then it contains a rainbow copy of every tree T with at most $(1 - \epsilon)n$ vertices and maximum degree Δ.

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Conjecture: There is a constant $c = c(\Delta)$ such that every cn-bounded edge colouring of K_n contains a rainbow copy of every *spanning* tree of K_n which has maximum degree at most Δ.

Our main tool is a theorem of Alon, Krivelevich and Sudakov:

Suppose that $\Delta \geq 2$ and $0 < \epsilon < 1/2$. Let H be a graph on N vertices with minimum degree δ_H and maximum degree Δ_H.

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Then H contains a copy of every tree with $\leq (1 - \epsilon)N$ vertices and maximum degree $\leq \Delta$.

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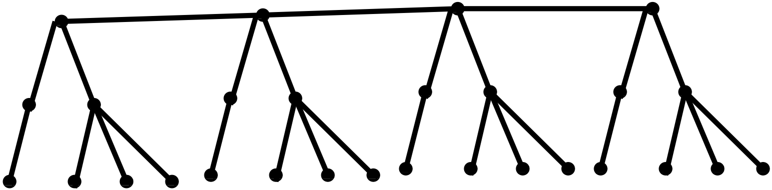
Existence of rainbow trees has now been demonstrated.

Using the (lop-sided) local lemma one can also prove:

Let T be an arbitrary rooted tree with ν vertices.

Let $T_1, T_2, \ldots, T_\nu$ be copies of T with roots $x_1, \ldots, x_\nu$.

Run a path through $x_1, \ldots, x_\nu$ to create the tree $T(\nu)$.



There exists an absolute constant $c > 0$ such that if an edge colouring of K_n is cn-bounded then there exists a rainbow copy of every possible $T(\nu)$.

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- Construct a polynomial time algorithm to find a rainbow Hamilton cycle in a cn-bounded coloring of K_n.
- Construct a polynomial time algorithm to find a random rainbow Hamilton cycle in a cn-bounded coloring of K_n.
- For what values of c, p does a cnp bounded coloring of $G_{n,p}$ contain a rainbow Hamilton cycle **whp**?

THANK YOU