Almost universal graphs

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If $\mathcal{H} = \mathcal{H}(c, n)$ is the class of graphs with vertex set $[n]$ and maximum degree $c$, then any $\mathcal{H}$-universal graph must contain $\Omega(n^{2-2/c})$ edges.

Alon, Capalbo, Kohayakawa, Rödl, Ruciński, Szemerédi
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An \( \mathcal{H}^*(c, n) \)-universal graph requires \( \Omega(n^{2-o(1)}) \) edges. It must contain all graphs with \((1 - \varepsilon)n\) isolated vertices and a \([c/\varepsilon]\)-regular graph on the remaining \(\varepsilon n\) vertices.
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We relax the notion of universal to almost universal.
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\( G \) is almost universal for \( \mathcal{H} \) if it contains a subgraph isomorphic to all but \( o(|\mathcal{H}|) \) graphs in \( \mathcal{H} \).

In particular we consider \( \mathcal{H} = \mathcal{H}^*(c, n) \). We estimate

\[
\Pr(G_{n,m} \text{ is almost universal for } \mathcal{H}^*(c, n)) \]

which we reduce to

\[
\Pi(c, m) = \Pr(G_{n,m} \supseteq G_{n,cn/2})
\]

where \( \supseteq \) denotes “contains a subgraph isomorphic to”. The two graphs \( G_{n,m}, G_{n,cn/2} \) are drawn independently.
Theorem
(a) Suppose that $c < 1$ is constant. Then if $A$ is constant,

$$
\Pi(c, m) \begin{cases} 
\leq 1 - (1 - e^{-c^3/6})e^{-A^3/6} + o(1), & m = An \\
= 1 - o(1), & m \geq \frac{C_0 \log \log n}{\log \log \log n} n 
\end{cases}
$$

for some sufficiently large $C_0$. 
Theorem

(a) Suppose that $c < 1$ is constant. Then if $A$ is constant,

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for some sufficiently large $C_0$.

(b) Suppose that $c > 1$ is constant. Then for constants $C_1, C_2$,

$$\Pi(c, m) = \begin{cases} 
o(1), & m \leq C_1 n^{2-2/(c+x_c)} \\
1 - o(1), & m \geq C_2 n^{2-1/(c-y_c)} 
\end{cases}$$

where $x_c, y_c \to 0$ as $c \to \infty$. 
In work on random graphs, it is usually more convenient to work in the independent model $G_{n,p}$ rather than in $G_{n,m}$. We therefore estimate

$$\Pi^\#(p_1, p_2) = \Pr(G_{n,p_2} \supseteq G_{n,p_1})$$

where $G_{n,p_1}$ and $G_{n,p_2}$ are generated independently and $p_2 = c/n$. 
\(c < 1, m = An, (p_2 = 2m/n).\)

\[\Pr(G_{n,p_1 - \frac{\log n}{n^{3/2}}} \text{ contains a triangle and } G_{n,p_2 + \frac{\log n}{n^{3/2}}} \text{ is triangle free}) = (1 - e^{-c^3/6})e^{-A^3/6} + o(1).\]
\( c < 1, \quad m = \omega n, \quad \omega = 21 \log \log n / \log \log \log n, \)

\[ G_i = G_{n,p_i}, \quad i = 1, 2, \quad (p_2 = 2m/n). \]

We assume that \( G_1 \) consists of isolated trees \( T_1, T_2, \ldots, T_s \) and unicyclic components \( K_1, K_2, \ldots, K_t \) of which \( > n(1 - e^{-1}) \) are isolated vertices.
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We keep embedding trees until either (i) we are done, or (ii) we have used $n(1 - e^{-1})$ vertices.
A vertex of $G_1$ is \textbf{large} if its degree is $\geq \omega/20$. For a tree $T$ of $G_1$, let

$$\sigma(T) = \prod_{i=1}^{a(T)} (d_i - 1)!$$

where $d_1, d_2, \ldots, d_{a(T)}$ are degrees of large vertices of $T$. 

\[\text{Lemma}\] The expected number of isolated trees $T$ in $G_1$ with $k$ vertices and $a(T) = a$ and $d_1 + d_2 + \ldots + d_{a(T)} = D$ is bounded by $A_1 a^2 D$. \[\text{p.10}\]
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**Lemma**

The expected number of isolated trees $T$ in $G_1$ with $k$ vertices and $\sigma(T) = \sigma$ and $a(T) = a$ and $d_1 + d_2 + \cdots + d_a = D$ is bounded by

$$An(ce^{1-c})^k(k e^{-1})^a 2^D \sigma^{-1}.$$
The expected number of vertices examined in trying to embed a tree $T$ with $k$ vertices and $d_1 + \cdots + d_a = D$ is at most

$$k(1 - e^{-\omega/10}) - k e^{2\omega a} 2^{D-a} \sigma(T) \omega^{-D+a}.$$
The expected number of vertices examined in trying to embed a tree $T$ with $k$ vertices and $d_1 + \cdots + d_a = D$ is at most

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So the expected number of vertices used to embed all trees with at least one large vertex is bounded by

$$A \sum_{\sigma,k,a,D} n (ce^{1-c})^k (ke^{-1})^a 2^D \sigma^{-1} k(1-e^{-\omega/10}) - k e^{2\omega a} 2^{D-a} \sigma \omega^{-D+a}$$

$$< An \sum_{\sigma,k,a,D} ((1 + o(1)) k(ce^{1-c}))^k (k\omega e^{2\omega - 1}/2)^a 2^{2D} \omega^{-D}$$

$$\leq n/(\log n)^{1/2}$$
A similar approach is used to embed the trees without large vertices and the unicyclic components.
$c > 1$, $m \leq C_1 n^{2-2/(c+x_c)}$

where $x = x_c$ be the unique solution in $(0, 1)$ to $xe^{-x} = ce^{-c}$. 
\( c > 1, \ m \leq C_1 n^{2-2/(c+x_c)} \)
where \( x = x_c \) be the unique solution in \((0, 1)\) to \( xe^{-x} = ce^{-c} \).

\[ \alpha = 1 - \frac{x}{c} \quad \text{and} \quad \beta = \frac{c}{2} \left( 1 - \frac{x^2}{c^2} \right). \]

Whp \( G_{n,cn/2} \) contains a giant component \( H \) with \( A \) vertices and \( B \) edges, where \( |A - \alpha n|, |B - \beta n| = O(n^{1/2} \log n) \) vertices.
\[
\Pr[G_{n,m} \supseteq G_1] \leq \Pr[G_{n,m} \supseteq H] \\
\leq (n)_A \left( (1 + o(1)) \frac{2m}{n^2} \right)^B \\
\leq n^{\alpha n + O(\sqrt{n} \log n)} \left( \frac{2m}{n^2} \right)^{\beta n + O(\sqrt{n} \log n)} \\
= \left[ n^{(1+o(1))\alpha} \left( \frac{2m}{n^2} \right)^{(1+o(1))\beta} \right]^n.
\]
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= \left[ n^{(1 + o(1))\alpha} \left( \frac{2m}{n^2} \right)^{(1 + o(1))\beta} \right]^n.
\]

If \( m \leq c_1 n^{2-2/(c+\epsilon_c)} \) for \( c_1 > 0 \) small, the above \( \to 0 \) as \( n \) grows, and \textbf{whp} \( G_{n,m} \) doesn’t contain almost all of the \( G_{n,cn/2} \).
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Whp the vertices of $G_{n,c/n}$ can be ordered as $v_1, v_2, \ldots, v_n$ so that $v_i$ has at most $d$ neighbours in $\{v_1, v_2, \ldots, v_{i-1}\}$.

$$d = c - \sqrt{c \log c} + O(\log c)$$ (follows from $k$-core paper of Pittel, Spencer, Wormald).

Also, $G_1$ is $K_{2,3}$-free whp.
Theorem

Let $\delta = \max \left\{ \frac{1}{d-1}, \frac{1}{d(d-3)} \right\}$. Let $p(n) = An^{(-1+\delta)/d}$, $A$ large.

$H$ is a $d$-degenerate $K_{2,3}$-free graph on $c_0n$, $c_0 < 1$ vertices, of maximum degree $\Delta(H) \leq \Delta_0 = 1/(4dp)$.

Whp the random graph $G_{n,p}$ contains a copy of $H$. 
Every $d$ vertices have $\geq A_1 n^\delta$ common neighbours
After adding some random edges to $H$ we can assume that if $U_i$ is the set of neighbours of $v_i$ in $\{u_1, u_2, \ldots, u_{i-1}\}$ then

(i) For each pair $A_1 n^\delta \leq i < j$, $U_i \setminus U_j \neq \emptyset$.

(ii) Each $U_k$ intersects at most $2d\Delta_0$ sets $U_j, j \neq k$. 
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(ii) Each $U_k$ intersects at most $2d\Delta_0$ sets $U_j, j \neq k$.

Now we try to embed the rest of $H$ in the order $v_{n_0+1}, \ldots, v_{c_0 n}$.
$W_i$ denotes the image of $U_i$ under the implied mapping.

The conditioning on $w$ is that $N_2(w) \nsubseteq W_i$, $i \in J'$ for some $J' \subseteq J = \{i < j : U_i \cap U_j \neq 0\}$.
\[ \Pr(N_2(w) \supseteq W_j \mid \text{history of process}) = \]
\[ \Pr(N_2(w) \supseteq W_j \mid N_2(w) \not\supseteq W_i, i \in J') \]
\[ \geq \Pr(N_2(w) \supseteq W_j \text{ and } x_i \not\in N_2(w), i \in J') \]
\[ \geq p^d (1 - p)^{2d\Delta_0} \]
\[ \geq p^d / 2. \]
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\[ \geq \Pr(N_2(w) \supseteq W_j \text{ and } x_i \notin N_2(w), i \in J') \]
\[ \geq p^d (1 - p)^{2d \Delta_0} \]
\[ \geq p^d / 2. \]

Thus,
\[ \Pr(\forall w : N_2(w) \supseteq W_j \mid \text{history of process}) \]
\[ \leq (1 - p^d / 2)^{(1-c_0)n/2} \leq e^{-An^\delta/4}. \]