Time allowed: 180 minutes.

1. Recite precisely the following definitions/facts/theorems/lemmas:
   (a) Tail $\sigma$-algebra. Kolmogorov’s $0 - 1$ law.
   (b) Kolmogorov’s three-series theorem on convergence of sums of IRVs.
   (c) Doob’s maximal $L_p$ inequalities for martingales.
   (d) Method of characteristic functions in weak convergence.

2. Let $(M_n)$ be a martingale bounded in $L_1$, that is, $\sup_n \mathbb{E}[|M_n|] < \infty$.
   (a) Will $(M_n)$ converge in distribution?
   (b) Will $\sup_n |M_n| < \infty$ (a.s.)?
   (c) Will $(M_n)$ be uniformly integrable?
   (d) Is it possible to find a random variable $M_\infty$ such that $M_n = \mathbb{E}[M_\infty | \mathcal{F}_n]$?

   Justify your answers.

3. Let $(X_n)$ be IRVs with zero mean such that $|X_n| \leq K$ for a positive constant $K$. Let
   
   $S_n = X_1 + \cdots + X_n, \quad S_0 = 0,$

   and suppose that $(S_n)$ converges (a.s.) to a RV $Y$.
   (a) Can we assert that $S_n = \mathbb{E}[Y | X_1, \ldots, X_n]$?
   (b) Is it true that $\mathbb{E}[\sup_n S_n^2] < \infty$?

   Justify your answers.
4. Let \((X_n)\) be IID Gaussian RVs with mean 0 and variance 1. Denote 
\[ S_n = X_1 + \cdots + X_n, \quad S_0 = 0. \]
Prove that 
\[ \mathbb{P}\left[ \max_{k \leq n} S_k \geq c \right] \leq e^{-\frac{c^2}{2n}}, \quad \forall c > 0. \]

5. Let \((X_n)\) be IID RVs with uniform distribution on \([0, 1]\). We denote 
\[ M_n = 2^n \prod_{k \leq n} X_k, \quad n \geq 1. \]
Show that there is a RV \(Y\) such that \(M_n \to Y\) (a.s.) and compute \(Y\).

6. Let \((X_n)\) be IID RVs with uniform distribution on \([0, 1]\). Compute the characteristic function \(\phi_n = \phi_n(\theta)\) of 
\[ Y_n = \mathbb{E}[X_1|X_1 + \cdots + X_n]. \]
Show that \(\phi_n(\theta) \to \phi(\theta), \ \theta \in \mathbb{R}\), and compute \(\phi = \phi(\theta)\).

7. A gambler throws dice until he gets \(N\) identical outcomes in a row. The cost of one throw is \(q\) and the reward at the end is \(A\). Find the relationship between \(q\), \(A\) and \(N\) for the game to be fair; that is, for the expected payoff to be 0.

8. Let \((X_n)\) be IID RVs with uniform distribution on \([0, 1]\). Denote 
\[ S_n = X_1 + \cdots + X_n, \quad S_0 = 0, \]
and 
\[ \tau = \min\{n \geq 0 : S_n > 1\}. \]
Compute \(\mathbb{E}[\tau]\) and \(\mathbb{E}[S_\tau]\).