1. Prove or find a counterexample: If \( f: (0, 1) \to (-\infty, \infty) \) is continuous and \( \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} f \, dm \) exists and is finite, then \( f \) is Lebesgue integrable on \( (0, 1) \).

2. Suppose \( f_n: X \to [0, \infty) \) is measurable for each \( n \in \mathbb{N} \) and \( \mu(X) < \infty \). Show that the sequence \( \{f_n\} \) converges to 0 in measure if and only if \( \int_X f_n \, d\mu \to 0 \) as \( n \to \infty \).

3. Suppose that for each \( n \in \mathbb{N} \), \( f_n = 1_{E_n} \) for some Lebesgue measurable set \( E_n \subset [0, 1]^d \), and \( \mu(A) \overset{\text{def}}{=} \lim_{n \to \infty} \int_A f_n \, dm \) exists and is finite for each Borel set \( A \subset [0, 1]^d \). Show that \( \mu \) is a Borel measure, and \( \mu \) is absolutely continuous with respect to Lebesgue measure \( m \) on \([0, 1]^d \), with Radon-Nikodym derivative \( d\mu/dm \) taking values in \([0, 1]\) a.e.

4. Suppose \( f \) and \( g \) are positive and measurable on \( X = [0, 1] \), and satisfy

\[
f(x)g(x) \geq 1 \quad \text{for all} \quad x \in [0, 1].
\]

(i) Show that \( \left( \int_X f \, dm \right) \left( \int_X g \, dm \right) \geq 1 \).

(ii) For which choices of \( p, q \in (0, \infty) \) does the assumption (1) imply

\[
\left( \int_X f^p \, dm \right)^{1/p} \left( \int_X g^q \, dm \right)^{1/q} \geq 1. \]

5. Let \( u: \mathbb{R} \to (0, \infty) \) be Lebesgue measurable and 1-periodic. Let \( X = [0, 1] \) and suppose \( \int_X u \, dm = 1 \). For each \( \varepsilon > 0 \), define \( u_\varepsilon(x) = u(x/\varepsilon) \), and define a measure \( \mu_\varepsilon \) on \( \mathcal{B}(X) \) by

\[
\mu_\varepsilon(A) \overset{\text{def}}{=} \int_A u_\varepsilon \, dm.
\]

(i) Prove that \( \mu_\varepsilon(A) \to m(A) \) as \( \varepsilon \to 0^+ \) for every \( A \in \mathcal{B}(X) \).

(ii) Let \( \nu_\varepsilon \overset{\text{def}}{=} |\mu_\varepsilon - m| \) denote the total variation of the signed measure \( \mu_\varepsilon - m \) on \( X \).

Show that if \( \nu_\varepsilon(X) \to 0 \) as \( \varepsilon \to 0^+ \), then \( u(x) = 1 \) a.e.