Basic Examination
General Topology
January 2017

Time allowed: 3 hours.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (20)</td>
<td></td>
</tr>
<tr>
<td>2 (20)</td>
<td></td>
</tr>
<tr>
<td>3 (20)</td>
<td></td>
</tr>
<tr>
<td>4 (20)</td>
<td></td>
</tr>
<tr>
<td>5 (20)</td>
<td></td>
</tr>
<tr>
<td>Total (100)</td>
<td></td>
</tr>
</tbody>
</table>
1. Consider $\mathbb{R}^N$ endowed with the box topology, that is the topology with the base

$$
B = \left\{ \prod_{n=1}^{\infty} U_i : U_i \text{ is open in } \mathbb{R} \right\}.
$$

Show that $f, g \in \mathbb{R}^N$ belong to the same connected component of $\mathbb{R}^N$ if and only if there exists $n_0$ such that for all $n \geq n_0$, $f_n = g_n$. 
2. Let \((X, \prec)\) be a well-ordered uncountable set. Recall that a set is well ordered if it is linearly ordered and every nonempty subset has a minimal element. We define the intervals

\[
(a, b) := \{ x \in X : a \prec x \prec b \} \\
[a, b] := \{ x \in X : a \preceq x \preceq b \}
\]

Let 0 be the smallest element of \(X\). Let \(\omega_1 = \min\{x \in X : [0, x] \text{ is uncountable}\}\). Consider the order topology on \(Y = [0, \omega_1)\), that is the topology with the basis

\[
B = \{(a, b) : a, b \in Y \} \cup \{[0, b) : b \in Y \}.
\]

Show that

(i) \(Y\) is not compact.

(ii) \(Y\) is countably compact, that is every countable open cover has a finite subcover.

Hint: To show (ii) it is useful to show that \(Y\) does not have a closed countable infinite subset on which the induced topology is discrete. To do so assume \(A\) is such a subset. Let \(\beta = \min\{x \in Y : [0, x] \cap A \text{ is infinite}\}\). Show that \(\beta\) is a limit point of \(A\).
3. (i) [15 points] Assume that $X$ is a $T_1$ space such that for every $C$ closed and every $W$ open such that $C \subseteq W$ there exists a sequence of open sets $\{W_n\}_{n=1,2,...}$ such that

$$C \subseteq \bigcup_{n=1}^{\infty} W_n \quad \text{and} \quad (\forall n) \ W_n \subseteq W.$$

Show that $X$ is $T_4$.

(ii) [5 points] Show that any 2nd countable $T_3$ space is $T_4$. 
4. Let $X = C([0, 1], \mathbb{R})$. Define for $f, g \in X$

$$
    d(f, g) = \int_0^1 \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} \, dx.
$$

(i) Show that $d$ is a metric on $X$.

(ii) Show that $(X, d)$ is not a complete metric space.
5. Let \{f_n\}_{n=1,2,...} be a sequence of continuous functions on \([0, 1]\) with values in \(\mathbb{R}\). Assume that \(f_n\) are differentiable on \((0, 1]\) and assume that for all \(n\) and all \(x \in (0, 1]\)

\[
|f_n'(x)| \leq \frac{1}{\sqrt{x}}
\]

Also assume \(\int_0^1 f_n(x) \, dx = 0\). Prove that \(\{f_n\}_{n=1,2,...}\) has a uniformly convergent subsequence.