Weak and Measure-Valued Solutions of the Incompressible Euler Equations

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Incompressible Euler Equations

The Cauchy problem for the incompressible Euler equations of inviscid fluid motion in d dimensions ($d \geq 2$) reads

$$\begin{aligned} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p &= 0 \\ \operatorname{div} v &= 0 \\ v(\cdot, 0) &= v_0 \end{aligned}$$

where the velocity v and the scalar pressure p are sought for and v_0 is a given initial velocity field with $\operatorname{div} v_0 = 0$.

Subsolutions

Following the framework of C. De Lellis and L. Székelyhidi, observe that the Euler equations are equivalent to

$$\begin{aligned} \partial_t v + \operatorname{div} u + \nabla q &= 0 \\ \operatorname{div} v &= 0 \end{aligned} \tag{1}$$

and $u = v \otimes v - \frac{1}{d}|v|^2\mathbb{I}_d$, $q = p + \frac{1}{d}|v|^2$ a.e. For a vector v and a traceless symmetric matrix u one defines the generalized energy

$$e(v,u) = \frac{d}{2}\lambda_{max}(v \otimes v - u).$$

A solution (v, u, q) of (1) is called a subsolution w.r.t initial data $v_0(x)$ and an energy density $\bar{e}(x,t)$ if

$$v(\cdot,0) = v_0, \;\; e(v,u) < \bar{e}.$$

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From Subsolutions to Exact Solutions

Theorem (De Lellis-Székelyhidi '10)

Let $v_0 \in L^2(\mathbb{R}^d)$ *be divergence-free and*

$$\bar{e} \in C(\mathbb{R}^d \times (0,\infty)) \cap C([0,\infty); L^1(\mathbb{R}^d)).$$

Suppose further that $(v,u,q) \in C^\infty(\mathbb{R}^d \times (0,\infty))$ *is a subsolution w.r.t.* v_0 *and* $\bar{e}$*, and that* $v \in C([0,\infty); L^2_w(\mathbb{R}^d))$*. Then, there exist infinitely many weak solutions for Euler in* $C([0,\infty); L^2_w(\mathbb{R}^d))$ *with initial data* v_0 *and energy density* $\bar{e}$*.*

Remark. The Theorem also holds in the case of periodic boundary conditions.

Global Existence of Weak Solutions

We now assume periodic boundary conditions.

Theorem (E. W. '11)

Let $v_0 \in L^2_{per}(\mathbb{R}^d)$ be divergence-free. Then there exist infinitely many global weak solutions in $C([0,\infty); L^2_w(\mathbb{R}^d))$ with initial data v_0 and bounded kinetic energy.

Remarks.

- This is the first global existence result for weak solutions of Euler.
- Although the energy of these solutions is bounded, it can not be chosen to be non-increasing. In fact, there will be a jump in the energy at time 0.

Sketch of Proof

Owing to De Lellis' and Székelyhidi's result, it suffices to construct subsolutions w.r.t. v_0 and some energy density yet to be chosen. To this end, consider the fractional heat equation

$$\begin{aligned} \partial_t v + (-\Delta)^{1/2} v &= 0 \\ \operatorname{div} v &= 0, \end{aligned}$$

which can easily be solved by Fourier transform. Since $(-\Delta)^{1/2} = -\operatorname{div}\mathcal{R}$ (where $\mathcal{R}$ denotes the Riesz transform), we see that $(v, -\mathcal{R}v, 0)$ is a subsolution. One can then choose $\bar{e}$ such that this subsolution has all the desired properties.

Young Measures

A (generalised) Young measure on $\mathbb{R}^d$ with parameters in $\mathbb{R}^d \times \mathbb{R}^+$ is a triple $(\nu_{x,t}, \lambda, \nu^\infty_{x,t})$, where

- $\nu_{x,t} \in \mathcal{P}(\mathbb{R}^d)$ for a.e. $(x,t) \in \mathbb{R}^d \times \mathbb{R}^+$ (oscillation measure)
- $\lambda \in \mathcal{M}^+(\mathbb{R}^d \times \mathbb{R}^+)$ (concentration measure)
- $\nu^\infty_{x,t} \in \mathcal{P}(S^{d-1})$ for λ-a.e. $(x,t) \in \mathbb{R}^d \times \mathbb{R}^+$ (concentration-angle measure)

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Compactness

Fundamental Theorem of Young Measures (DiPerna-Majda '87, Alibert-Bouchitté '97)

If (v_n) is a bounded sequence in $L^2(\mathbb{R}^d \times \mathbb{R}^+; \mathbb{R}^d)$, then there exists a subsequence which generates some Young measure $(\nu_{x,t}, \lambda, \nu^\infty_{x,t})$, i.e.

$$f(v_n)dxdt \overset{*}{\rightharpoonup} \left(\int_{\mathbb{R}^d} f(z)d\nu_{x,t}(z) \right) dxdt + \left(\int_{S^{d-1}} f^\infty(\zeta)d\nu^\infty_{x,t}(\zeta) \right) \lambda$$

for every continuous $f : \mathbb{R}^d \to \mathbb{R}$ for which the L^2-recession function

$$f^\infty(\zeta) = \lim_{s\to\infty} \frac{f(s\zeta)}{s^2}$$

exists and is continuous on S^{d-1}.

Measure-Valued Solutions for Euler

Definition

A Young measure $(\nu_{x,t}, \lambda, \nu^\infty_{x,t})$ is called a *measure-valued solution (mvs)* of the Euler equations if

$$\partial_t \langle \nu_{x,t}, z \rangle + \operatorname{div}\left(\langle \nu_{x,t}, z \otimes z \rangle + \langle \nu^\infty_{x,t}, \zeta \otimes \zeta \rangle \lambda \right) + \nabla p(x,t) = 0$$

$$\operatorname{div} \langle \nu_{x,t}, z \rangle = 0$$

in the sense of distributions.

It is possible to define a notion of initial data and of kinetic energy for a mvs. We omit details. If $E(t)$ is the energy of a mvs with initial data v_0, and if

$$E(t) \leq \frac{1}{2} \int_{\mathbb{R}^d} |v_0|^2 dx$$

for a.e. $t > 0$, then we call the mvs admissible.

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The Main Result

It is easy to see that a sequence of admissible weak solutions of Euler generates an admissible mvs. Is the converse also true?

Theorem (L. Székelyhidi - E. W. '11)

Let $(\nu_{x,t}, \lambda, \nu^\infty_{x,t})$ *be an admissible mvs with initial data* v_0*. Then there exists a sequence* (v_n) *of weak solutions that generates* $(\nu, \lambda, \nu^\infty)$ *as a Young measure. In addition,*

$$\|v_n(t=0) - v_0\|_{L^2(\mathbb{R}^d)} < \frac{1}{n}$$

and

$$\sup_{t\geq 0} \frac{1}{2}\int_{\mathbb{R}^d} |v_n(x,t)|^2 dx \leq \frac{1}{2}\int_{\mathbb{R}^d} |v_n(x,0)|^2 dx.$$

Discussion of the Result

- A priori, mvs seem to be a much weaker concept than weak solutions. The Theorem shows however that they are in a sense the same.
- DiPerna and Majda constructed explicit examples for the development of oscillations and concentrations in sequences of weak solutions. The Theorem shows that in fact *any* conceivable oscillation/concentration behavior can be realized by a sequence of weak solutions.
- The result gives an example of a characterization of Young measures generated by constrained sequences where the constant rank property does not hold.
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An Existence Assertion

Corollary

Let $H = \{v \in L^2(\mathbb{R}^d) : \operatorname{div} v = 0\}$. *There exists a dense subset* $\mathcal{E} \in H$ *such that for all* $v_0 \in \mathcal{E}$ *there exists a weak solution with initial data* v_0 *such that*

$$\sup_{t \geq 0} \frac{1}{2} \int_{\mathbb{R}^d} |v(x,t)|^2 dx \leq \frac{1}{2} \int_{\mathbb{R}^d} |v_0(x)|^2 dx.$$