Self-propulsion in viscous fluids through shape deformation

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joint work with Gianni Dal Maso and Antonio DeSimone (SISSA)

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Motivations and definition

We are interested in the mathematical study of the motion of a micro-swimmer in a viscous fluid.
We give the following definition of swimming:

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Swimming is the ability of an organism to perform a variation of its spatial position caused by the variation of its shape, under the self-propulsion constraint.

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Self-propulsion means no external forces or momenta.

A brief (biased) history

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In our model, we build on these results to construct a framework with no constraints on the number of shape parameters and symmetry.

Movie time

(movie by Luca Heltai (SISSA))

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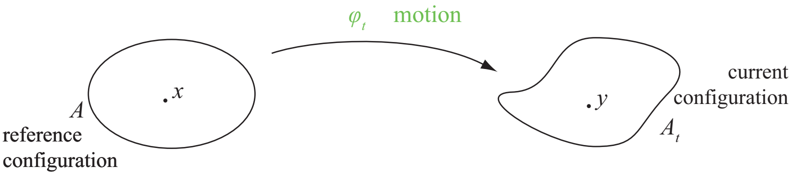
Two major properties:

- linearity, of the dependance of the solution on the boundary data (good);
- time reversibility, see the Scallop theorem; need of symmetry breaking to swim (can be dramatically bad).

The model II: the swimmer



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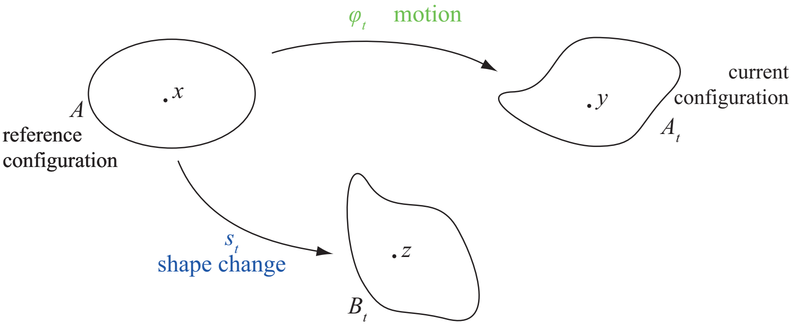

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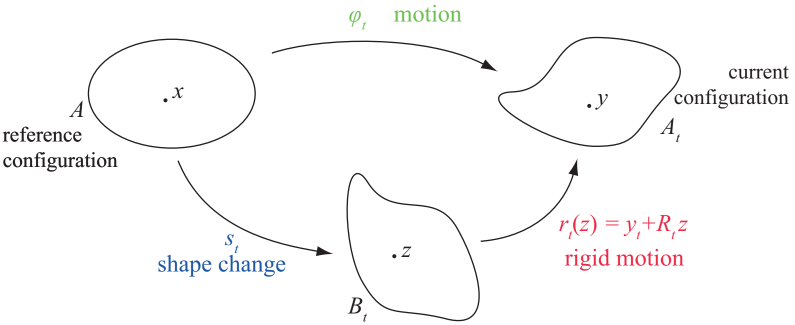

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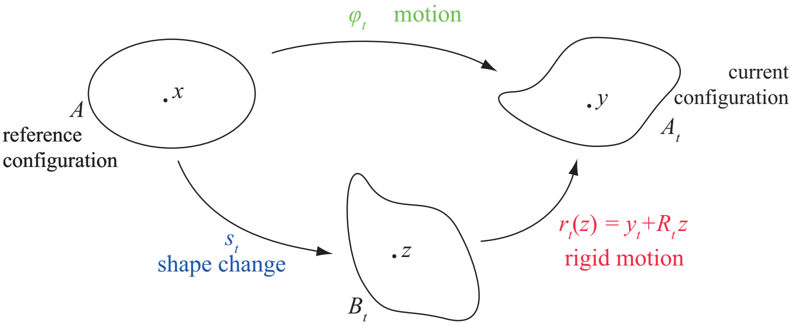

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Data are infinite dimensional, while the unknowns are finite dimensional.

Main existence, uniqueness, and regularity result

Theorem

For every sufficiently smooth shape change $t \mapsto s_t$ *, the position functions* $t \mapsto r_t$ *are uniquely determined by the initial conditions at* $t = 0$*. More precisely, there exists a unique family of rigid motions* $t \mapsto r_t$ *such that the state functions* $t \mapsto \varphi_t := r_t \circ s_t$ *satisfy the equations of motion, and* φ_t *(or equivalently* r_t*) takes a prescribed value at* $t = 0$.

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The equations of motion are derived from Newton's second law $\boldsymbol{F}_t = m\boldsymbol{a}_t$, where $\boldsymbol{F}_t = \boldsymbol{F}_t^{\text{ext}} + \boldsymbol{F}_t^{\text{visc}}$.

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$$F_t^{\text{visc}} = \int_{\partial A_t} \sigma_t n_t \, \mathrm{ds}(y) = 0, \qquad M_t^{\text{visc}} = \int_{\partial A_t} y \times \sigma_t n_t \, \mathrm{ds}(y) = 0.$$

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Write Stokes equations in the reference frame of the swimmer $B_t := s_t(A)$. The new boundary velocity is

$$\begin{aligned} V_t(z) &= R_t^\top \dot{y}_t + R_t^\top \omega_t \times z + R_t^\top \dot{s}_t \circ s_t^{-1} \\ &= V_t^{\text{rigid}}[\dot{y}_t\,, \omega_t] + V_t^{\text{shape}}[\dot{s}_t \circ s_t^{-1}], \end{aligned}$$

ω_t being the axial vector of $\dot{R}_t R_t^\top$.

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ω_t being the axial vector of $\dot{R}_t R_t^\top$. Then,

$$\begin{bmatrix} F_t^{\text{visc}} \\ M_t^{\text{visc}} \end{bmatrix} = - \begin{bmatrix} K_t & C_t^\top \\ C_t & J_t \end{bmatrix} \begin{bmatrix} R_t^\top & 0 \\ 0 & R_t^\top \end{bmatrix} \begin{bmatrix} \dot{y}_t \\ \omega_t \end{bmatrix} + \begin{bmatrix} F_t^{\text{shape}} \\ M_t^{\text{shape}} \end{bmatrix},$$

F_t^{shape} and M_t^{shape} being the viscous drag force and torque of a Stokes fluid with boundary velocity $\dot{s}_t \circ s_t^{-1}$.

$[\cdots]$ is referred to as the *grand resistance matrix*.

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The equations of motion

A simple manipulation yields

$$\begin{bmatrix} \dot{y}_t \\ \omega_t \end{bmatrix} = \begin{bmatrix} R_t & 0 \\ 0 & R_t \end{bmatrix} \begin{bmatrix} H_t & D_t^\top \\ D_t & L_t \end{bmatrix} \begin{bmatrix} F_t^{\text{shape}} \\ M_t^{\text{shape}} \end{bmatrix}$$

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We make the dependance on the data explicit

$$\dot{y}_t = R_t \big(H_t\, F_t^{\mathrm{shape}} + D_t^\top\, M_t^{\mathrm{shape}} \big),$$
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$$\begin{aligned} \dot{y}_t &= R_t\big(H_t(s_t)\, F_t^{\text{shape}}(s_t \qquad\qquad) + D_t^\top(s_t)\, M_t^{\text{shape}}(s_t \qquad\qquad)\big), \\ \dot{R}_t &= R_t \mathcal{A}\big(D_t(s_t)\, F_t^{\text{shape}}(s_t \qquad\qquad) + L_t(s_t)\, M_t^{\text{shape}}(s_t \qquad\qquad)\big). \end{aligned}$$

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Now, use classical results from ODE theory to get existence, uniqueness, and regularity of the solutions y_t and R_t.
What do we need more? Regularity for the coefficients $b_t(s_t\,, \dot{s}_t)$ and $\Omega_t(s_t\,, \dot{s}_t)$.

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Both cases are solved via a variational technique, which in case (2) is not straightforward.

How we solve the problem

The dependence of F_t^{shape} and M_t^{shape} on s_t and on $\dot{s}_t \circ s_t^{-1}$ simultaneously makes it hard to compare the external Stokes flows at different instants of time: there are difficulties in building a solenoidal velocity field that fits for these two different times.

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The dependence of $F_t^{\rm shape}$ and $M_t^{\rm shape}$ on s_t and on $\dot{s}_t \circ s_t^{-1}$ simultaneously makes it hard to compare the external Stokes flows at different instants of time: there are difficulties in building a solenoidal velocity field that fits for these two different times. We drop the request of continuity and prove that $F_t^{\rm shape}$ and $M_t^{\rm shape}$ depend in a measurable fashion on time t. This is enough to have existence and uniqueness to the equations of motion.

Could we ask for more regularity? Of course, we could; but cases when $\dot{s}_t$ is not even continuous are interesting in the optimal controllability problems.

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- Equations of motion: change of position and orientation in space achieved by change of shape, by exploiting the viscous resistance of the fluid.
- Solution of the resulting linear system of ODEs under minimal regularity assumptions on the data.
- We *proved* that the motion of a micro-swimmer is uniquely determined by the history of its shapes.

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Approximate theories for 1-d swimmers

We focus now on a mono-dimensional swimmer $\chi(s,t) \in \mathbb{R}^2$, $(s,t) \in [0,L]\times[0,T]$, performing a planar motion in a three-dimensional infinite viscous fluid. Approximate theories have been proposed to avoid the dimensional gap when stating the boundary conditions.

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\begin{aligned}
f(s,t) &= C_{\parallel}\dot{\chi}_{\parallel}(s,t)\chi'(s,t) + C_{\perp}\dot{\chi}_{\perp}(s,t)J\chi'(s,t), \\
m(s,t) &= \chi(s,t)\times[C_{\parallel}\dot{\chi}_{\parallel}(s,t)\chi'(s,t) + C_{\perp}\dot{\chi}_{\perp}(s,t)J\chi'(s,t)].
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The global drag force and momentum are given by

$$F(t) = \int_0^L K_\chi(s,t)\dot{\chi}(s,t)\,\mathrm{d}s, \quad M(t) = \int_0^L \chi(s,t)\times K_\chi(s,t)\dot{\chi}(s,t)\,\mathrm{d}s$$

Existence of an optimal swimming strategy

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$$\mathcal{P}(\chi) := \int_0^L \int_0^T \langle f(s,t), \dot{\chi}(s,t) \rangle \, \mathrm{d}s\mathrm{d}t = \int_0^L \int_0^T \langle K_\chi(s,t)\dot{\chi}(s,t), \dot{\chi}(s,t) \rangle \, \mathrm{d}s\mathrm{d}t.$$

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We prove the following result

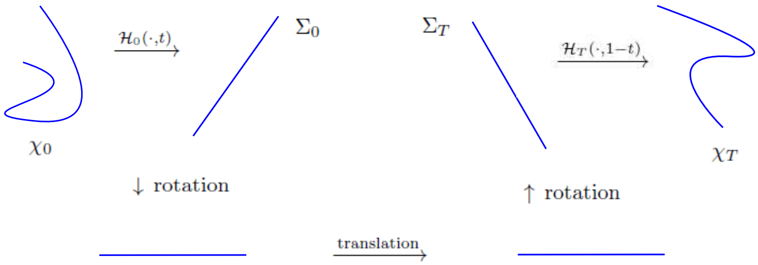
Theorem

The minimum problem

$$\min\{\mathcal{P}(\chi) : \chi \in \Xi, \chi(\cdot,0)=\chi_0(\cdot), \chi(\cdot,T)=\chi_T(\cdot), \textit{(SP)},\dots\},$$

where χ_0 and χ_T are assigned states, has a solution.

How to control the system



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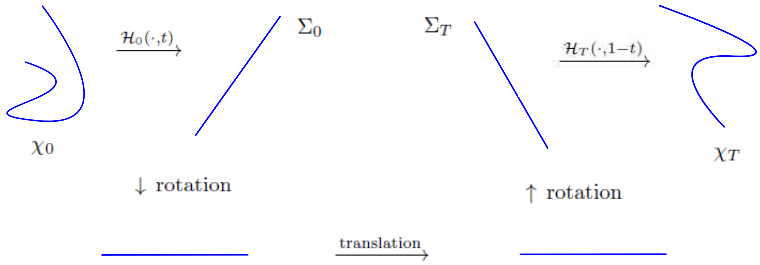

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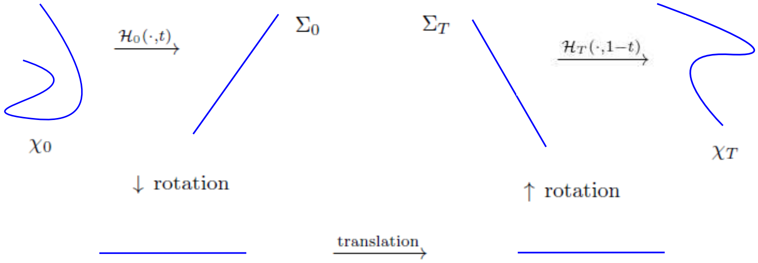
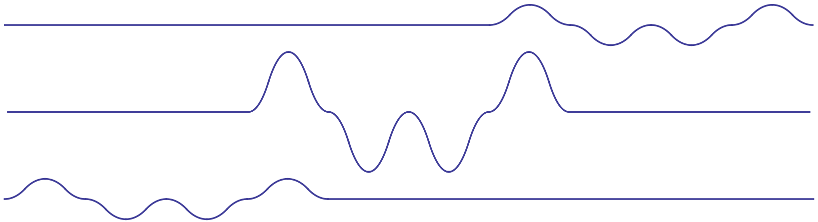
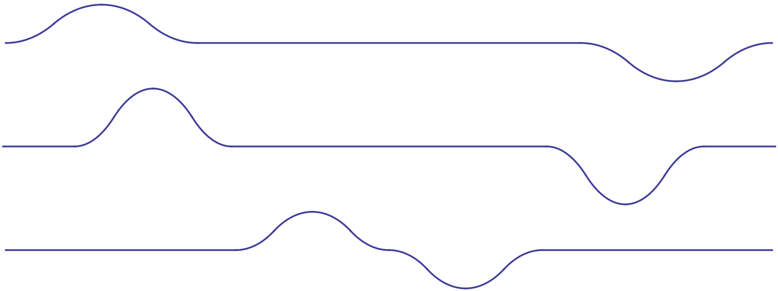


The maps $\mathcal{H}_0$ and $\mathcal{H}_T$ exist by virtue of the equations of motion. We build a way to make a straight rod translate along its centerline and rotate around its center.

Translation



Rotation



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Thank you very much for your attention!