Relative Entropy and the Stability of Shocks for Systems of Conservation Laws

Nicholas Leger
Carnegie Mellon University

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Outline

(I) Scalar Conservation Laws

- L^1 Stability: Kružkov's Estimate
- An L^2 result

(II) Systems of Conservation Laws

- Background: Shocks and Entropy
- Hypotheses and Definitions
- Statement of the Result
- Main Ideas of the Proof

Scalar Conservation Laws

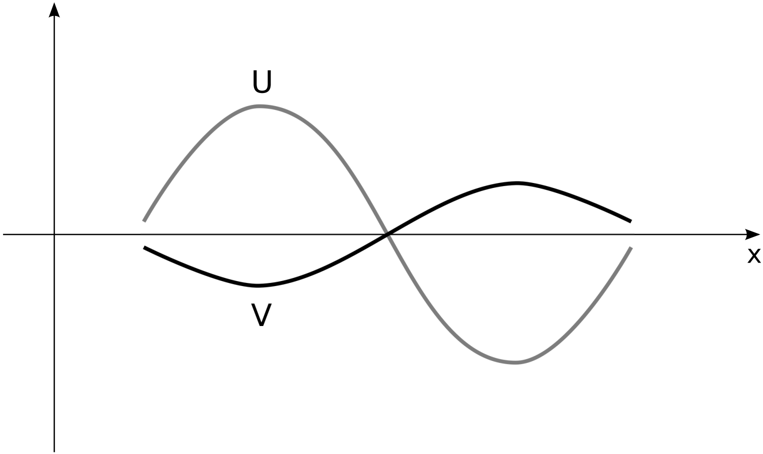
We consider the initial value problem

$$\begin{cases} \partial_t U + \partial_x A(U) = 0, \\ U(x,0) = U^0(x). \end{cases}$$

- $U : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$
- $A : \mathbb{R} \to \mathbb{R}$ smooth and strictly convex

L^1 Stability: Kružkov's Estimate

$$\|U(\cdot,t) - V(\cdot,t)\|_{L^1(\mathbb{R})} \leq \|U^0 - V^0\|_{L^1(\mathbb{R})}$$



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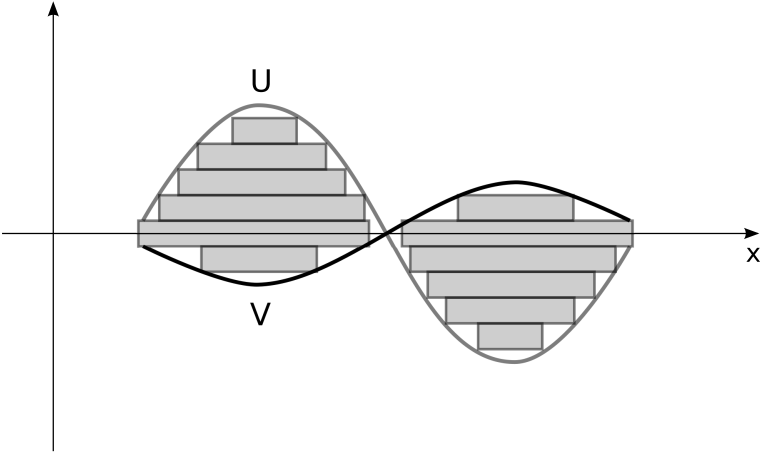

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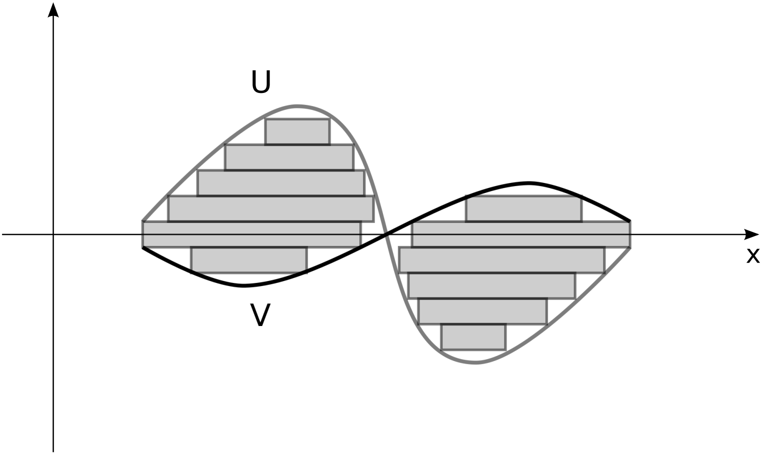

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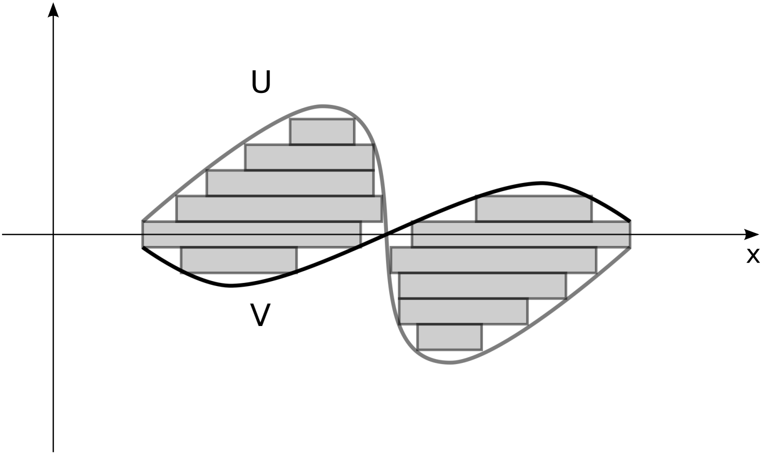

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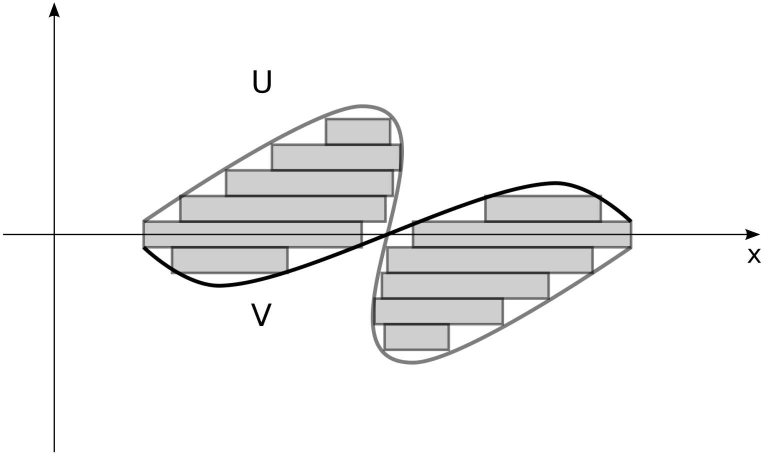

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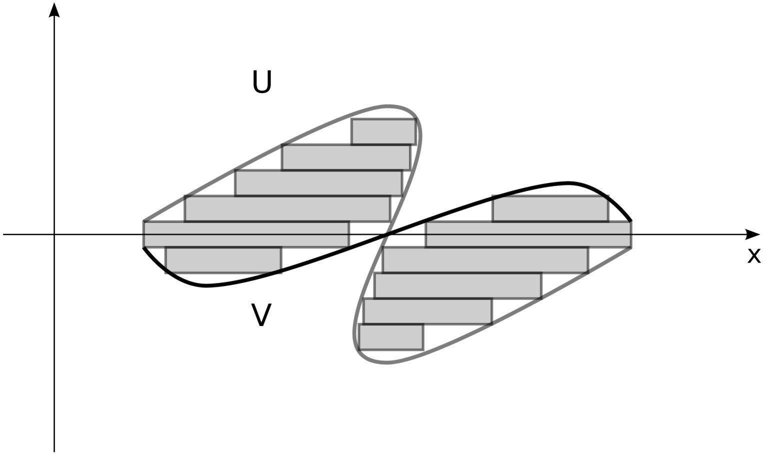

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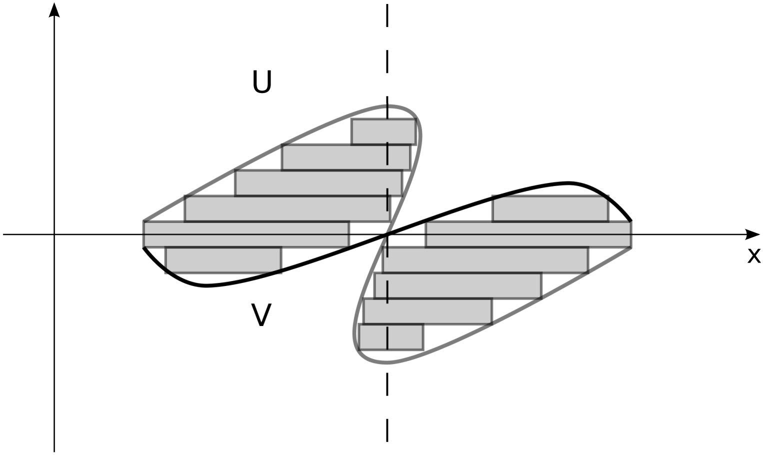

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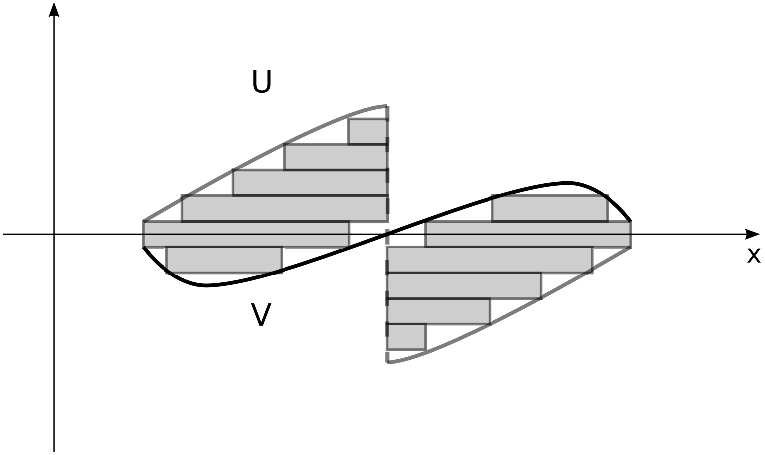

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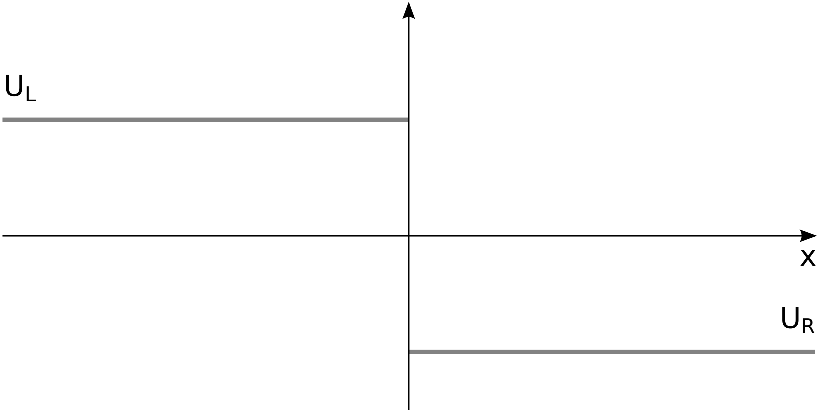

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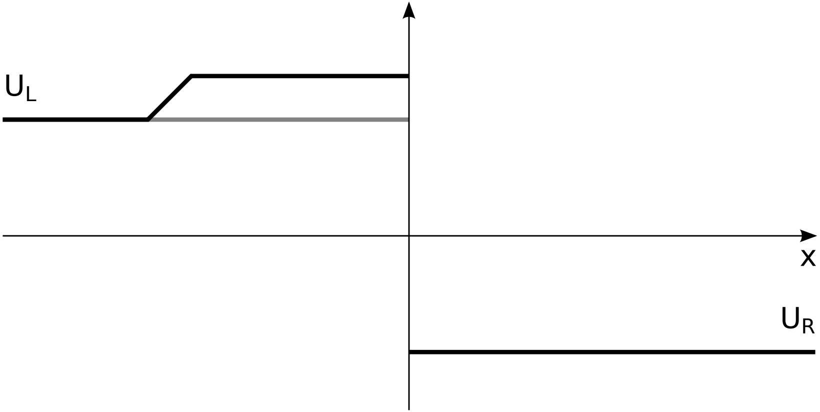

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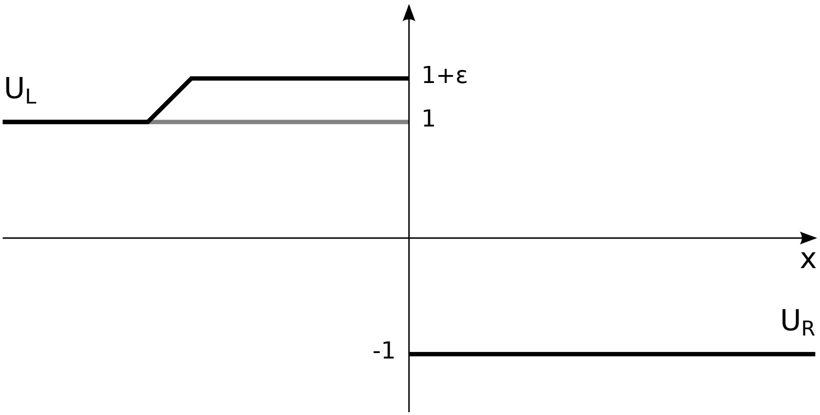

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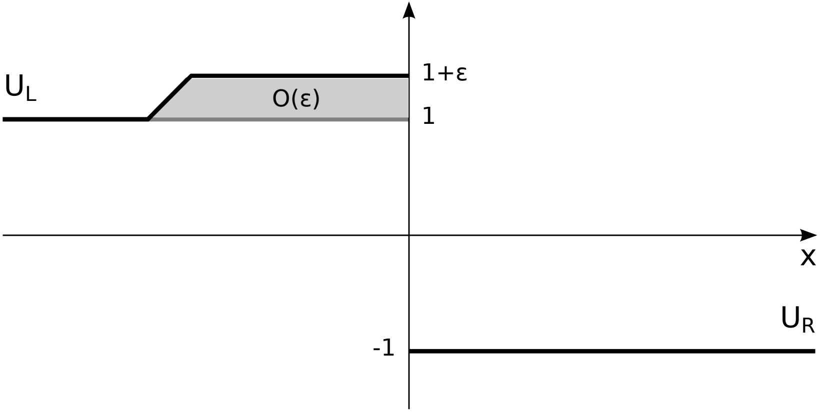

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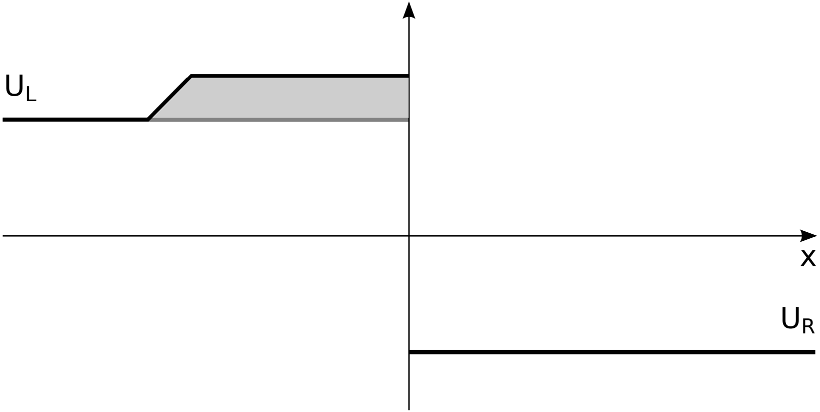

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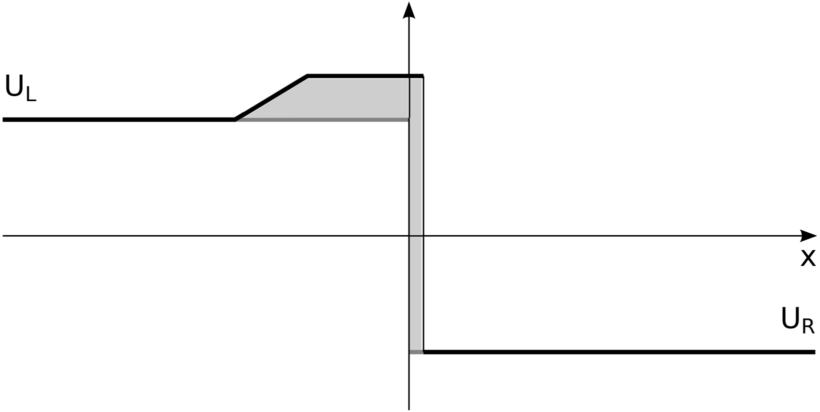

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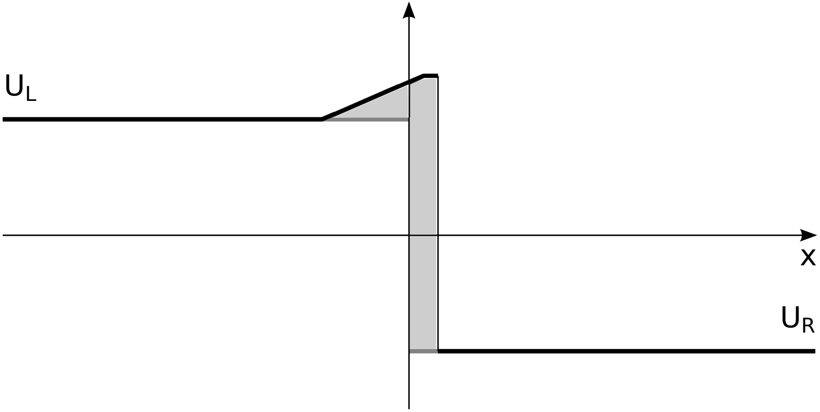

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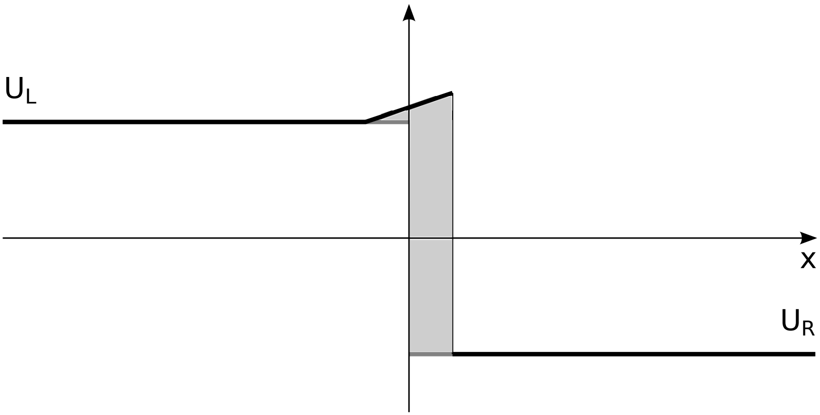

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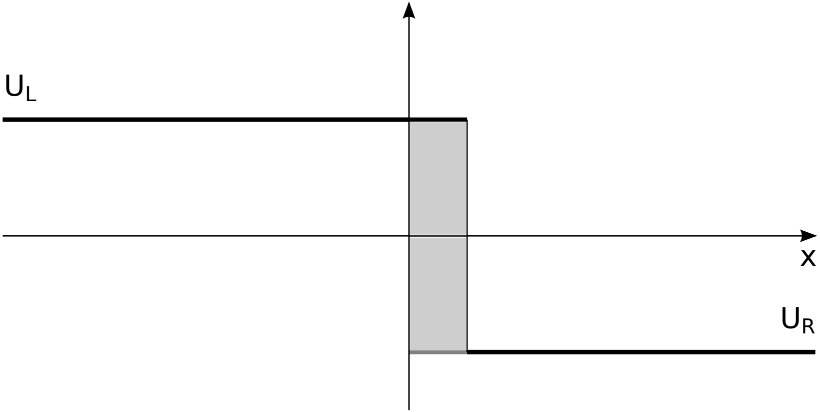

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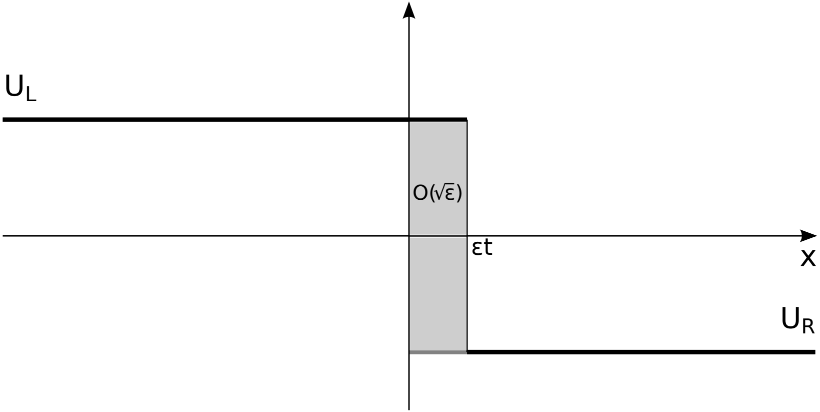

An L^2 Stability Estimate for Shocks

Theorem (L. 2010). Let $U^0 \in L^\infty(\mathbb{R})$ and assume $U^0 - \phi \in L^2(\mathbb{R})$ where

$$\phi(x) = \begin{cases} U_L, & \text{if } x < 0; \\ U_R, & \text{if } x > 0, \end{cases}$$

with $U_L > U_R$. Further, assume U is the unique entropy solution of (IVP). Then there exists a Lipschitz continuous function $x : [0, \infty) \to \mathbb{R}$ and a constant $\lambda(\|U^0\|_{L^\infty}; \phi; A) > 0$ such that

$$\|U(\cdot, t) - \phi(\cdot - \sigma t - x(t))\|_{L^2(\mathbb{R})} \leq \|U^0 - \phi\|_{L^2(\mathbb{R})}$$

and

$$|x(t)| \leq \lambda \|U^0 - \phi\|_{L^2(\mathbb{R})} \sqrt{t}$$

for all $t \geq 0$, where σ is given by the Rankine-Hugoniot relation.

Systems of Conservation Laws

$$\begin{cases} \partial_t U^i + \partial_x A^i(U) = 0, & \text{i = 1, 2,..., n} \\ U(x,0) = U^0(x). \end{cases}$$

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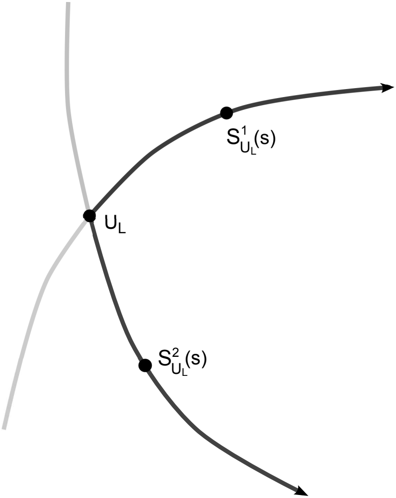
- For example, existence of solutions is known (via the Glimm scheme, wave-tracking methods, etc.) when the initial data has sufficiently small total variation.
- The L^1 stability theory of Bressan et al. relies on similar assumptions. In particular, Kružkov's estimate fails.

Hugoniot Curves

$$A(S_{U_L}(s)) - A(U_L) = \sigma(S_{U_L}(s) - U_L)$$



Entropy Solutions

Consider the system

$$\partial_t U + \partial_x A(U) = 0, \tag{1}$$

and assume U takes values in $\mathcal{V} \subset \mathbb{R}^n$. Then,

$$\eta : \mathcal{V} \to \mathbb{R}$$

is called an entropy of (1) if there exists

$$G : \mathcal{V} \to \mathbb{R}$$

such that

$$\partial_j G = \nabla \eta \cdot \partial_j A.$$

Entropy Solutions

- A weak solution U is called an entropy solution if

$$\partial_t \eta(U) + \partial_x G(U) \leq 0,$$

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- We say that (U_L, U_R) is an entropic Rankine-Hugoniot discontinuity if there exists $\sigma \in \mathbb{R}$ such that

$$A(U_R) - A(U_L) = \sigma(U_R - U_L),$$

$$G(U_R) - G(U_L) \leq \sigma(\eta(U_R) - \eta(U_L)).$$

The Main Result

Assumptions

- $A \in C^2(\mathcal{V})$ where $\mathcal{V} \subset \mathbb{R}^m$ is open, bounded, and convex.

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- $U_L \in \mathcal{V}$ satisfies (H1)-(H3)

The Main Result

Theorem (L. - Vasseur, ARMA, 2011). Assume $U_L, U_R \in \mathcal{V}$ form a 1-shock (or 1-contact discontinuity) with velocity σ. Then $\exists\ C > 0$, $\varepsilon_0 > 0$ such that for any $0 \leq \varepsilon < \varepsilon_0$ and any weak entropic solution $U \in L^\infty(0, T; \mathcal{U})$ with the strong trace property (STP) verifying

$$\int_{-\infty}^{0} |U_0(x) - U_L|^2\, dx \leq \varepsilon^4, \qquad \int_{0}^{\infty} |U_0(x) - U_R|^2\, dx \leq \varepsilon,$$

there exists a Lipschitz curve $t \to x(t)$ such that for any $0 < t < T$:

$$\int_{-\infty}^{0} |U(x + x(t), t) - U_L|^2\, dx \leq \varepsilon^4, \qquad \int_{0}^{\infty} |U(x + x(t), t) - U_R|^2\, dx \leq C(1 + t)\varepsilon.$$

Moreover,

$$|x(t) - \sigma t| \leq C\sqrt{\varepsilon t(1 + t)}.$$

Structural Hypotheses

Let $U \in L^\infty(\mathbb{R} \times \mathbb{R}^+)$. We say that U verifies the *strong trace property* if for any Lipschitz curve $t \to X(t)$, there exists two bounded functions $U_-, U_+ \in L^\infty(\mathbb{R}^+)$ such that for any $T > 0$

$$
\begin{aligned}
0 &= \lim_{\varepsilon \to 0} \int_0^T \operatorname*{ess\,sup}_{y \in (0,\varepsilon)} |U(t, x(t) + y) - U_+(t)| \, dt \\
&= \lim_{\varepsilon \to 0} \int_0^T \operatorname*{ess\,sup}_{y \in (-\varepsilon,0)} |U(t, x(t) + y) - U_-(t)| \, dt
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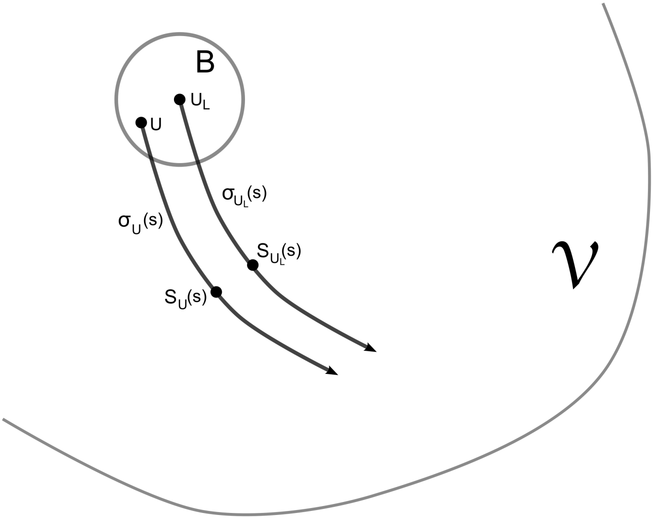
- All functions $U \in L^\infty \cap BV_{loc}$ verify the strong trace property.

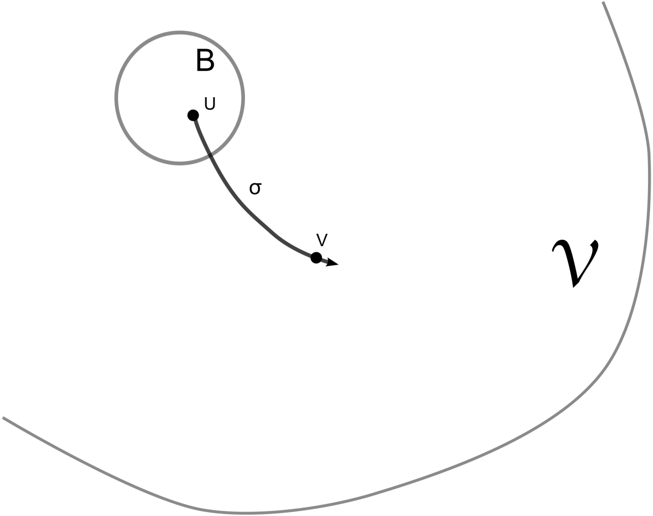
Structural Hypotheses

(H1) $$\varepsilon_0^2 = \inf_{U \notin B} \eta(U \mid U_L)$$



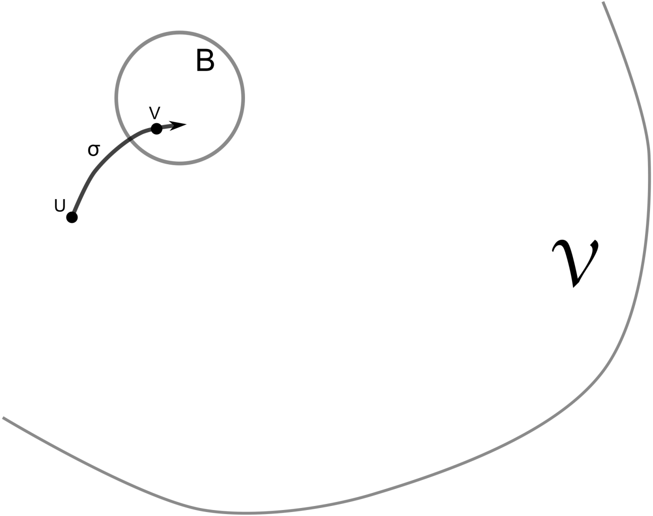
Structural Hypotheses

$$\text{(H2)} \quad U \in B \text{ and } \sigma < \lambda^-(U) \quad \Longrightarrow \quad V = S_U(s) \;\; \text{(1-shock)}$$



Structural Hypotheses

(H3) $V \in B \implies \sigma \geq \lambda^-(V)$



Relative Entropy

For any fixed state $V \in \mathcal{V}$, entropy solutions verify

$$\partial_t \eta(U \mid V) + \partial_x F(U, V) \leq 0,$$

where

$$\eta(U \mid V) = \eta(U) - \eta(V) - \nabla\eta(V) \cdot (U - V)$$

is the quadratic part of η at V

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is the quadratic part of η at V, and $F(U, V)$ is defined by

$$F(U, V) = G(U) - G(V) - \nabla\eta(V) \cdot (A(U) - A(V)).$$

Relative Entropy

When η is strictly convex, i.e., $D^2\eta$ is positive definite, we have

$$\frac{1}{C}|U - V|^2 \leq \eta(U \mid V) \leq C|U - V|^2$$

A Relative Entropy Technique For Shocks

We would like to control the quantity

$$\int_{-\infty}^{\infty} \eta(U(x,t) \mid \phi(x - x(t))\, dx = \int_{-\infty}^{x(t)} \eta(U(x,t) \mid U_L)\, dx + \int_{x(t)}^{\infty} \eta(U(x,t) \mid U_R)\, dx.$$

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Formally,

$$\frac{d}{dt}\left\{\int_{-\infty}^{x(t)} \eta(U(x,t) \mid U_L)\, dx\right\} \;\leq\; \dot{x}(t)\, \eta(U(x(t),t) \mid U_L) - F(U(x(t),t), U_L),$$

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$$\frac{d}{dt}\left\{\int_{x(t)}^{\infty} \eta(U(x,t) \mid U_R)\, dx\right\} \le F(U(x(t),t), U_R) - \dot{x}(t)\, \eta(U(x(t),t) \mid U_R).$$

We may define, for instance,

$$\dot{x}(t) = \frac{F(U(x(t),t), U_L)}{\eta(U(x(t),t) \mid U_L)}.$$

An Important Formula

An explicit formula for the loss of entropy across a shock is given by

$$G(S_{U_L}(s)) - G(U_L) = \sigma_{U_L}(s)\,(\eta(S_{U_L}(s)) - \eta(U_L))$$
$$+ \int_0^s \dot{\sigma}_{U_L}(\tau)\,\eta(U_L \mid S_{U_L}(\tau))\,d\tau.$$

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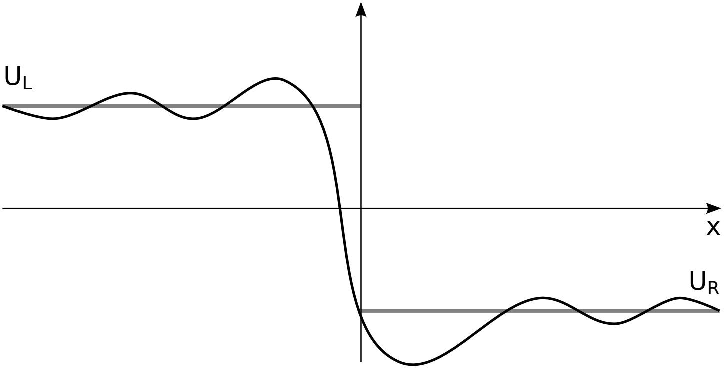
Equivalently,

$$F(S_{U_L}(s), U_R) - \sigma_{U_L}(s)\,\eta(S_{U_L}(s) \,|\, U_R) = \int_{s_R}^s \dot{\sigma}_{U_L}(\tau)[\eta(U_L \,|\, S_{U_L}(\tau)) - \eta(U_L \,|\, U_R)]\,d\tau$$

A Relative Entropy Technique For Shocks



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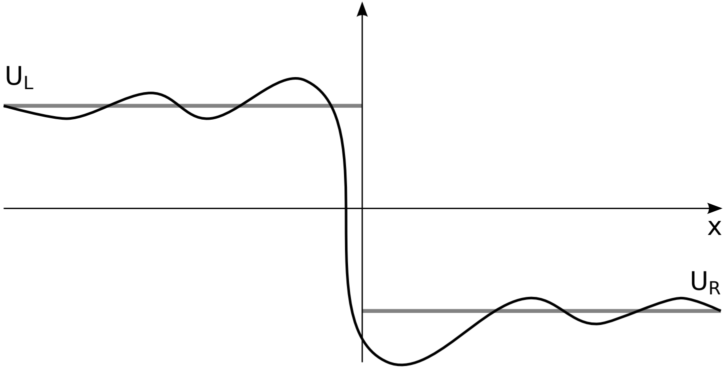

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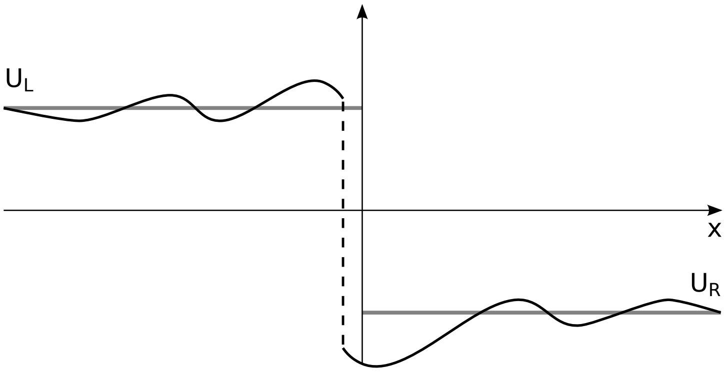

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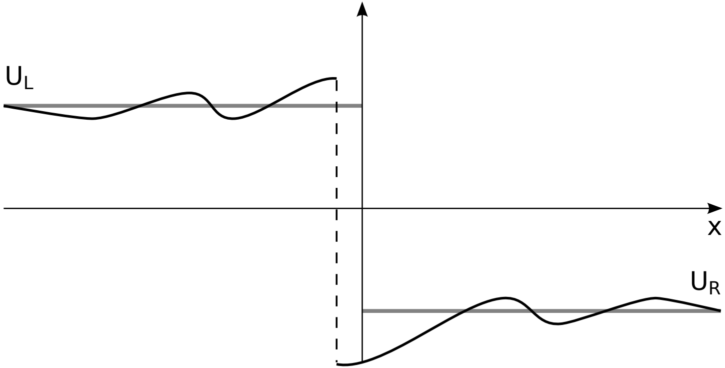

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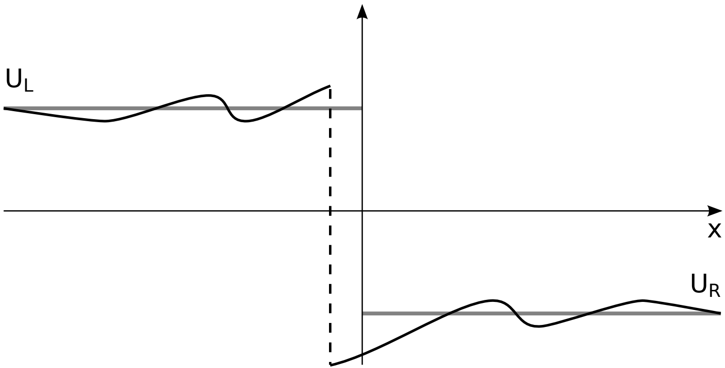

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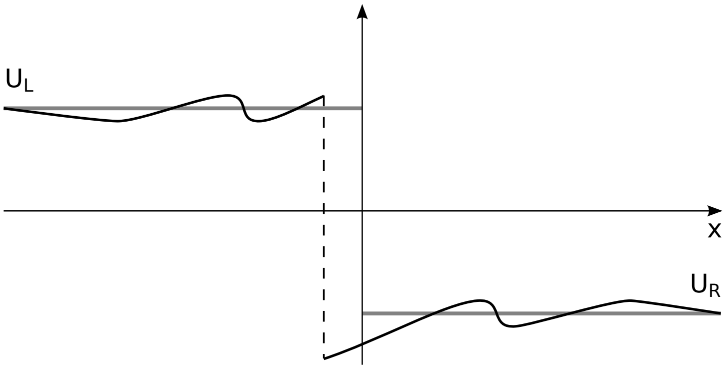

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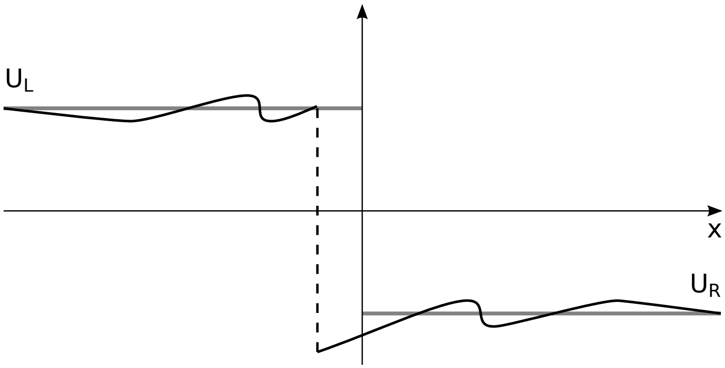

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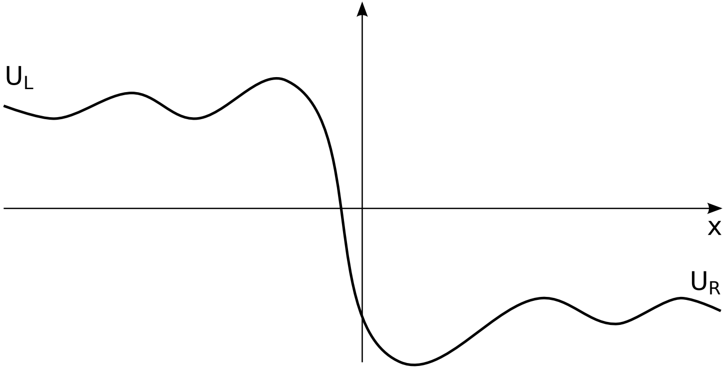

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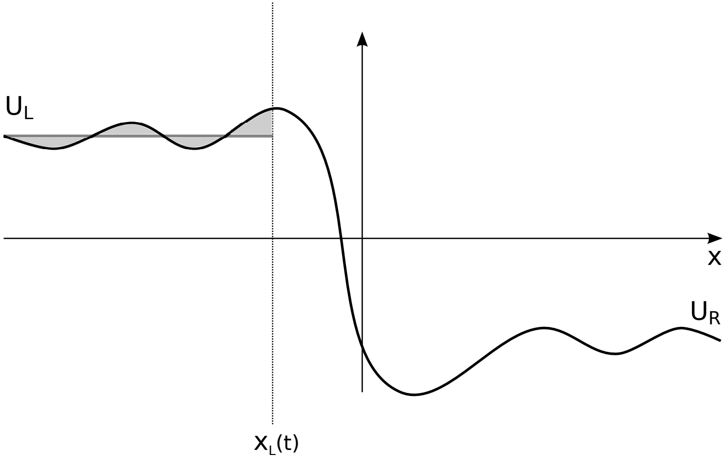

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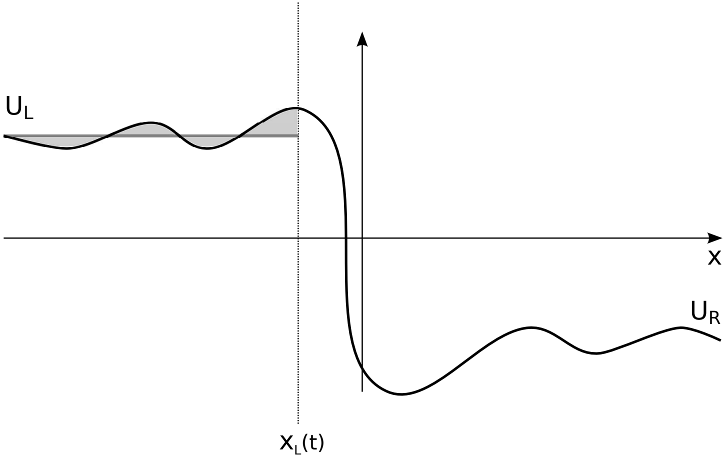
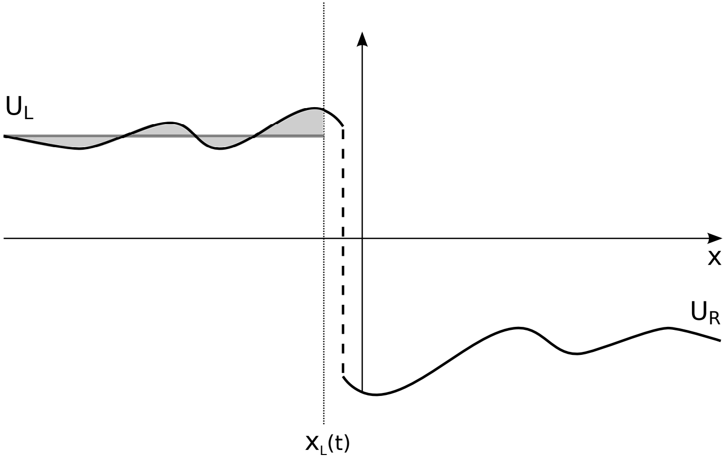
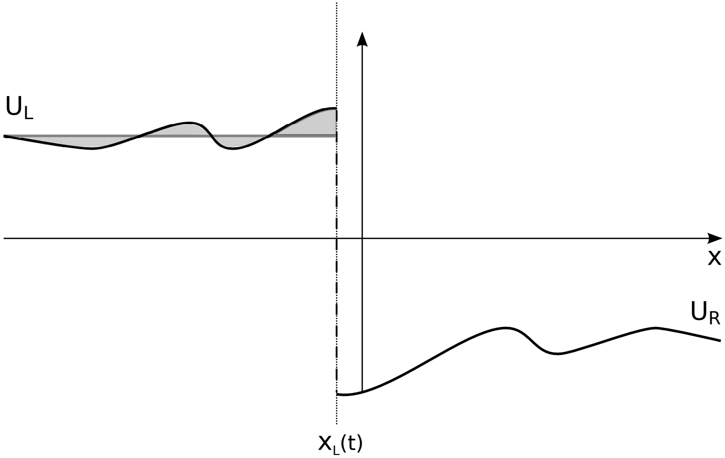


A Relative Entropy Technique For Shocks



A Relative Entropy Technique For Shocks



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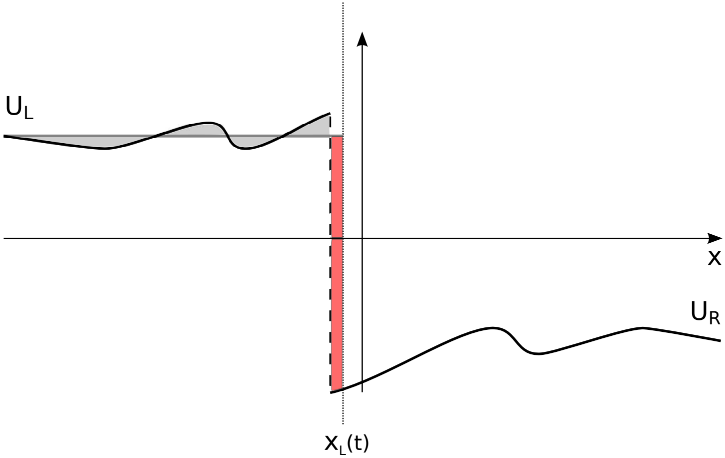

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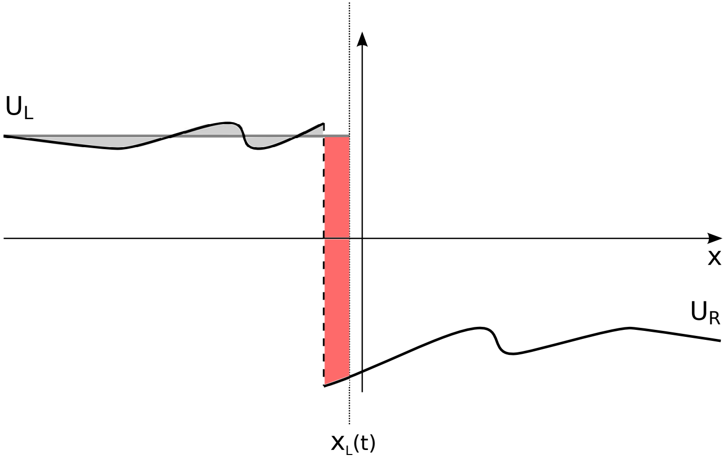

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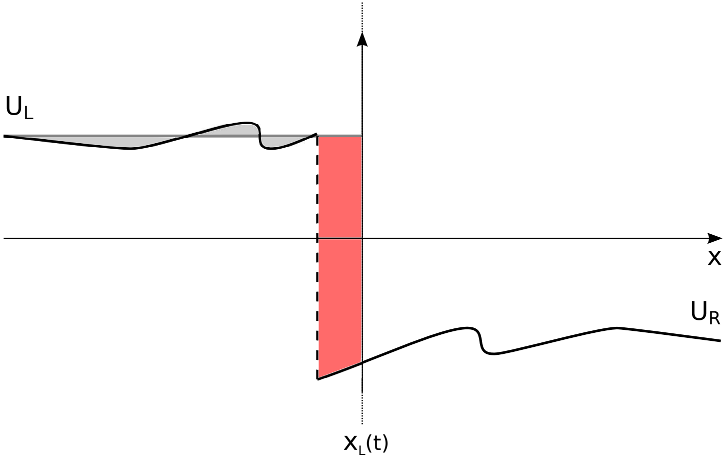

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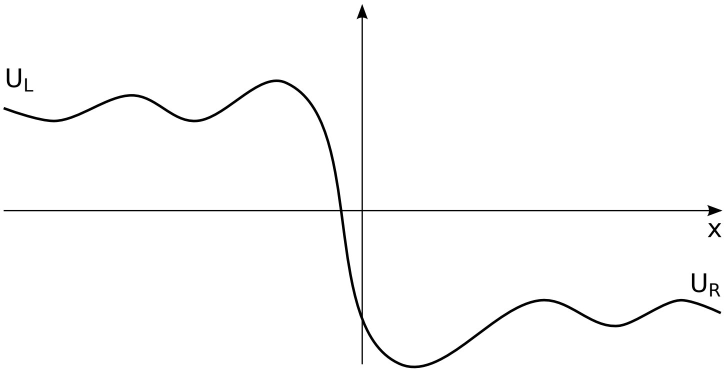

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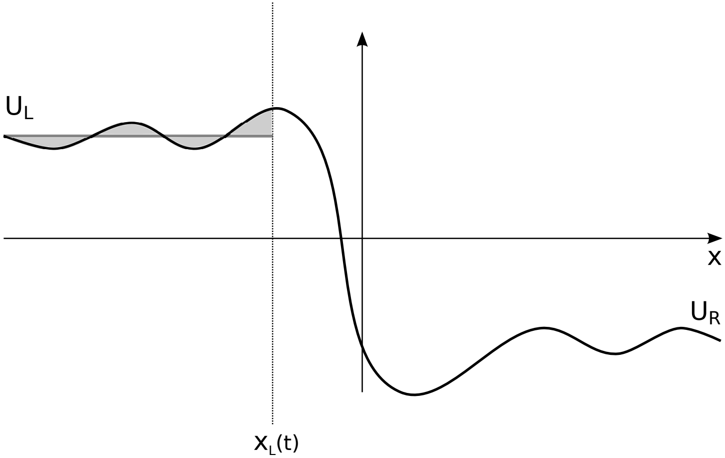

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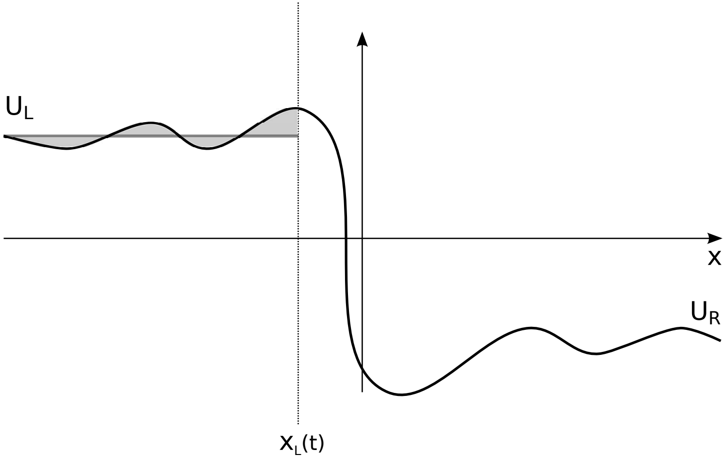

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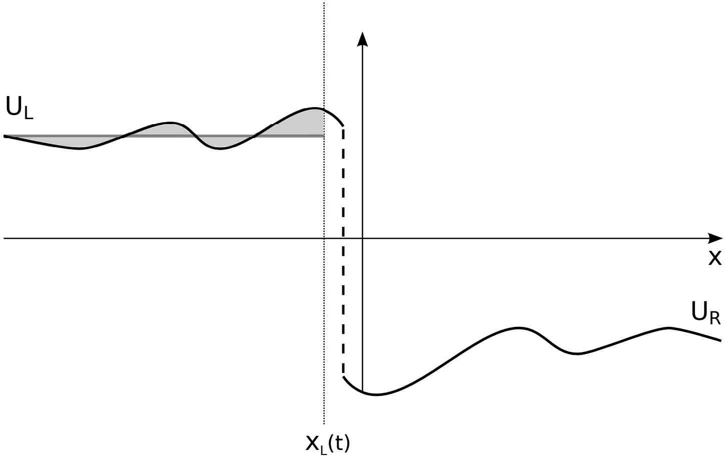

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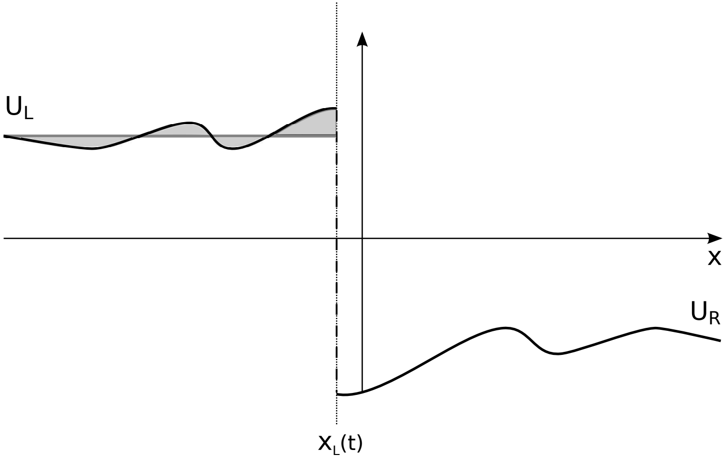

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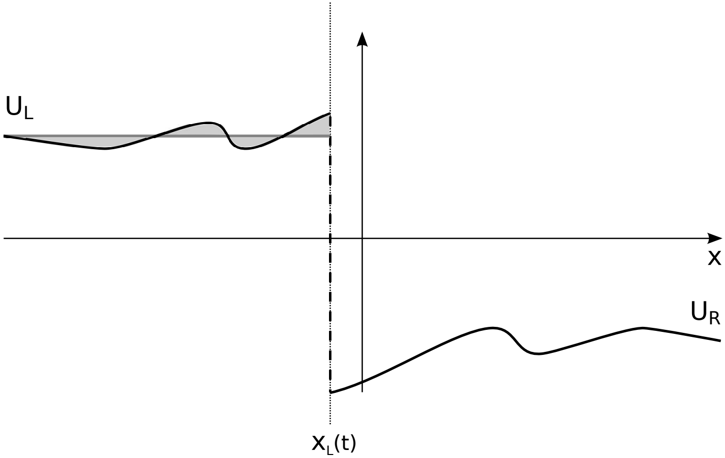

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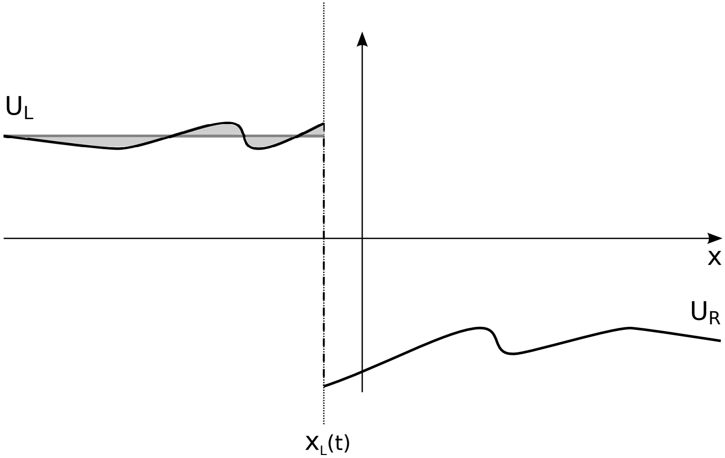

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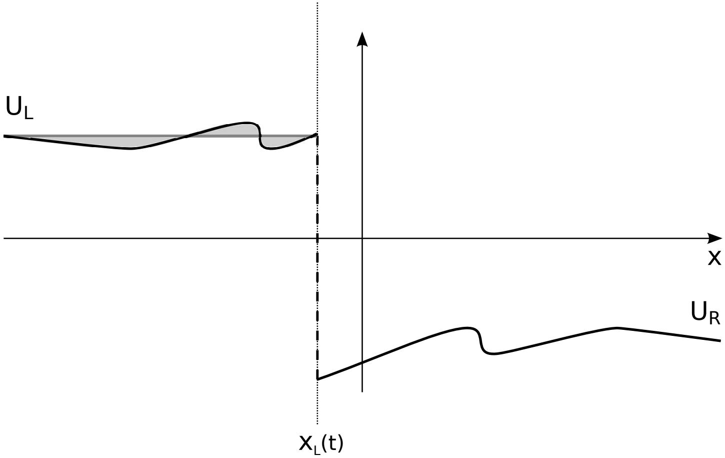

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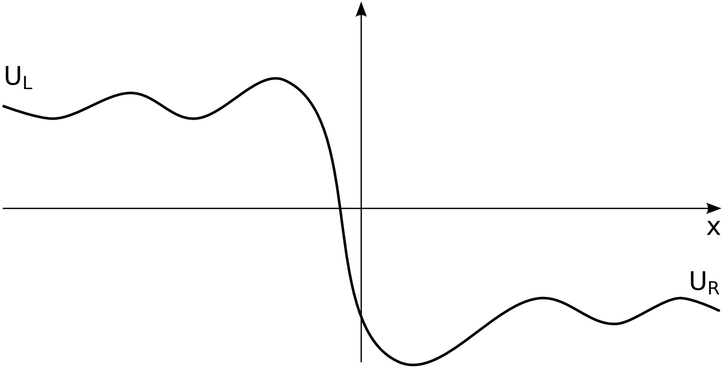

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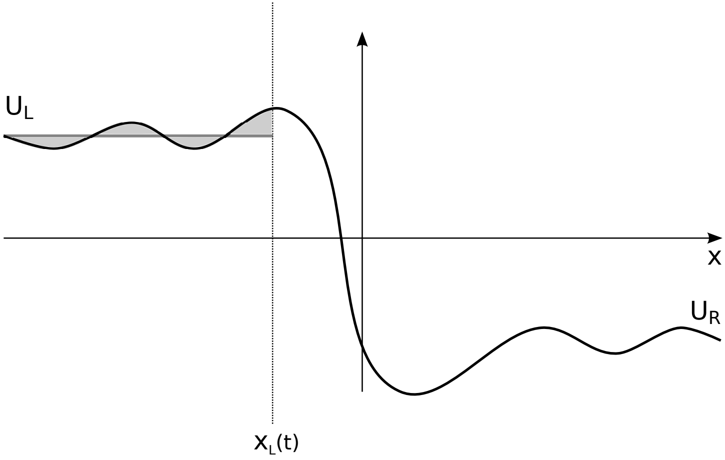

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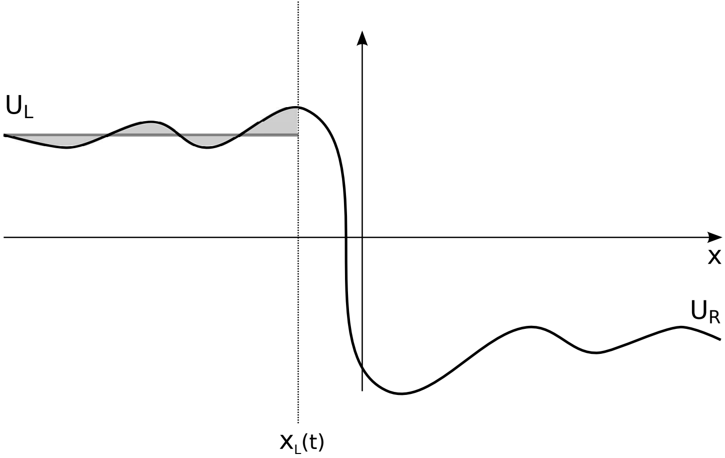

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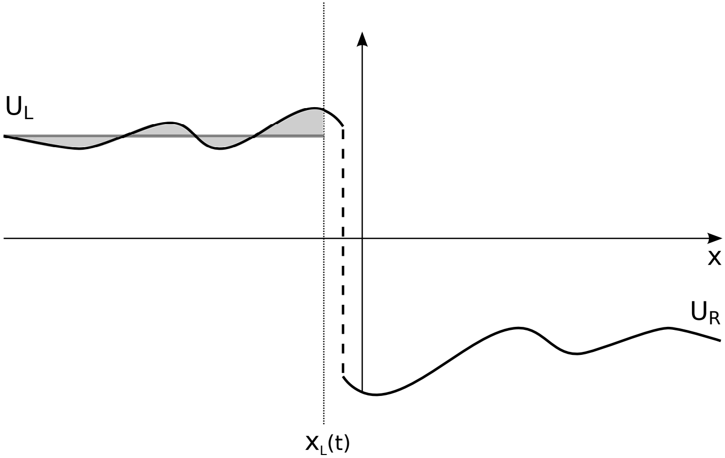

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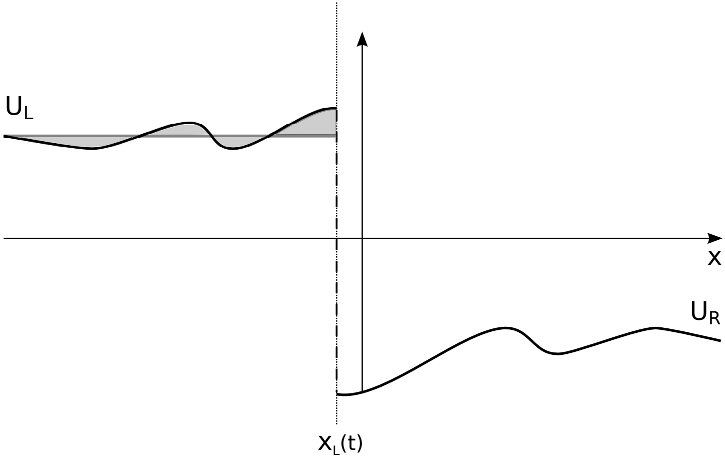

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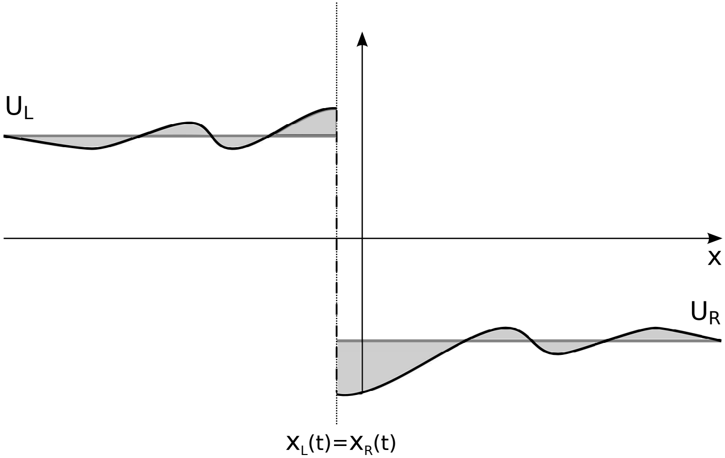

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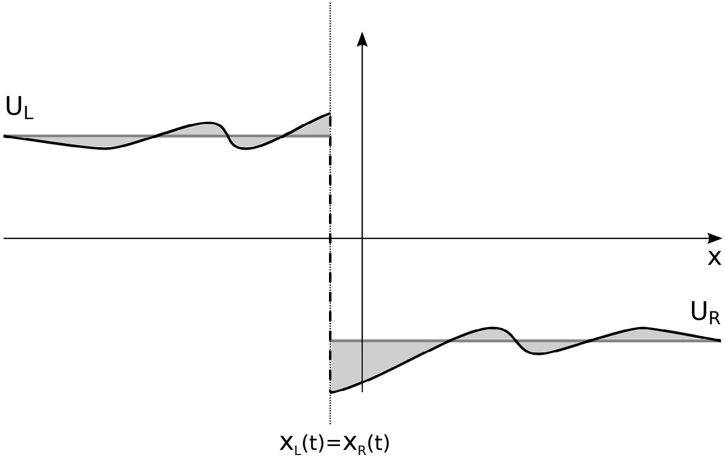

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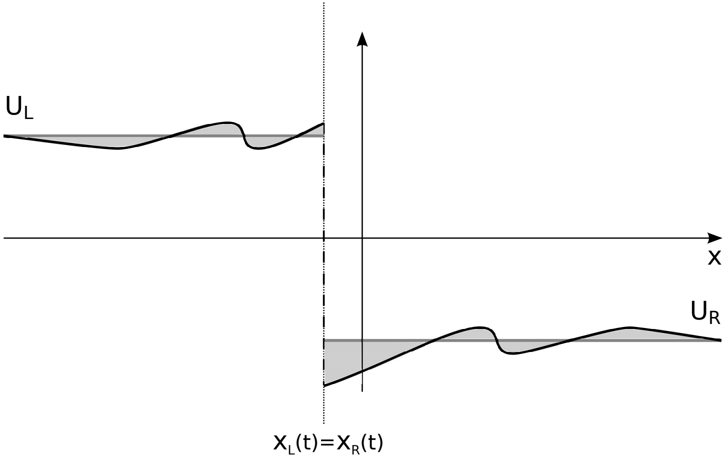

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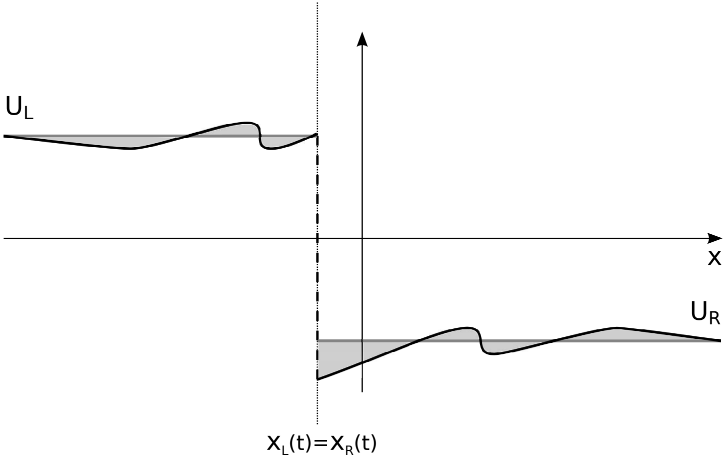

2×2 Euler system

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2 + P(\rho)) = 0. \end{cases}$$

- $P'(\rho) > 0, \qquad [\rho P(\rho)]'' \geq 0.$

2×2 Euler system

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2 + P(\rho)) = 0. \end{cases}$$

- $P'(\rho) > 0, \qquad [\rho P(\rho)]'' \geq 0.$
- $\eta(\rho, \rho u) = \dfrac{(\rho u)^2}{2\rho} + S(\rho), \qquad S''(\rho) = \dfrac{P'(\rho)}{\rho}$

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- $G(\rho, \rho u) = \dfrac{(\rho u)^3}{2\rho^2} + \rho u\, S'(\rho)$

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- $G(\rho, \rho u) = \dfrac{(\rho u)^3}{2\rho^2} + \rho u\, S'(\rho)$
- $\mathcal{V} = \left\{ (\rho, \rho u) \in \mathbb{R}^+ \times \mathbb{R} \;\middle|\; 0 < \|(\rho, u)\|_{L^\infty} < K \right\}.$

Thank You!