Incompressible Boussinesq Equations in Borderline Besov Spaces

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The Boussinesq equations
Basic definitions

In 2D, the incompressible Boussinesq equations are given by:

$$(B_{\kappa,0}) \begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla P = \begin{pmatrix} 0 \\ \rho \end{pmatrix} \\ \partial_t \rho + (u \cdot \nabla)\rho = \kappa \Delta \rho \\ \text{div } u = 0 \\ u(x,0) = u_0(x), \;\; \rho(x,0) = \rho_0(x). \end{cases}$$

- The scalar ρ, can be thought of as density or temperature.
- For $\rho \equiv 0$, this system is the incompressible Euler equations.

The Boussinesq equations

The main result is that for initial data (u_0, ρ_0) in a Besov-type space, B_Γ, there exists $T > 0$ and a unique solution to $(B_{\kappa,0})$ such that the vorticity, $\omega = \text{ curl } u$, and $\nabla\rho$ remain bounded in the related space B_{Γ_1} for $t \in [0, T]$.

- B_Γ is based on the Besov space $B^0_{\infty,1}$.
- $\Gamma(\xi)$ is a function which grows at infinity like $\log^\beta(\xi)$, $\beta \in (0, 1]$.
- Under stronger assumptions on $\Gamma(\xi)$ global-in-time existence follows.

Background

Littlewood-Paley decomposition

The **Littlewood-Paley decomposition** is used to break a function into a sum of functions with localized frequencies:

$$f = \sum_{j \geq -1} \Delta_j f$$

where $\Delta_j f = \varphi_j * f$, and $\varphi_j = 2^{jn}\varphi(2^j \cdot\ )$ is a Schwartz-class function whose Fourier transform is supported near the annulus of radius 2^j (for $j \geq 0$) or within the unit ball (for $j = -1$).

Background

Besov spaces

Using the Littlewood-Paley decomposition, the Besov space $B^s_{p,q}$ is defined as the set of $f \in \mathcal{S}'$ such that:

$$\left(\sum_{j \geq -1} 2^{jqs} \left\| \Delta_j f \right\|_p^q \right)^{\frac{1}{q}} < \infty$$

In this talk, we'll predominantly consider the Besov space $B^0_{\infty,1}$.

Background
B_Γ Spaces

From the definition of $B^0_{\infty,1}$, we define the space B_Γ as the set of $f \in \mathcal{S}'$ such that

$$\sup_{N \geq -1} (\Gamma(N))^{-1} \sum_{j=-1}^{N} \|\Delta_j f\|_\infty \leq C < \infty.$$

- $\Gamma : \mathbb{R} \to [0, \infty)$ grows like $\log^\beta(\xi)$ for $\beta \in (0, 1]$.
- $\Gamma_1(\xi) = (\xi + 2)\Gamma(\xi)$, i.e. Γ_1 grows like $\xi \log^\beta \xi$.

Related Results

R. Danchin and M. Paicu

In 2009, R. Danchin and M. Paicu proved[1] a Yudovich-type result in $\mathbb{R}^2$ for $(B_{\kappa,0})$ under the assumption that $u_0 \in \mathrm{L}^2$, $\omega_0 \in \mathrm{L}^r \cap \mathrm{L}^\infty$ $(r \geq 2)$ and $\rho_0 \in \mathrm{L}^2 \cap B^{-1}_{\infty,1}$.

Motivating question: Can this result be extended to more general spaces than L^∞ and $\mathrm{L}^2 \cap B^{-1}_{\infty,1}$?

[1] Raphaël Danchin and Marius Paicu, *Global well-posedness issues for the inviscid Boussinesq system with Yudovich's type data*, Comm. Math. Phys. **290** (2009), no. 1, 1–14

Related Results
M. Vishik

The B_Γ spaces were first introduced by M. Vishik in 1999[2], when he proved the existence and uniqueness of a solution to the Euler equations in B_{Γ_1} with initial vorticity $\omega_0 \in B_\Gamma \cap \mathrm{L}^{p_0} \cap \mathrm{L}^{p_1}$ for $1 < p_0 < 2 < p_1 < \infty$, and Γ a function which grows like $\log^\beta$ for $\beta \in (0, 1]$.

Motivating question: Can Vishik's result be extended to the Boussinesq system $(B_{\kappa,0})$?

[2] Misha Vishik, *Incompressible flows of an ideal fluid with vorticity in borderline spaces of Besov type*, Ann. Sci. École Norm. Sup. (4) **32** (1999), no. 6, 769–812

Main Result

Theorem (G-L)

For $1 < p_0 < 2 < p_1 < \infty$, let curl $u_0 = \omega_0 \in B_\Gamma \cap \mathrm{L}^{p_0} \cap \mathrm{L}^{p_1}$ and $\rho_0 \in \mathrm{W}^{1,p_0} \cap \mathrm{W}^{1,p_1}$ such that $\nabla \rho_0 \in B_\Gamma$. Assume that $(\xi + 2)\Gamma'(\xi) \leq C$ for a.e. $\xi \in [-1, \infty)$. Then there exists a $T > 0$ and a unique solution (u, ρ) to the system of equations $(B_{\kappa,0})$, such that

$$\omega(\cdot) \in \mathrm{L}^\infty([0, T]; \mathrm{L}^{p_0} \cap \mathrm{L}^{p_1}) \cap C_{w^*}([0, T]; B_{\Gamma_1}),$$
$$\nabla\rho(\cdot) \in \mathrm{L}^\infty([0, T]; \mathrm{L}^{p_0} \cap \mathrm{L}^{p_1}) \cap C_{w^*}([0, T]; B_\Gamma).$$

Note that $u = \mathcal{K} * \omega$, where $\mathcal{K}$ is the Biot-Savart kernel.

Sketch of proof

B_Γ spaces

One of the more useful properties of B_Γ spaces is their relation to B_{Γ_1} spaces via the flow map:

Theorem (Vishik, 1999)

For $t \in [0, T]$, let u be a solution to the Euler equations with inital data $f = \omega_0 \in B_\Gamma \cap \mathrm{L}^{p_0} \cap \mathrm{L}^{p_1}$. Let X_u be the associated flow map. Then X_u maps B_Γ to B_{Γ_1}, i.e. for any $t \in [0, T]$,

$$\left\| f \circ X_u^{-1}(t) \right\|_{\Gamma_1} \leq C \left\| f \right\|_\Gamma .$$

Sketch of proof

Generalizations

A closer inspection of this last theorem shows that:

- It can be extended to Euler-type systems such as the Boussinesq equations.
- The function f can be decoupled from the initial data.

Sketch of proof
Boussinesq equations

Taking the curl of the first equation in $(B_{\kappa,0})$ gives the vorticity equation:

$$\partial_t \omega + (u \cdot \nabla)\omega = \partial_1 \rho.$$

Integration along flow lines then gives:

$$\|\omega(t)\|_p \leq \|\omega_0\|_p + \int_0^t \|\nabla \rho\|_p \, \mathrm{d}\tau, \quad p \in [1, \infty)$$

$$\|\omega(t)\|_{\Gamma_1} \leq \left\|\omega_0(X_u^{-1}(t;0))\right\|_{\Gamma_1} + \int_0^t \left\|\nabla \rho(X_u^{-1}(t;\tau), \tau)\right\|_{\Gamma_1} \mathrm{d}\tau$$

To use the action of the flow map on B_Γ, we need to show that $\nabla \rho(t) \in B_\Gamma$ for $t \in [0, T]$.

A priori estimate
Statement

Theorem (G-L)

Assume u_0, ρ_0 are smooth. In addition, let $\omega_0 \in B_\Gamma \cap \mathrm{L}^{p_0} \cap \mathrm{L}^{p_1}$, and let $\rho_0 \in \mathrm{W}^{1,p_0} \cap \mathrm{W}^{1,p_1}$ such that $\nabla \rho_0 \in B_\Gamma$. Let (u, ρ) be the associated regular solution to $(B_{\kappa,0})$. Then there is a constant $C > 0$ such that

$$\int_0^t \|\nabla \rho(\tau)\|_{\mathrm{L}^{p_0} \cap B_\Gamma} \, d\tau \leq C(t) < \infty$$

for any $t \in [0, T]$.

A priori estimate
Sketch of proof

To prove the a priori bound on density, we use a result of J.-Y. Chemin[3] on the interaction of the heat kernel and Littlewood-Paley operators to prove that:

$$\int_0^t \|\rho(\tau)\|_{B^1_{\infty,1}} \, d\tau \leq C\alpha \left(\|\rho_0\|_{B^{-1}_{\infty,1}} + \int_0^t \|(u \cdot \nabla)\rho\|_{B^{-1}_{\infty,1}} \, d\tau \right),$$

where $\alpha = \left(\frac{1+\kappa t}{\kappa}\right)$.

- $\|\nabla\rho\|_{B_\Gamma} \leq C \|\rho\|_{B^1_{\infty,1}}$
- Proof of a priori bound follows from paradifferential calculus, L^p estimates and Gronwall's inequality.

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[3] Jean-Yves Chemin, *Théorèmes d'unicité pour le système de Navier-Stokes tridimensionnel*, J. Anal. Math. **77** (1999), 27–50

For uniqueness and existence, apply the Δ_j operator to the first two equations in $(B_{\kappa,0})$ and use estimates on the commutator

$$\Delta_j(u \cdot \nabla f) - (S_{j-2}u \cdot \nabla)\Delta_j f,$$

for f equal to velocity and density in the respective equations.

- For uniqueness, use Osgood Uniqueness Theorem, properties of $\Gamma(\xi)$ and Littlewood-Paley theory to show that uniqueness holds for $t \in [0, T]$.
- For existence, standard approximation argument with Sobolev-regular solutions[4] to $(B_{\kappa,0})$.

[4] Dongho Chae, *Global regularity for the 2D Boussinesq equations with partial viscosity terms*, Adv. Math. **203** (2006), no. 2, 497–513

Thanks to the conference organizers and thank you for your time!

More information available at:
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