Estimates on Fractional Higher Derivatives of Weak Solutions for the Navier-Stokes Equations

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Joint work with Alexis Vasseur

Analysis of incompressible fluids, Turbulence and Mixing
In honor of Peter Constantin's 60th birthday.
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Outline

We consider 3D Navier-Stokes equations.

$$\begin{aligned}\partial_t u + (u\cdot\nabla)u + \nabla P - \Delta u = 0 \quad &\text{and} \\ \operatorname{div} u = 0, \qquad t\in(0,\infty), \quad & x\in\mathbb{R}^3\end{aligned} \tag{1}$$

with L^2 initial data

$$u_0 \in L^2(\mathbb{R}^3), \quad \operatorname{div} u_0 = 0. \tag{2}$$

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  The estimate depends only on $\|u_0\|_{L^2(\mathbb{R}^3)}$.

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- For fracional derivatives, we encounter nonlocality of **pressure**.

Our main theorem(C., A. Vasseur'11) is the following.

Theorem

There exist universal constants $C_{d,\alpha}$ which depend only on integer $d \geq 1$ and real $\alpha \in [0,2)$ with the following two properties (I) and (II):
(I) Suppose that we have a smooth solution u of (1) *on $(0,T) \times \mathbb{R}^3$ for some $0 < T \leq \infty$ with some initial data* (2)*. Then it satisfies*

$$\|(-\Delta)^{\frac{\alpha}{2}} \nabla^d u\|_{L^{p,\infty}(t_0,T;L^{p,\infty}(K))} \leq C_{d,\alpha}\Big(\|u_0\|^2_{L^2(\mathbb{R}^3)} + \frac{|K|}{t_0}\Big)^{\frac{1}{p}}$$

for any $t_0 \in (0,T)$, any integer $d \geq 1$, any $\alpha \in [0,2)$ and any bounded open subset K of $\mathbb{R}^3$, where $p = \frac{4}{d+\alpha+1}$ and $|\cdot|$ = the Lebesgue measure in $\mathbb{R}^3$.
(II) For any initial data (2)*, we can construct a suitable weak solution u of* (1) *on $(0,\infty) \times \mathbb{R}^3$ such that $(-\Delta)^{\frac{\alpha}{2}} \nabla^d u$ is locally integrable in $(0,\infty) \times \mathbb{R}^3$ for $d = 1,2$ and for $\alpha \in [0,2)$ with $(d+\alpha) < 3$. Moreover, the estimate* (1) *holds with $T = \infty$ under the same setting of the above part (I) as long as $(d+\alpha) < 3$.*

Outline

1. Introduction and the main result
   - Navier-Stokes and previous estimates about higher derivatives
   - Our main result : $\nabla^\alpha u \in$ weak-$L^{4/(\alpha+1)}_{loc}$
2. Local to global
   - Why $p = 4/(d+1)$ and why weak-L^p with $\nabla^d u \in$ weak-L^p?
   - CKN theorem and its quantitative variation
3. Local study for smooth solutions
   - Converting into a problem with a right parabolic cylinder
4. Nonlocality : weak solutions
   - We need a smooth approximation scheme.
   - Difficulities from convective velocity
5. More nonlocality : fractional derivatives of weak solutions
   - Difficulity from pressure

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- We want to use full power of the scaling factor $\frac{1}{\epsilon}$ of $|\nabla u|^2$:
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- $p = 4/(\alpha+1)$ is optimal if we use only $|\nabla u|^2 \in L^1_{(t,x)}$

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- **weak**-L^p is natural.

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- Let $p > 1$.
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We improve it for $p = 1$ by adopting a new pressure decomposition, which will be used to get the limit case : weak-L^p instead of $L^{p-\epsilon}$.

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We achieve a quantitative theorem following a flow

Theorem

Let $0 < \epsilon^2 < t$. Then there exist a function $F_{u,P}(\cdot,\cdot) \in L^1((0,\infty) \times \mathbb{R}^3)$ and a flow $X_u^{(\epsilon,t,x)}(\cdot)$ depending on (ϵ, t, x) such that

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(II) If $\frac{1}{\epsilon} \iint\limits_{Q_\epsilon(t,x)} F_{u,P}(s, y + X_u^{(\epsilon,t,x)}(s))\, dyds \le \delta$,

then $|\nabla^\alpha u| \le C_\alpha/\epsilon^{(\alpha+1)}$ in $Q_{\frac{\epsilon}{2}}(t,x)$ for real $\alpha \ge 1$.

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- $\int_0^t \int_{\mathbb{R}^3} F_{u,P}(s, y + X_u^{(\epsilon,t,x)}(s)) dsdy = \int_0^t \int_{\mathbb{R}^3} F_{u,P}(s,y) dsdy < \infty$ due to incompressibility of X.

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To prove the following:

$\frac{1}{\epsilon}\iint_{Q_\epsilon(t,x)} F_{u,P}(s,y+X_u^{(\epsilon,t,x)}(s))\,dyds \le \delta \Rightarrow |\nabla^d u| \le C_d/\epsilon^{(d+1)}$ in $Q_{\frac{\epsilon}{2}}(t,x)$ for integer $d \ge 1$,

we reformulate the problem via

translation $\begin{cases} v_1(s,y) = u(t+s,x+y) \\ Q_1(s,y) = P(t+s,x+y), \end{cases}$

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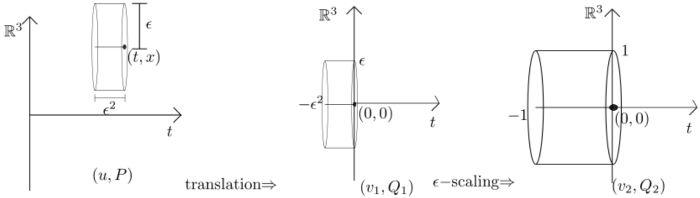

(cont'd)

translation, ϵ-scaling,

and Galilean invariance $$\begin{cases} v_3(s,y) = v_2(s, y + X_{v_2}(s)) - \dot{X}_{v_2}(s) \\ Q_3(s,y) = Q_2(s,y) + y\ddot{X}_{v_2}(s) \end{cases}$$

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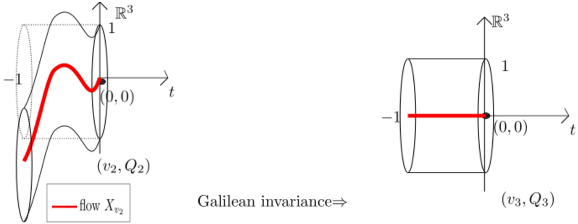

where $\begin{cases} \dot{X}_{v_2}(s) = (v_2 * \phi)(s, X_{v_2}(s)) \\ X_{v_2}(0) = 0. \end{cases}$. Note, v_3 is mean zero w.r.t. ϕ :

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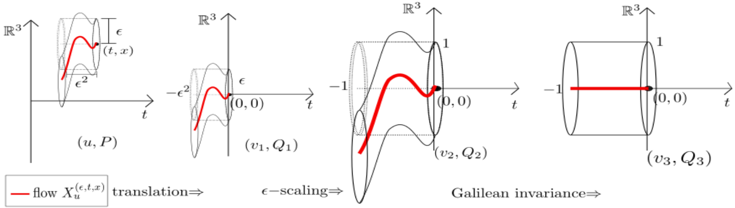

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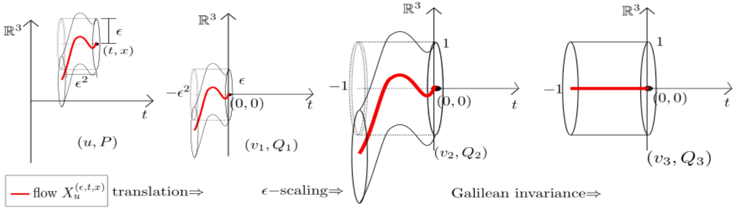

Hence, it is enough to show that

if (v,Q) is a solution with mean-zero $\int_{\mathbb{R}^3} v(x)\phi(x)dx = 0$

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- Thanks to mean-zero property, we can control $\|v\|_{L^\infty L^2(Q_{2/3})}$ and $\|Q\|_{L^\infty L^1(Q_{2/3})}$ so small that we can apply the local regularity theorem
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- It finishes the proof for the case u:smooth.

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Instead we consider an approximation scheme.

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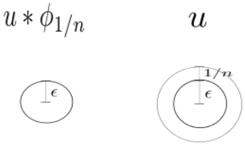

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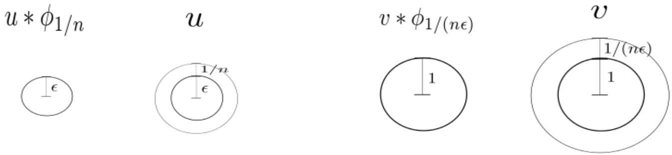

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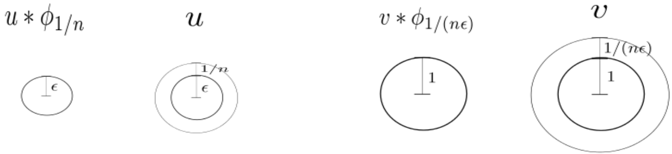

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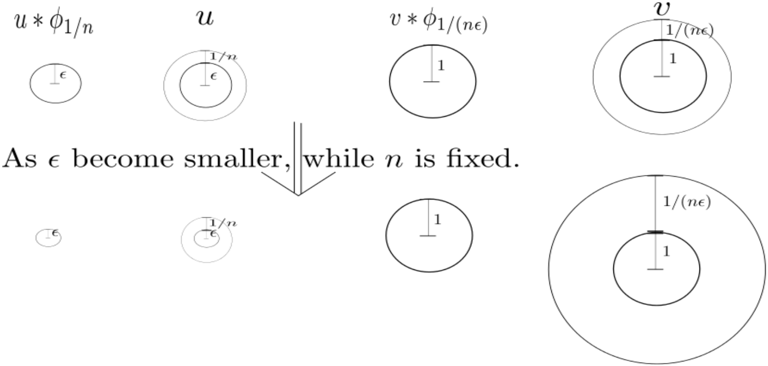

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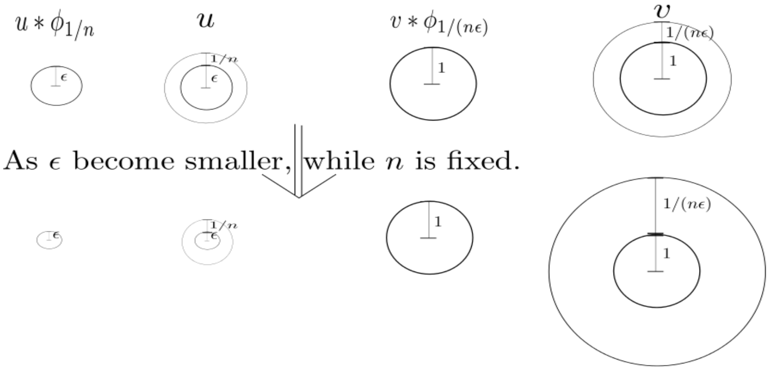

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- While the scaled velocity v is mean-zerp, the scaled convective velocity $[v * \phi_{(\frac{1}{(n\epsilon)})}]$ is not mean-zero.

- For local study (De Giorgi type), we need a uniform local estimate of the scaled convective velocity $w_v := [v * \phi_{(\frac{1}{(n\epsilon)})}]$.
- The scaled convective velocity depends on v too much nonlocally as ϵ goes to zero.

- We take $F_{u,P} = |\nabla u|^2 + |\nabla^2 P| + \Big[\mathcal{M}_x(|\nabla u|)\Big]^2$.
- While the scaled velocity v is mean-zerp, the scaled convective velocity $[v * \phi_{(\frac{1}{(n\epsilon)})}]$ is not mean-zero.
- We need to find a different flow.

Outline

- The fractional Laplacian $(-\Delta)^{\frac{\beta}{2}}$ for $0 < \beta < 2$:

$$\left[(-\Delta)^{\frac{\beta}{2}} f\right](x) = P.V. \int_{\mathbb{R}^3} \frac{f(t,x) - f(t,y)}{|x-y|^{3+\beta}} dy$$

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- As a result , non-local information of $\nabla^2 P$ is required.

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- It plays a similar role of Maximal function.

Thank you.