Boundary effect and Turbulence.

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Analysis of incompressible fluids, Turbulence and Mixing.
In honor of Peter Constantins 60th birthday.

Introduction

- Discuss the relation between boundary effect and "turbulence, singularities, anomalies" in the 0 viscosity limit.
- Use the notion of dissipative solutions as introduced by A. Majda, P.L. Lions and R. Di Perna.
- Show that the convergence/non convergence to the solution of the Euler equation is an issue independent of the appearance of singularities.
- Therefore I will consider smooth initial data $u_0(x)$ generating smooth solutions $u_\nu(x,t)$ and $u(x,t)$ of the Navier Stokes (with boundary conditions) in a domain $\Omega \subset \mathbb{R}^n$ with $n=2$ or $n=3$ and of the Euler equation with the impermeability condition for $t \in [0,T]$.

The problem

$$\overline{u}_\nu(x,t) = \mathrm{weak}\lim_{\nu\to 0} u_\nu(x,t) = (\mathrm{or}) \neq u(x,t)$$

seems to be related to all the issues of turbulence (Lax, Tartar).

- To support this remark I want to show that the discussion is similar when the Navier-Stokes limit is replaced by the Boltzmann limit (joint work with F. Golse and L.Paillard).

The Navier-Stokes Equations

$$\partial_t u_\nu + (u_\nu \cdot \nabla) u_\nu - \mu^* \Delta u_\nu + \nabla p_\nu = 0$$
$$\nabla \cdot u_\nu = \sum_{1 \le i \le d} \partial_{x_i} (u_\nu)_i = 0 \,, \; u_\nu \cdot \nabla u_\nu = \sum_{1 \le i \le d} (u_\nu)_i \partial_{x_i} u_\nu \,.$$

Called incompressible because of the relation $\nabla \cdot u = 0$.
But are also equations for fluctuations of mass, density and velocity around some reference state.
In particular ϵ the Mach number is the ratio between the fluctuation of velocity and the sound speed.

The Navier-Stokes Equations

$$u = \epsilon \tilde{u} \quad \theta = 1 + \epsilon \tilde{\theta}\,, \rho = 1 + \epsilon \widetilde{\rho}$$
$$\nabla_x \cdot \tilde{u} = 0\,, \quad \partial_t \tilde{u} + (\tilde{u} \cdot \nabla_x) \tilde{u} + \nabla_x \tilde{p} = \mu^* \Delta \tilde{u}\,,$$

Density and temperature fluctuations $\widetilde{\rho}\,, \tilde{\theta}$ are passive scalars:

$$\widetilde{\rho} + \tilde{\theta} = 0\,, \qquad \text{Boussinesq approximation}$$
$$\tfrac{d+2}{2} (\partial_t \tilde{\theta} + \tilde{u} \cdot \nabla_x \tilde{\theta}) = \kappa^* \Delta \tilde{\theta} \qquad \text{Fourier Law.}$$

Phenomenological derivation or consequence of the Boltzmann equation Hilbert 6th problem.

ν in Navier-Stokes is not the real viscosity of the fluid, but is the inverse of the Reynolds number, a rescaled viscosity adapted to the size of the fluctuations of the velocity is given by the formula:

$$\mathcal{R}e = \frac{UL}{\mu^*}$$

In all practical applications $\mathcal{R}e$, is very large, therefore ν is very small. Bicycle 10^2, Industrial fluids (pipes, ships...) 10^4, Wings of airplanes 10^6, Space Shuttle 10^8, Weather Forcast, Oceanography 10^{10}, Astrophysic 10^{12}. *It would be natural to study the limit $\nu \to 0$ in the Navier-Stokes equations or even to put $\nu = 0$ and then consider the Euler equations...*

With convenient (below given) boundary conditions:

$$\frac{d}{dt}\int_\Omega \frac{|u(x,t)|^2}{2} + \nu \int_\Omega |\nabla u(x,t)|^2 dx = O(\nu) \to 0$$

Therefore (modulo subsequences) $u_\nu \to \overline{u}$ in $\mathrm{weak} L^\infty((0,T);L^2(\Omega))$

- However in presence of boundary things are not so simple but very useful to consider.
- Intuition is that in general, in the presence of boundary, convergence does hold: Existence of wake and d'Alembert Paradox.

Figure: Euler, D'Alembert, Navier and Stokes

Statistical theory or weak convergence for Turbulence

$< .,. >$ statistical average $\simeq\simeq \overline{u_\nu}$ weak limit ,

$(\langle u_\nu \otimes u_\nu \rangle - \langle u_\nu \rangle \otimes \langle u_\nu \rangle)$ Reynolds stresses tensor ,

$0 \leq \lim_{\nu \to 0} (u_\nu - \overline{u_\nu}) \otimes (u_\nu - \overline{u_\nu}) = \lim_{\nu \to 0} (u_\nu \otimes u_\nu - \overline{u_\nu} \otimes \overline{u_\nu})$ Reynolds s.t. .

$w(k) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{iky} \langle u_\nu(x + \frac{y}{2})) \otimes u_\nu(x - \frac{y}{2}) dy \rangle$ Turbulence (spectra)

$$W_\nu(x, k, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{iky} (u_\nu(x + \frac{\sqrt{\nu}}{2}) \otimes u_\nu(x - \frac{\sqrt{\nu} y}{2}) - \overline{u}(x) \otimes \overline{u}(x)) dy .$$ Wigner Transform

$\overline{W}_\nu(x, t, k) = \lim_{\nu \to 0} W_\nu(x, k, t)$. Wigner Measure

Remarks

- My sources for statistical theory (Peter Constantin, Uriel Frisch)
- Law is in average with a forcing term.
- In the statistical theory hypothesis of isotropy and homogeneity appear.
- One of the consequence is the Kolmogorov law:

$$\epsilon = \nu\langle|\nabla u_\nu|^2\rangle \simeq \frac{\nu}{T}\int_0^T\int_\Omega |\nabla u_\nu|^2 dxdt \text{ Kolmogorov hypothesis},$$

$$\langle|u(x+r)-u(x)|^2\rangle^{\frac{1}{2}} \simeq (\nu\langle|\nabla u|^2\rangle)^{\frac{2}{3}}|r|^{\frac{1}{3}} \quad \text{Kolmogorov law}.$$

More remarks

• As seen below a deterministic version of $\epsilon > 0$ rules out strong convergence to the smooth solution
• A deterministic version of the $1/3$ law implies convergence to the smooth solution and consistent with the conservation of energy
Constantin, E, Titi and als..
In the present case a look for weak convergence (not strong) and dissipation of energy!

More remarks

- In the weak formulation all the objects are (x, t) local (integral can be done after localisation)
- The Wigner transform is not positive but the Wigner measure is a local positive object.
- The Kolmogorov law rules out weak convergence so it should not be uniform in ν. On the other hand it may appear in the spectra when there is a non trivial Wigner measure which may behave like

$$E(k) = \mathrm{Trace}(\lim_{\nu\to 0} W_\nu(x,k,t).|k|^{-2}) \simeq (\lim_{\nu\to 0} \nu \int |\nabla u_\nu|^2\rangle)^{\frac{2}{3}}|k|^{-\frac{5}{3}})$$

With Boundary conditions the following non trivial equivalent criteria define what Turbulence is NOT

- Convergence to a " up to the boundary" weak solution" $\Rightarrow$ No non trivial Reynolds stresses tensor.
- Convergence weakly to the regular solution
- Strong convergence to this solution.
- No anomalous dissipation of energy.
- No production of the vorticity at the physical boundary.
- No production of vorticity at a boundary layer of size ν

The Prandlt equations of the boundary layer are not valid.
There is a non trivial Reynolds stress tensor related to a Kolmogorov-Heisenberg spectra by a non trivial Wigner Measure.

A-priori estimates

A "general family" of boundary conditions containing the "classical":

$$u_\nu \cdot \vec{n} = 0 \quad \text{and}\ \nu(\partial_{\vec{n}} u_\nu + (C(x)u_\nu)_\tau + \lambda(\nu)u_\nu = 0 \ \text{ on } \partial\Omega \qquad (1)$$

$$\text{with } \lambda(\nu, x) \geq 0 \quad \text{and} \quad C(x) \in C(\mathbb{R}^n, \mathbb{R}^n) \qquad (2)$$

$$u_\nu \cdot \vec{n} = 0 \Rightarrow ((\nabla^\perp u_\nu) \cdot \vec{n})_\tau = (C(x)u_\nu)_\tau$$

Hence with $u_\nu \cdot \vec{n} = 0$ are of the type (1):

Dirichlet with $\lambda(\nu) = \infty$,

Dirichlet-Neumann with $\lambda(\nu) = C(x) = 0$,

Fourier with $C(x)(u_\nu) = (\nabla^\perp u_\nu) \Rightarrow \nu(S(u_\nu)\vec{n})_\tau + \lambda(\nu)u_\nu = 0$,

With $\lambda(x) = 0$ No stresses $(S(u_\nu)\vec{n})_\tau = 0$,

With vorticity $\nu((\nabla(u_\nu) - \nabla^\perp(u_\nu))\vec{n})_\tau + \lambda(\nu)u_\nu = 0$.

Energy estimates

$$\frac{1}{2}\int_\Omega |u_\nu(x,T)|^2 dx + \int_0^T (\nu \int_\Omega |\nabla u_\nu|^2 dx - \nu \int_0^T \int_{\partial\Omega} (\partial_{\vec n} u_\nu)\cdot u_\nu d\sigma = 0\,,$$

$$-\nu \int_{\partial\Omega} (\partial_{\vec n} u_\nu)\cdot u_\nu d\sigma = -\nu \int_{\partial\Omega} (C(x)u_\nu, u_\nu) d\sigma + \lambda(\nu)\int_{\partial\Omega} |u_\nu(x,t)|^2 d\sigma\,,$$

$$|\nu \int_{\partial\Omega} (C(x)u_\nu, u_\nu) d\sigma| \le C\nu (\int_\Omega |\nabla u_\nu|^2 dx)^{\frac12} (\int_\Omega |u_\nu|^2 dx)^{\frac12}\,,$$

$$\frac{1}{2}\int_\Omega |u_\nu(x,T)|^2 dx \le \frac{1}{2}(\int_\Omega |u_\nu(x,0)|^2 dx) e^{C\nu T}\,,$$

$$\frac{1}{2}\int |u_\nu(x,T)|^2 dx + \int_0^T (\nu \int_\Omega |\nabla u_\nu|^2 dx + \int_{\partial\Omega} \lambda(\nu)|u_\nu(x,t)|^2 d\sigma) dt = \frac{1}{2}\int |u_\nu(x,0)|^2 dx + o(\nu)\,.$$

Dissipative Solutions and Viscosity Solutions

Di Perna (1979), Dafermos (1979) Majda- Di Perna (1987) P.L. Lions (1996), with boundary CB Titi (2007).

$$S(w) = \frac{1}{2}(\nabla w + (\nabla w)^t)\,, \quad \partial_t w + (w\cdot\nabla w) = E(x,t) = E(w)$$
$$\partial_t u + \nabla\cdot(u\otimes u) + \nabla p = 0\,, \nabla u = 0\,, u\cdot\vec{n} = 0 \text{ with } u \,\text{smooth}\,,$$
$$\partial_t w + w\cdot\nabla w + \nabla q = E(w)\,, \quad \nabla\cdot w = 0\,.$$
$$\frac{1}{2}\int_\Omega |u(x,t)-w(x,t)|^2 \le \int_0^t\int |(E(x,s),u(x,s)-w(x,s))|dxds$$
$$+\int_0^t\int_\Omega |(u(x,s)-w(x,s))S(w)(u(x,s)-w(x,s))|dxds$$
$$+\frac{1}{2}\int_\Omega |u(x,0)-w(x,0)|^2dx\,. \qquad (3)$$

A dissipative solution is as a divergence free tangent to the boundary vector field which for any test function w as introduced above satisfies the relation (3).

Hence the stability of dissipative solutions with respect to smooth solutions and, in particular, the fact that whenever exists a smooth solution $u(x,t)$ any dissipative solution which satisfies $w(.,0) = u(.,0)$ coincides with u for all time.
However, it is important to notice that to obtain this property one needs to include in the class of test functions w vector fields that may have non zero tangential component on the boundary.

Viscosity limit

$$\partial_t u_\nu + u_\nu \cdot \nabla u_\nu - \nu\Delta u_\nu + \nabla p_\nu = 0\,, \quad \overline{u} = \text{weak} - \lim_{\nu\to 0} u\,,$$

$$\partial_t w + w \cdot \nabla w + \nabla q = E(w)\,,$$

$$\frac{1}{2}\frac{d}{dt}|u_\nu(x,t) - w(x,t)|^2_{L^2(\Omega)} + \nu|\nabla u_\nu(t)|^2_{L^2(\Omega)}$$

$$\leq |(S(w) : (u_\nu - w)\otimes(u_\nu - w))| + |(E(w), u_\nu - w)|$$

$$+\nu(\nabla u_\nu, \nabla w)_{L^2(\Omega)} + \nu(\partial_{\vec{n}} u_\nu, u_\nu - w)_{L^2(\partial\Omega))}\,,$$

$$\frac{1}{2}\int_\Omega |\overline{u}(x,t) - w(x,t)|^2 \leq \int_0^t \int |(E(x,s), \overline{u}(x,s) - w(x,s))|dxds$$

$$+\int_0^t \int_\Omega |(\overline{u}(x,s) - w(x,s)S(w)\overline{u}_\nu(x,s) - w(x,s))|dxds$$

$$+\frac{1}{2}\int_\Omega |u(x,0) - w(x,0)|^2 dx + \lim_{\nu\to 0} \nu \int_0^t \int_{\partial\Omega} (\partial_{\vec{n}} u_\nu, u_\nu - w)d\sigma dt$$

With no boundary convergence (modulo subsequence) to a dissipative solution is always true.

If there exists a smooth solution $u(x,t)$ on $[0,T]$ with the same initial data then $\overline{u}(x,t) = u(x,t)$.

$$\frac{1}{2}\int |\overline{u}(x,0)|^2 dx = \frac{1}{2}\int |\overline{u}(x,t)|^2 dx \leq \frac{1}{2}\lim_{\nu\to 0}\int |\overline{u}_\nu(x,t)|^2 dx \leq \frac{1}{2}\int |\overline{u}(x,0)|^2 d$$

- In the absence of boundary and with the existence of a smooth solution of the Euler equations there is no anomalous energy dissipation, no w.Reynolds stresses tensor.

Proof Peter Constantin Periodic Boundary Conditions and Kato in the whole space.

About wilde solutions of DeLellis and Székelyhidi

Even without boundary in the absence of regular solutions (loss of regularity for Euler solution or wild initial data) $\overline{u}$ is still a dissipative, but may be not a weak solution (Reynolds stresses tensor $\neq 0$ and may not be the unique solution.
In particular when u_0 is the initial data of a wilde solution in the sense of DeLellis and Székelyhidi.
However this is not the situation considered below.

Direct results with boundary

Theorem In the presence of a smooth Euler solution.

- Weak convergence to a dissipative solution.
- Convergence to a weak solution (up to the boundary) or with $C^{0,\alpha}\,,\alpha > 1/3\,,.$
- It is the solution.
- The sum of the kinetic and friction energy go to 0.

$$\lim_{\nu\to 0}\frac{1}{T}\int_0^T(\nu\int_\Omega|\nabla u_\nu(x,t)|^2dx+\int_{\partial\Omega}\lambda(x)|u_\nu(x,t)|^2dx)dt\to 0\,.$$

Conversely

Theorem In the presence of a smooth Euler solution Convergence to a dissipative solution:

1 In any case, in particular Dirichlet $(\nu \frac{\partial u_\nu}{\partial \vec{n}})_\tau \to 0$ in $\mathcal{D}'(\partial\Omega \times]0, T[)$,

2 For Fourier-Navier $\lambda(\nu) u_\nu \to 0$: in $\mathcal{D}'(\partial\Omega \times]0, T[) \to 0$,

3 $\lambda(\nu) \to 0$ or $\lambda(\nu)$ bounded and $\int_{\partial\Omega \times]0,T[} \lambda(\nu) |u_\nu(x,t)|^2 d\sigma dt \to 0$,

4 In any case Kato $\lim_{\nu \to 0} \nu \int_0^T \int_{\Omega \cap \{d(x,\partial\Omega) < \nu\}} |\nabla u_\nu(x,t)|^2 dx dt \to 0$.

No turbulence in the presence of physical boundary

In the presence of a smooth solution u for Euler equation on $[0, T]$ with the same initial data the following facts are equivalents

- Weak convergence to a " up to the boundary" weak solution $\Rightarrow$ No w.Reynold stresses tensor.
- $u_\nu \rightharpoonup u$. Weak convergence to the solution of the Euler equations.
- $\forall 0 < t < T$ $\frac{1}{2}\int_\Omega |w.\lim u_\nu(x,t)|^2 dx = \frac{1}{2}\int_\Omega |u_0(x)|^2 dx$. Energy conservation.
- $u_\nu \to u$. Strong convergence
- $\lim_{\nu\to 0} \nu \frac{\partial u_\nu}{\partial \vec{n}} = 0$ in $\mathcal{D}'(\Omega)$. No anomalous vorticity production at the boundary.
- $\lim_{\nu\to 0} \int_0^T (\int_\Omega \nu |\nabla u_\nu(x,t)|^2 dx + \lambda(\nu)\int_{\partial\Omega} |u_\nu|^2 d\sigma) dt = 0$. No anomalous energy dissipation.
- $\lim_{\nu\to 0} \int_0^T \int_{d(x,\partial\Omega)<\nu} |\nabla u_\nu(x,t)|^2 dx = 0$. No anomalous "order ν" boundary layer energy dissipation.

Remarks

• The existence of a Prandlt boundary layer (and in particular the analytic configuration considered by Asano, Caflish and Sanmartino (1998)) implies Kato hypothesis. Converse may not be true.

• In the case of slip boundary condition (of the type $\lambda(\nu) \to 0$ and with more regularity constraints many results concerning strong (in higher norms) convergence have already been obtained (Yudovich (1963), JL Lions (1969), Bardos (1972), Clopeau-Mikelic-Robert (1998), Beirao da Veiga and Crispo (2010), Xiao and Xin (2007)).

• If one of the above equivalent fact is not satisfied one would expect generation of turbulence.

The limit is not a solution of the Euler equations, there is no energy conservation, there is anomalous energy dissipation, the weak Reynolds stresses tensor is not 0 . *etc...*

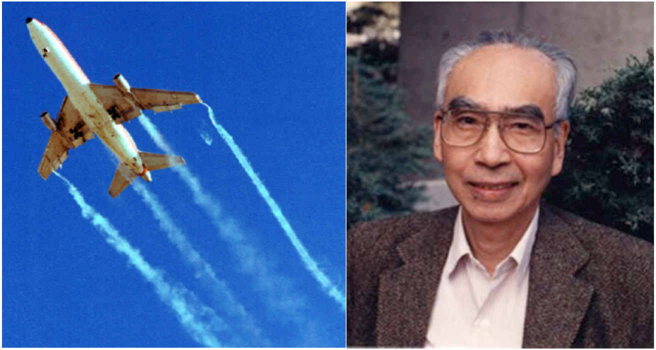

Figure: Kato: Prandlt..Boundary layer, Kelvin Helmholtz, Von Karman vortex street.

Proof of Kato argument

For any $w \in rmT(\partial\Omega\times]0,T[)$ introduce a sequence $w_\nu(s,\tau,t)$ (in geodesic coordinates near $\partial\Omega$) with

$$\mathrm{support}(w_\nu) \subset \Omega_\nu\times]0,T[\,, \nabla\cdot w_\nu = 0, \text{ and on } \partial\Omega\times]0,T[\quad w_\nu = w\,,$$
$$|\nabla_{\tau,t} w_\nu|_{L^\infty} \le C\,, \quad |\partial_s w_\nu|_{L^\infty} \le \frac{C}{\nu}\,.$$

From

$$(0, w_\nu) = ((\partial_t u_\nu + \nabla(u_\nu\otimes u_\nu) - \Delta u_\nu + \nabla p_\nu) w_\nu) =$$
$$-(u_\nu, \partial_t w_\nu) + ((u_\nu\otimes u_\nu):\nabla w_\nu) + \nu(\nabla u_\nu, \nabla w_\nu) - (\nu\partial_{\vec n} u_\nu w)_{L^2(\partial\Omega\times]0,T[}$$
$$\Rightarrow |(\nu\partial_{\vec n} u_\nu w)_{L^2(\partial\Omega\times]0,T[)}| = |((u_\nu\otimes u_\nu):\nabla w_\nu)| + o(\nu)$$

Poincaré estimate and a priori estimate

$$\Rightarrow |((u_\nu\otimes u_\nu):\nabla w_\nu)| \le C\int_0^T\int_{\Omega_\nu} \nu|\nabla u_\nu|^2 dxdt \to 0\,.$$

Boltzmann →Euler limit with boundary effect

To consolidate the fact that Kato approach may be the correct point of view and that the boundary condition

$$\nu(\partial_{\vec{n}} u_\nu + (C(x)u_\nu))_\tau + \lambda(\nu)u_\nu = 0$$

(which contains Dirichlet and Neumann) is the good one, one can argue that the introduction of a microscopic derivation based on the Boltzmann equation leads to the same results.

$F_\epsilon(x,v,t)\geq 0$: Density distribution of particles which at the point $x\in\Omega$ and the time t do have the velocity $v\in\mathbb{R}^n_v$) of the (rescaled in time) Boltzmann equation:

$$\epsilon\partial_t F_\epsilon + v\cdot\nabla_x F_\epsilon = \frac{1}{\epsilon^{1+q}}\mathcal{B}(F_\epsilon,F_\epsilon) \text{ Quadratic operator in } \mathbb{R}^n_v$$

with Maxwell Boundary Condition for $v\cdot\vec{n}<0$ in term of $v\cdot\vec{n}>0$.

$$F^-_\epsilon(x,v)=(1-\alpha(\epsilon))F^+_\epsilon(x,v^*)+\alpha(\epsilon)M(v)\sqrt{2\pi}\int_{v\cdot\vec{n}<0}|v\cdot\vec{n}|F^+_\epsilon(x,v)dv\,,$$

$$0\leq\alpha(\epsilon)\leq 1\,,\, v^*=v-2(v\cdot\vec{n})\vec{n}=\mathcal{R}(v)\,,$$

$$M(v)=\frac{1}{(2\pi)^{\frac{n}{2}}}e^{-\frac{|v|^2}{2}}\,,\quad \Lambda(\phi)=\sqrt{2\pi}\int_{\mathbb{R}^n_v}(v\cdot\vec{n})_+\phi(v)M(v)dv,$$

$$\Lambda(1)=1(\mathrm{proba!})\quad F^-_\epsilon(x,v)=(1-\alpha(\epsilon))F^+_\epsilon(x,\mathcal{R}(v))+\alpha(\epsilon)\Lambda(\frac{F_\epsilon}{M})\,,$$

$$F_\epsilon(x,v,0)=M(v)(1+\epsilon g(v))\quad \lim_{\epsilon\to 0}u_\epsilon=\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\mathbb{R}^n_v}\vec{v}F(\epsilon(x,v,t)dv\,.$$

• For $q = 0$, $u_\epsilon = \frac{1}{\epsilon} \int_{\mathbb{R}^n_v} vF_\epsilon dv$ converges to a Leray solution of Navier-Stokes with the boundary condition:

$$u \cdot \vec{n} = 0 \quad \text{and} \quad \nu((\nabla u + \nabla^t u) \cdot n)_\tau + \lambda(\nu) u = 0$$
$$\lambda(\nu) = \frac{1}{\sqrt{2\pi}} \lim_{\epsilon \to 0} \frac{\alpha(\epsilon)}{\epsilon} \quad \text{Dirichlet} \Leftrightarrow \lim_{\epsilon \to 0} \frac{\alpha(\epsilon)}{\epsilon} = \infty\,.$$

• Aoki, Inamuro, Onishi (1979) Stationary solution linearized regime and Hilbert expansion;

• Masmoudi-Saint Raymond (2003) for Mischler solutions towards Leray solutions.

• General formal proof C.B., Golse, Paillard (2011).

Entropy Dissipation versus Energy Balance

$$H(F|G) = \int_{\Omega\times\mathbb{R}^n_v} (F\log(\frac{F}{G}) - F + G)dxdv \text{ Relative entropy},$$

$$\frac{1}{\epsilon^2}\frac{d}{dt}H(F_\epsilon(t)|M) + \frac{1}{\epsilon^{q+4}}\int_\Omega\int_{\mathbb{R}^3_v} DE(F_\epsilon)dvdv_1d\sigma + \frac{1}{\epsilon^3}\int_{\partial\Omega}\mathrm{DG} = 0$$

$$\mathrm{DE}(F)(v,v_1,\sigma) = \frac{1}{4}(F'F_1' - FF_1)\log(F'F_1' - FF_1)b(|v-v_1|,\sigma) \text{ En. diss}$$

$$\mathrm{DG}(F) = \int_{\mathbb{R}^3_v} v\cdot\vec{n}H(F_\epsilon|M)d\sigma dv \text{ The Darrozes-Guiraud local entropy}.$$

$$h(z) = (1+z)\log(1+z) - z)$$

$$\sqrt{2\pi}\mathrm{DG} = \int_{\mathbb{R}^3_v} v\cdot\vec{n} H(F_\epsilon|M) d\sigma dv =$$

$$\sqrt{2\pi}\int_{\mathbb{R}^3_v} v\cdot\vec{n} H(M(1+\epsilon g_\epsilon)|M)dv = \sqrt{2\pi}\int_{\mathbb{R}^3_v} v\cdot\vec{n} M(v)h(1+\epsilon g_\epsilon)dv$$

$$= \sqrt{2\pi}\int_{\mathbb{R}^3_v}(v\cdot\vec{n})_+ M(v)h(\epsilon g_\epsilon(v))dv - \sqrt{2\pi}\int_{\mathbb{R}^3_v}(v\cdot\vec{n})_+ M(v)h(\epsilon g_\epsilon(\mathcal{R}v))$$

$$= \Lambda(h(\epsilon g_\epsilon)) - \Lambda(h[(1-\alpha(\epsilon))\epsilon g_\epsilon + \alpha(\epsilon)\Lambda(\epsilon g_\epsilon)])$$

$$\geq \alpha(\epsilon)\left[\Lambda(h(\epsilon g_\epsilon(v))) - h(\Lambda(\epsilon g_\epsilon(v)))\right] \geq 0$$

Hence the final entropy estimate:

$$\frac{1}{\epsilon^2}\frac{d}{dt}H(F_\epsilon(t)|M) + \frac{1}{\epsilon^{q+4}}\int_\Omega\int_{\mathbb{R}^3_v} DE(F_\epsilon)dvdv_1d\sigma$$
$$+\frac{1}{\epsilon^2}\frac{\alpha(\epsilon)}{\epsilon}\frac{1}{\sqrt{2\pi}}\int_{\partial\Omega}[\Lambda(h(\epsilon g_\epsilon(v))) - h(\Lambda(\epsilon g_\epsilon(v))))]d\sigma \leq 0\,.$$

Compare formally to energy with $g_\epsilon = \epsilon^{-1}(F_\epsilon - M)/M \to u \cdot v$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\nu(x,t)|^2 dx + \nu \int_\Omega |\nabla u_\nu|^2 dx + \int_{\partial\Omega} \lambda(\nu)|u_\nu(x,t)|^2 d\sigma \to 0$$

$$\frac{1}{\epsilon^2}\frac{d}{dt}H(F_\epsilon(t)|M) \to \frac{1}{2}\frac{d}{dt}\int_\Omega |u(x,t)|^2 dx$$

$$\frac{1}{\epsilon^{q+4}}\int_\Omega\int_{\mathbb{R}^3_v} DE(F_\epsilon) dv dv_1 d\sigma \simeq \epsilon^q \nu \int_\Omega |\nabla u + \nabla^\perp u|^2 dx$$

$$\frac{1}{\epsilon^2}\int_{\partial\Omega}[\Lambda(h(\epsilon g_\epsilon(v))) - h(\Lambda(\epsilon g_\epsilon(v)))] d\sigma \simeq \int_{\partial\Omega} |u_\epsilon(x,t)|^2 d\sigma$$

$$\frac{\alpha(\epsilon)}{\epsilon}\frac{1}{\sqrt{2\pi}} \simeq \lambda(\epsilon^q \nu)$$

Entropic convergence to a regular Euler solution $\Rightarrow$

$$\frac{1}{\epsilon^{q+4}}\int_\Omega\int_{\mathbb{R}^3_v} DE(F_\epsilon)dvdv_1d\sigma$$
$$+\frac{1}{\epsilon^2}\frac{\alpha(\epsilon)}{\epsilon}\frac{1}{\sqrt{2\pi}}\int_{\partial\Omega}[\Lambda(h(\epsilon g_\epsilon(v)))-h(\Lambda(\epsilon g_\epsilon(v))))]d\sigma\to 0$$

Theorem Sufficient condition for the convergence to Euler:

$$\lim_{\epsilon\to 0}\frac{\alpha(\epsilon)}{\epsilon}=0 \text{ or}$$
$$\frac{\alpha(\epsilon)}{\epsilon}\le C<\infty \text{ and } \frac{1}{\epsilon^2}\int_{\partial\Omega\times]0,T[}[\Lambda(h(\epsilon g_\epsilon(v)))-h(\Lambda(\epsilon g_\epsilon(v))))]d\sigma dt\to 0$$

Conjecture (Kato!)

$$\frac{1}{\epsilon^{q+4}}\int_0^T\int_{\Omega\cap\{d(x,\partial\Omega)\le\epsilon^q)\}}\int_{\mathbb{R}^3_v} DE(F_\epsilon)dvdv_1d\sigma dt\to 0\,.$$

Some details in proof

Proof uses Laure Saint Raymond argument. Simpler assuming local conservation of moment. Focus on the terms coming from the boundary. Introduce a divergence free tangent to the boundary smooth vector fields $w(x,t)$.

$$\frac{1}{\epsilon^2} H(M_{(1,\epsilon u_0,1)}|M_{(1,\epsilon w,1)}) = \frac{1}{2}\int_\Omega |u_{in} - w(x,0)|^2 dx$$

$$\frac{1}{\epsilon^2} H(F_\epsilon|M_{(1,\epsilon w,1)})(t) = \frac{1}{\epsilon^2} H(F_\epsilon|M)(t) + \int_{\Omega\times\mathbb{R}^3_v} (\frac{w^2}{2} - \frac{v}{\epsilon} w) F_\epsilon(t,x,v) dxdv$$

$$\frac{1}{2\epsilon^2}\frac{d}{dt}\int_\Omega\int F_\epsilon(t,x,v)(\epsilon^2 w^2 - 2\epsilon v\cdot w) dxdv$$
$$= \int_\Omega\int \partial_t w\cdot(w - \frac{1}{\epsilon}v) F_\epsilon(t,x,v) dxdv$$
$$+ \int_\Omega \left(\frac{w^2}{2}\partial_t \int F_\epsilon(t,x,v) dv - \frac{w}{\epsilon}\cdot\int \partial_t F_\epsilon(t,x,v) v dv\right) dx\,.$$

For $\partial_t \int F_\epsilon(t,x,v)dv$ and $\partial_t \int F_\epsilon(t,x,v)vdv$ use the local conservation laws : In the first term appears the conservation of mass:

$$\int_{\partial\Omega}\int_{\mathbb{R}^d_v} v\cdot\vec{n}F_\epsilon(t,x,v)dvd\sigma = 0 :$$

$$\int_\Omega \frac{1}{2}w^2\partial_t \int F_\epsilon(t,x,v)dx = -\frac{1}{\epsilon}\int_\Omega \frac{1}{2}w^2\nabla_x\cdot\int vF_\epsilon(t,x,v)vdvdx$$
$$= \frac{1}{\epsilon}\int_\Omega\int (v\cdot\nabla_x w)\cdot wF_\epsilon(t,x,v)dvdx$$
$$-\frac{1}{\epsilon}\int_{\partial\Omega} d\sigma\frac{1}{2}w^2\int_{\mathbb{R}^d_v} v\cdot\vec{n}F_\epsilon(t,x,v)dv =$$
$$= \int_\Omega\int \frac{1}{\epsilon}(v\cdot\nabla_x w)\cdot wF_\epsilon(t,x,v)dvdx\,.$$

In the second term appear the boundary effects:

$$-\int_\Omega \frac{w}{\epsilon}\cdot\int \partial_t F_\epsilon(t,x,v)vdv = \int_\Omega\int_{\mathbb{R}^3}\frac{w}{\epsilon^2}\cdot\int \nabla_x F_\epsilon(t,x,v)v\otimes vdv =$$
$$-\frac{1}{\epsilon^2}\int_\Omega\int (v\cdot\nabla_x)w\cdot vF_\epsilon(t,x,v)dvdx + \int_{\partial\Omega}\frac{1}{\epsilon^2}\int F_\epsilon(t,x,v)(w\cdot v)(\vec{n}\cdot v$$

Since w is tangent to the boundary one has for $x\in\partial\Omega$:

$$\frac{1}{\epsilon^2}\int F_\epsilon(t,x,v)(w\cdot v)(\vec{n}\cdot v) =$$
$$\frac{\alpha(\epsilon)}{\epsilon^2}\int F_\epsilon(t,x,v)(w\cdot v)(\vec{n}\cdot v)_+ dv =$$
$$\frac{1}{\sqrt{2\pi}}\frac{\alpha(\epsilon)}{\epsilon^2}\Lambda(\epsilon g_\epsilon(x,v,t)(w\cdot v))\,.$$

Therefore one obtains:

$$\frac{1}{\epsilon^2}\frac{d}{dt}H(F_\epsilon|M_{(1,\epsilon w,1)})(t)+$$
$$\frac{1}{\epsilon^{4+q}}\mathrm{DE}(F_\epsilon)+\frac{1}{\sqrt{2\pi}}\frac{\alpha(\epsilon)}{\epsilon^3}\int_{\partial\Omega}\Big[\Lambda(h(\epsilon g_\epsilon(v)))-h(\Lambda(\epsilon g_\epsilon(v)))\Big]d\sigma$$
$$\leq\int_\Omega\int(\partial_t w+w\cdot\nabla w)(w-\frac{v}{\epsilon})F_\epsilon(t,x,v)dxdv-$$
$$\int_\Omega\int(w-\frac{v}{\epsilon})\nabla_x w(w-\frac{v}{\epsilon})F_\epsilon(t,x,v)dxdv$$
$$+\frac{1}{\sqrt{2\pi}}\frac{\alpha(\epsilon)}{\epsilon^2}\int_{\partial\Omega}\Lambda(\epsilon g_\epsilon(x,v,t)(w\cdot v))d\sigma\,.$$

The exotic terms coming from the boundary are

$$\text{Good} \quad \frac{1}{\sqrt{2\pi}}\frac{\alpha(\epsilon)}{\epsilon^3}\int_{\partial\Omega}\Big[\Lambda(h(\epsilon g_\epsilon(v))) - h(\Lambda(\epsilon g_\epsilon(v)))\Big]d\sigma$$

$$\text{Bad} \quad \frac{1}{\sqrt{2\pi}}\frac{\alpha(\epsilon)}{\epsilon^2}\int_{\partial\Omega}\Lambda(\epsilon g_\epsilon(x,v,t)(w\cdot v))d\sigma\,.$$

The bad has to be balanced by the good.

Proposition

$$\forall \eta > 0$$

$$\int_{\partial\Omega} \Lambda(\epsilon g_\epsilon(t,x,v))(w\cdot v))d\sigma \;\le\; (\frac{1}{\eta} + \frac{\eta C(w)}{\epsilon}) \int_{\partial\Omega} \Lambda(h(\epsilon g_\epsilon) - h(\epsilon \Lambda g_\epsilon) d\sigma$$
$$+ C_2 \eta \int_{\partial\Omega} \int_{\mathbb{R}^3} F_\epsilon (v\cdot \vec{n}_x)^2 dv d\sigma$$

With $\eta = 2\epsilon$

$$\frac{\alpha(\epsilon)}{\epsilon^2} \int_{\partial\Omega} \Lambda(\epsilon g_\epsilon(t,x,v))(w\cdot v))d\sigma$$
$$\le (1 + 2\epsilon C(w)) \frac{\alpha(\epsilon)}{2\epsilon^3} \int_{\partial\Omega} \Lambda(h(\epsilon g_\epsilon) - h(\epsilon\Lambda g_\epsilon) d\sigma$$
$$+ C_2 \frac{\alpha(\epsilon)}{\epsilon} \int_{\partial\Omega} \int_{\mathbb{R}^3} F_\epsilon (v\cdot \vec{n}_x)^2 dv d\sigma$$

With $\frac{\alpha(\epsilon)}{\epsilon} \to 0$

$$\frac{1}{\epsilon^2}\frac{d}{dt}H(F_\epsilon|M_{(1,\epsilon w,1)})(t) \leq \int_\Omega\int(\partial_t w + w\cdot\nabla w)(w-\frac{v}{\epsilon})F_\epsilon(t,x,v)dxdv$$
$$-\int_\Omega\int(w-\frac{v}{\epsilon})\nabla_x w(w-\frac{v}{\epsilon})F_\epsilon(t,x,v)dxdv + o(\epsilon)$$

Then (cf. Saint Raymond) for

$$u = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\mathbb{R}^3_v} vF_\epsilon(x,v,t)dv$$

$$\frac{1}{2}\partial_t\int_\Omega|u(x,t)-w(x,t)|^2 \quad + \quad \int(u(x,t)-w(x,t)S(w)u(x,t)-w(x,t))d$$
$$\leq \quad \int(E(x,t),u(x,t)-w(x,t))dx\,.$$

Proof of the Proposition 2 steps

- Symmetry: $\Lambda(\Lambda(g_\epsilon)(w\cdot v))=0$
- Legendre duality between

$$
\begin{aligned}
&I(\epsilon g_\epsilon-\Lambda(\epsilon g_\epsilon)))\\
&=h((\epsilon g_\epsilon-\Lambda(\epsilon g_\epsilon))+\Lambda(\epsilon g_\epsilon))-h(\Lambda(\epsilon g_\epsilon))-h'(\Lambda(\epsilon g_\epsilon))(g_\epsilon-\Lambda(\epsilon g_\epsilon))
\end{aligned}
$$

and its Legendre transform:

$$
I^*(p)=(1+\Lambda(\epsilon g_\epsilon))(e^p-p-1)
$$

$$
(\epsilon g_\epsilon(t,x,v)-\Lambda(\epsilon g_\epsilon))(w\cdot v))=\frac{1}{\eta}(\epsilon g_\epsilon(t,x,v)-\Lambda(\epsilon g_\epsilon))(\eta w\cdot v))
$$

$$
\leq\frac{1}{\eta}\left(h((\epsilon g_\epsilon-\Lambda(\epsilon g_\epsilon))+\Lambda(\epsilon g_\epsilon))-h(\Lambda(\epsilon g_\epsilon))-h'(\Lambda(\epsilon g_\epsilon))(g_\epsilon-\Lambda(\epsilon g_\epsilon))\right)
$$

$$
+(1+\Lambda(\epsilon g_\epsilon))\frac{(e^{\eta|w||v|}-\eta|w||v|-1)}{\eta}
$$

$\Lambda(h'(\Lambda(\epsilon g_\epsilon))(g_\epsilon-\Lambda(\epsilon g_\epsilon)))=0$ Proba!

Step 2

$$\int_{\partial\Omega}(1+\Lambda(\epsilon g_\epsilon)d\sigma$$
$$\leq C_1\int_{\partial\Omega}\Lambda(h(\epsilon g_\epsilon)-h(\epsilon\Lambda g_\epsilon)d\sigma + C_2\int_{\partial\Omega}\int_{\mathbb{R}^3}F_\epsilon(v\cdot\vec{n}_x)^2 dvd\sigma$$

Proof With $G_\epsilon = F_\epsilon/M$ and $c=\int(v\cdot\vec{n})_+^2\wedge 1Mdv$

$$c\int_{\partial\Omega}(1+\Lambda(\epsilon g_\epsilon)d\sigma = \int_{\partial\Omega}\Lambda(G_\epsilon)\int(v\cdot\vec{n})_+^2\wedge 1dvM(v)d\sigma_x$$
$$= I_1 + I_2$$
$$\int_{\partial\Omega}\int_{\mathbb{R}^3_v}\Lambda(G_\epsilon)\mathbf{1}_{|G_\epsilon/\Lambda(G_\epsilon)-1|>\beta}(v\cdot\vec{n})_+^2\wedge 1M(v)d\sigma_x dv$$
$$+$$
$$\int_{\partial\Omega}\int_{\mathbb{R}^3_v}\Lambda(G_\epsilon)\mathbf{1}_{|G_\epsilon/\Lambda(G_\epsilon)-1|\leq\beta}(v\cdot\vec{n})^2\wedge 1M(v)d\sigma_x dv$$

$h(z)=(z+1)\log(z+1)-z\,,\, h(z)\geq h(|z|)$ and h is increasing on $\mathbb{R}_+$

$$
\begin{aligned}
I_1 &\leq \frac{1}{h(\beta)}\int_{\partial\Omega}\int_{\mathbb{R}^3_v}\Lambda(G_\epsilon)h\Big(|G_\epsilon/\Lambda(G_\epsilon)-1|\Big)(v\cdot\vec{n})_+^2\wedge 1M(v)d\sigma_x dv\\
&\leq \frac{1}{h(\beta)}\int_{\partial\Omega}\int_{\mathbb{R}^3_v}\Lambda(G_\epsilon)h\Big(G_\epsilon/\Lambda(G_\epsilon)-1\Big)(v\cdot\vec{n})_+M(v)d\sigma_x dv\\
&\leq \frac{1}{h(\beta)}\int_{\partial\Omega}\int_{\mathbb{R}^3_v}\Big(G_\epsilon\log(\frac{G_\epsilon}{\Lambda(G_\epsilon)})-G_\epsilon+\Lambda(G_\epsilon)\Big)(v\cdot\vec{n})_+M(v)d\sigma_x dv\\
&= \frac{1}{h(\beta)}\int_{\partial\Omega}\Lambda(h(\epsilon g_\epsilon)-h(\epsilon\Lambda(g_\epsilon))d\sigma
\end{aligned}
$$

For I_2 with $\beta < 1$

$$|G_\epsilon/\Lambda(G_\epsilon) - 1| \leq \beta \Rightarrow (\Lambda(G_\epsilon)) \leq \frac{1}{1-\beta} G_\epsilon$$

Hence

$$\begin{aligned} I_2 &= \int_{\partial\Omega}\int_{\mathbb{R}^3_v} \Lambda(G_\epsilon)\mathbf{1}_{|G_\epsilon/\Lambda(G_\epsilon)-1|\leq\beta}(v\cdot\vec{n})^2 \wedge 1 M(v)d\sigma_x dv \\ &\leq \frac{1}{1-\beta}\int_{\partial\Omega}\int_{\mathbb{R}^3_v} G_\epsilon(v\cdot\vec{n})^2_+ \wedge 1 M(v)d\sigma_x dv \\ &\leq \frac{1}{1-\beta}\int_{\partial\Omega}\int_{\mathbb{R}^3_v} F_\epsilon(v\cdot\vec{n})^2_+ d\sigma_x dv \end{aligned}$$

Use trace theorems introduced by Mischler!!!

Remarks

• **Proposition** With $\frac{\alpha(\epsilon)}{\epsilon} \to \lambda < \infty$ the convergence to zero of the Darrozes Guiraud entropy implies the convergence to a dissipative solution.

• The Maxwell boundary condition with the hypothesis $\alpha(\epsilon)/\epsilon \to 0$ which appears above has been generalized by Golse (2011). The analysis remains the same confirming the validity of the discussion

Conclusion

In the presence of boundary that the analogy between the notion of weak convergence and the statistical theory of turbulence is the most striking. A series of equivalent criteria for the absence of turbulence. Which at contrario would define turbulence as a situation where any of this effects is present:
More precisely there is no turbulence if one of the following effect is present:
1 No anomalous dissipation of energy.
2 No non trivial Reynolds stress tensor. With a spectra for the Wigner-measure that may fit some idea of statistical theory of turbulence.
3 No production of the vorticity at the boundary.
4 No production of vorticity in a region of size ν
5 No detachement .
Comparison with the analysis of the convergence from Boltzmann to Euler confirms the universality of the issues raised by the boundary.

Thanks for the invitation,

Thanks for listening

Happy Birthday Peter.