Arbitrage-Free Pricing, Optimal Investment and Utility-Based Valuation

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Part I

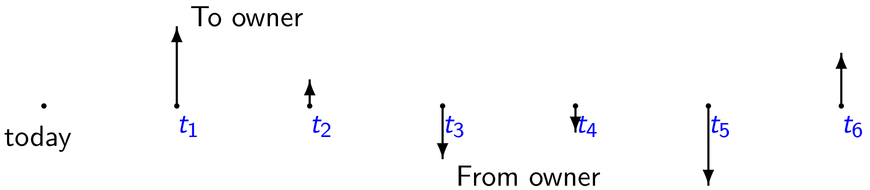
Arbitrage-free pricing

Outline

Financial security

Financial Security = Cash Flow

Example (Interest Rate Swap)



Pricing problem: compute “fair” value of the security **today**.

Classification of financial securities

We classify all financial securities into 2 groups:

1. **Traded securities**: the price is *given* by the market.

Financial model = All traded securities

2. **Non-traded securities**: the price has to be *computed*.

Remark

This "black-and-white" classification is quite idealistic. Real life securities are usually "gray".

In this tutorial we shall deal with two types of pricing methodologies:

1. **Arbitrage-free pricing**,
2. **Utility-based pricing**.

Arbitrage-free price

Inputs:

1. Financial model (collection of all traded securities)
2. A non-traded security.

Arbitrage strategy (intuitive definition):

1. start with zero capital (*nothing*)
2. end with positive and non zero wealth (*something*)

Assumption

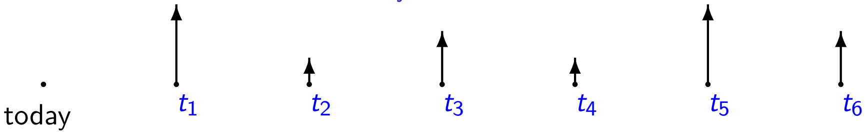
The financial model is arbitrage free.

Definition

An amount p is called an **arbitrage-free price** if, given an opportunity to trade the non-traded security at p, one is not able to construct an arbitrage strategy.

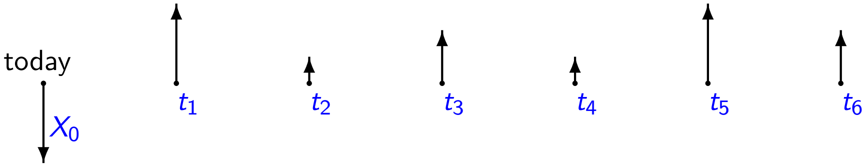
Replication

Cash flow of non-traded security:



Replicating strategy:

1. starts with some initial capital X_0
2. generates *exactly* the same cash flow in the future



Methodology of arbitrage-free pricing

Theorem

An arbitrage-free price p is unique if and only if there is a replicating strategy. In this case,

$$p = X_0,$$

where X_0 is the initial capital of a replicating strategy.

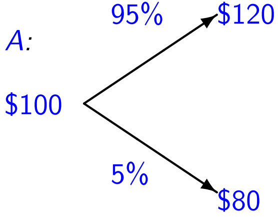
Main Principle:

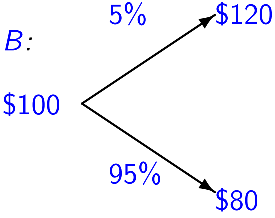
(Unique) Arbitrage-Free Pricing = Replication

Problem on two calls

Problem

Consider two stocks: A and B. Assume that





Consider call options on A and B with the same strike $K = \$100$.
Assume that $T = 1$ and $r = 5\%$.
Compute the difference $C^A - C^B$ of their arbitrage-free prices.

Pricing in Black and Scholes model

There are two traded assets: a savings account and *a stock*.
We assume that the interest rate is zero:

$$r = 0.$$

The price of the stock:

$$dS_t = S_t \left(\mu dt + \sigma dW_t \right).$$

Here $W = (W_t)_{t \geq 0}$ is a Wiener process and
$\mu \in \mathbf{R}$: drift
$\sigma > 0$: volatility

Problem (Black and Scholes, [BS73])

Compute arbitrage-free price V_0 *of European put option with maturity* T *and payoff*

$$\Psi = \max(K - S_T, 0).$$

Replication in Black and Scholes model

Basic principle: **Pricing = Replication**

Replicating strategy:

1. has wealth evolution:

$$X_t = X_0 + \int_0^t \Delta_u dS_u,$$

   where X_0 is the initial capital and Δ_t is the number of shares at time t;

2. generates *exactly* the same payoff as the option:

$$X_T(\omega) = \Psi(\omega) = \max(K - S_T(\omega), 0), \quad \mathbb{P}\text{-a.s.}.$$

Two standard methods: "direct" (PDE) and "dual" (martingales).

PDE method

Since $X_T = f(S_T)$ we look for replicating strategy in the form:

$$X_t = v(S_t, t)$$

for some deterministic $v = v(s, t)$. By Ito's formula,

$$dX_t = v_s(S_t, t)dS_t + (v_t(S_t, t) + \frac{1}{2}\sigma^2 S_t^2 v_{ss}(S_t, t))dt.$$

But, (since X is a wealth process)

$$dX_t = \Delta_t dS_t.$$

Hence, $v = v(s, t)$ solves PDE:

$$\begin{cases} v_t(s,t) + \frac{1}{2}\sigma^2 s^2 v_{ss}(s,t) & = & 0 \\ v(s,T) & = & \max(K - s, 0) \end{cases}$$

Martingale method

Observation: replication problem is defined "almost surely" and, hence, is invariant with respect to an equivalent change of probability measure.

Convenient choice: **martingale measure** $\mathbb{Q}$ for S. We have

$$dS_t = S_t \sigma dW_t^{\mathbb{Q}},$$

where $W^{\mathbb{Q}}$ is a Brownian motion under $\mathbb{Q}$.

Replication strategy: (by Martingale Representation Theorem)

$$X_t = X_0 + \int_0^t \Delta dS = \mathbb{E}^{\mathbb{Q}}[\Psi|\mathcal{F}_t].$$

Risk-neutral valuation: (no replication!)

$$V_0 = \mathbb{E}^{\mathbb{Q}}[\Psi].$$

Arbitrage-free pricing: general financial model

There are $d + 1$ *traded* or *liquid* assets:

1. a *savings account* with zero interest rate.
2. d *stocks*. The price process S of the stocks is a *semimartingale* on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$.

Question

Is the model **arbitrage-free**?

Question

Is the model **complete**? In other words, does it allow replication of any non-traded derivative?

Fundamental Theorems of Asset Pricing

Let $\mathcal{Q}$ denote the family of **martingale measures** for S, that is,

$$\mathcal{Q} = \{\mathbb{Q} \sim \mathbb{P} : \quad S \text{ is a local martingale under } \mathbb{Q}\}$$

Theorem (**1st FTAP**)

$$\textbf{Absence of arbitrage} \quad \Longleftrightarrow \quad \mathcal{Q} \neq \emptyset.$$

Theorem (**2nd FTAP**)

$$\textbf{Completeness} \quad \Longleftrightarrow \quad |\mathcal{Q}| = 1.$$

Free Lunch with Vanishing Risk

For 1st FTAP to hold true the following definition of arbitrage is needed (Delbaen and Schachermayer [DS94]):

1. There is a set $A \in \Omega$ with $\mathbb{P}[A] > 0$.
2. For any $\epsilon > 0$ there is a strategy X such that
   2.1 X is *admissible*, that is, for some constant $c > 0$,
   $$X \geq -c.$$
   2.2 $X_0 \leq \epsilon$ (start with almost nothing)
   2.3 $X_T \geq 1_A$ (end with something)

Risk-Neutral Valuation

Consider a European option with payoff Ψ at maturity T. The formula

$$V_0 = \mathbb{E}^{\mathbb{Q}}[\Psi],$$

where $\mathbb{Q} \in \mathcal{Q}$ is called **Risk-Neutral Valuation**.

Arbitrage-free models:

Unique Arbitrage-Free Pricing = Replication

Complete models: (no replication!)

Arbitrage-Free Pricing = Risk-Neutral Valuation

References

Fischer Black and Myron Scholes.
The pricing of options and corporate liabilities.
Journal of Political Economy, 81:637–654, 1973.

Freddy Delbaen and Walter Schachermayer.
A general version of the fundamental theorem of asset pricing.
Math. Ann., 300(3):463–520, 1994.

Part II

Optimal investment

Outline

Introduction to optimal investment

Consider an economic agent (an investor) in an *arbitrage-free* financial model.

x: initial capital

Goal: invest x *"optimally"* up to maturity T.

Question

How to compare two investment strategies:

1. $x \longrightarrow X_T = X_T(\omega)$
2. $x \longrightarrow Y_T = Y_T(\omega)$

Clearly, we would prefer 1st to 2nd if $X_T(\omega) \geq Y_T(\omega)$, $\omega \in \Omega$. However, as the model is arbitrage-free, in this case, $X_T(\omega) = Y_T(\omega)$, $\omega \in \Omega$.

Introduction to optimal investment

Classical approach (Von Neumann - Morgenstern, Savage): an investor is *"quantified"* by

$\mathbb{P}$: "scenario" probability measure

$U = U(x)$: *utility function*

"Quality" of a strategy

$$x \longrightarrow X_T = X_T(\omega)$$

is then measured by *expected utility*: $\mathbb{E}[U(X_T)]$.

Given two strategies: $x \longrightarrow X_T$ and $x \longrightarrow Y_T$ the investor will prefer the 1st one if

$$\mathbb{E}[U(X_T)] \geq \mathbb{E}[U(Y_T)]$$

Introduction to optimal investment

Inputs:

1. Arbitrage-free financial model (all traded securities)
2. Risk-averse investor:

x: initial wealth

$\mathbb{P}$: "real world" probability measure

$U = U(x)$: strictly increasing and strictly concave utility function

Output: the optimal investment strategy $x \longrightarrow \widehat{X}_T$ such that

$$\mathbb{E}[U(\widehat{X}_T)] = u(x) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)].$$

Here $\mathcal{X}(x)$ is the set of strategies with initial wealth x.

Merton's problem

First papers in continuous time finance: Merton [Mer69].

Black and Scholes model: *a savings account* and *a stock*.

1. We assume that the interest rate is 0.
2. The price of the stock:

$$dS_t = S_t \left(\mu dt + \sigma dW_t \right).$$

Here $W = (W_t)_{t \geq 0}$ is a Wiener process and

$\mu \in \mathbf{R}$: drift

$\sigma > 0$: volatility

Merton's problem

The problem of optimal investment

$$u(x) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)]$$

becomes in this case a *stochastic control problem*:

$$u(x,t) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_{T-t})] = \sup_{\pi} \mathbb{E}[U(X^{\pi}_{T-t})],$$

where the *controlled* process X^{π} is the wealth process:

$$dX^{\pi} = X^{\pi}\pi(\mu dt + \sigma dW) \quad X^{\pi}_0 = x$$

and the *control* process π is the proportion of the capital invested in stock.

Merton's problem

Bellman equation:

$$u_t + \sup_{\pi} \left[\pi x \mu u_x + \frac{1}{2}\pi^2\sigma^2 x^2 u_{xx} \right] = 0.$$

It follows that

$$\begin{cases} u_t(x,t) & = & \frac{\mu^2 u_x^2}{2\sigma^2 u_{xx}}(x,t) \\ u_{xx}(x,t) & < & 0 \\ u(x,T) & = & U(x) \end{cases}$$

and the optimal proportion:

$$\widehat{\pi}(x,t) = -\frac{\mu u_x}{\sigma^2 x u_{xx}}(x,t).$$

Merton's problem

Merton [Mer69] solved the system for the case, when

$$U(x,\alpha) = \frac{x^\alpha - 1}{\alpha} \quad (\alpha < 1).$$

Here

$$-\frac{U'(x)}{xU''(x)} = \frac{1}{1-\alpha} \quad (= \text{const!})$$

This key property is "inherited" be the solution:

$$\frac{u_x}{xu_{xx}}(x,t) = \text{const}.$$

Merton's problem

After this substitution the first equation in the system becomes

$$u_t = \text{const}\ x^2 u_{xx}$$

and could be solved analytically.
The optimal strategy (Merton's point):

$$\widehat{\pi} = \frac{\mu}{(1-\alpha)\sigma^2}.$$

Surprisingly, the problem has not been solved for general utility function U for quite a long time.

Merton's problem

In general case, we define the conjugate function

$$v(y,t) = \sup_{x>0}\left[u(x,t) - xy\right]$$

The function v satisfies

$$\begin{aligned} v_t &= \text{const } y^2 v_{yy} \\ v(y,T) &= V(y) := \sup_{x>0}[U(x) - xy] \end{aligned}$$

Methodology: compute v first and then find u from the inverse duality relationship:

$$u(x,t) = \inf_{y>0}\left[v(y,t) + xy\right]$$

Model of a financial market

There are $d+1$ *traded* or *liquid* assets:

1. a *savings account* with zero interest rate.
2. d *stocks*. The price process S of the stocks is a semimartingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$.

Assumption (No Arbitrage)

$$\mathcal{Q} \neq \emptyset$$

where $\mathcal{Q}$ is the family of martingale measures for S.

Economic agent or investor

x: initial capital

U: utility function for consumption at the maturity T such that

1. $U : (0, \infty) \to \mathbf{R}$
2. U is strictly increasing
3. U is strictly concave
4. The Inada conditions hold true:

$$U'(0) = \infty \quad U'(\infty) = 0$$

Problem of optimal investment

The goal of the investor is to maximize **the expected utility of terminal wealth**:

$$u(x) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)], \quad x > 0$$

Here $\mathcal{X}(x)$ is the set of strategies with initial wealth x.

Assumption

The value function is finite:

$$u(x) < \infty, \quad x > 0.$$

Two main approaches

1. **Bellman equation**.
2. **Duality and martingales**. Basic idea: as

$$\mathbb{E}[U(\widehat{X}_T(x))] = \max_{X \in \mathcal{X}(0)} \mathbb{E}[U(\widehat{X}_T(x) + X_T)]$$

we have that for any $X \in \mathcal{X}(0)$

$$\mathbb{E}[U'(\widehat{X}_T(x))X_T] = 0$$

Hence, there is $\mathbb{Q} \in \mathcal{Q}$ such that

$$U'(\widehat{X}_T(x)) = \text{const}\frac{d\mathbb{Q}}{d\mathbb{P}}$$

Investment in complete models

Complete model: $|\mathcal{Q}| = 1$

Define the functions

$$V(y) = \max_{x>0} \left[U(x) - xy \right], \quad y > 0.$$

$$v(y) = \mathbb{E}\left[V\left(y(\frac{d\mathbb{Q}}{d\mathbb{P}}) \right) \right], \quad y > 0$$

Theorem

$$u(x) = \inf_{y>0} [v(y) + xy]$$

Investment in complete models

Theorem

The following conditions are equivalent:

1. *The dual value function* $v = v(y)$ *is finite:*

$$v(y) < \infty, \quad y > 0$$

2. *The primal value function* $u = u(x)$ *is strictly concave and satisfies the Inada conditions.*

Moreover, in this case, $\widehat{X}(x)$ *exists for any* $x > 0$ *and*

$$\widehat{X}_T(x) = -V'\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right), \quad y = u'(x).$$

Investment in complete markets

The optimal terminal wealth $\widehat{X}_T(x)$ is uniquely determined by the equations:

$$\widehat{X}_T(x) = -V'(y\frac{d\mathbb{Q}}{d\mathbb{P}})$$

$$\mathbb{E}_{\mathbb{Q}}[\widehat{X}_T(x)] = x$$

The optimal number of stocks $\widehat{H}_t(x)$ at time t is given by the integral representation formula:

$$\widehat{X}_t(x) = \mathbb{E}_{\mathbb{Q}}[\widehat{X}_T(x)|\mathcal{F}_t] = x + \int_0^t \widehat{H}_u(x)dS_u.$$

Back to Merton's problem

For Black and Scholes model we have

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp(-\frac{\mu}{\sigma} W_T - \frac{1}{2}\frac{\mu^2}{\sigma^2} T) = \exp(-\frac{\mu}{\sigma} W_T^{\mathbb{Q}} + \frac{1}{2}\frac{\mu^2}{\sigma^2} T),$$

where

$$W_t^{\mathbb{Q}} = W_t + \frac{\mu}{\sigma} t,$$

is the $\mathbb{Q}$-Brownian motion. We deduce

$$\widehat{H}_t(x) S_t = \frac{\mu}{\sigma^2} R_t(x),$$

where $R(x)$ is the *risk-tolerance wealth process* defined as the wealth process replicating the payoff:

$$R_T(x) := -\frac{U'(\widehat{X}_T(x))}{U''(\widehat{X}_T(x))}.$$

Basic questions for incomplete models

1. Does the optimal investment strategy $X(x)$ exist?
2. Does the value function $u = u(x)$ satisfy the *standard* properties of a utility function? In other words,
    2.1 Is u strictly concave?
    2.2 Do Inada conditions

$$u'(0) = \infty, \quad u'(\infty) = 0$$

hold true?

Basic questions for incomplete models

3. Does the conjugate function

$$v(y) = \sup_{x>0}\{u(x) - xy\}, \quad y > 0,$$

have the representation:

$$v(y) = \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}[V(y\frac{d\mathbb{Q}}{d\mathbb{P}})],$$

where

$$V(y) = \sup_{x>0}\{U(x) - xy\}, \quad y > 0?$$

Asymptotic elasticity

Recall that the *elasticity* for U is defined as

$$E(U)(x) = \frac{xU'(x)}{U(x)}$$

The crucial role is played by the *asymptotic elasticity*:

$$AE(U) = \limsup_{x \to \infty} \frac{xU'(x)}{U(x)}.$$

We always have $AE(U) \leq 1$.

Assumption

$$AE(U) < 1.$$

Minimal market independent condition

Theorem (K.& Schachermayer [KS99])

The following conditions are equivalent :

1. $AE(U) < 1$.
2. *For any financial model the "qualitative" properties 1–3 hold true.*

In addition, in this case

$$AE(u) \leq AE(U) < 1.$$

Remark

The condition $AE(U) < 1$ is similar to Δ_2-condition in the theory of Orlicz spaces.

Necessary and sufficient conditions

Theorem (K.& Schachermayer [KS03])

The following conditions are equivalent for given financial model:

1. *For any* $y > 0$ *there is* $\mathbb{Q} \in \mathcal{Q}$ *such that*

$$\mathbb{E}[V\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)] < \infty.$$

2. *The "qualitative" properties 1–3 hold true.*

Dual space of supermartingales

The lower bound in

$$v(y) = \inf_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}\left[V\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right]$$

is, in general, not attained. However, if we extend the space of density processes of martingale measures to the space $\mathcal{Y}(y)$ of strictly positive supermartingales Y such that

1. $Y_0 = y$
2. XY is a supermartingale for any $X \in \mathcal{X}(x)$

then (without any extra assumptions!) we have

$$v(y) = \inf_{Y\in\mathcal{Y}(y)} \mathbb{E}[V(Y_T)]$$

and the lower bound above is attained by $\widehat{Y}(y) \in \mathcal{Y}(y)$. *This is even more convenient for computations!*

References

Robert C. Merton.
Lifetime portfolio selection under uncertainty: the continuous-time case.
Rev. Econom. Statist., pages 247–257, 1969.

D. Kramkov and W. Schachermayer.
The asymptotic elasticity of utility functions and optimal investment in incomplete markets.
Ann. Appl. Probab., 9(3):904–950, 1999.

D. Kramkov and W. Schachermayer.
Necessary and sufficient conditions in the problem of optimal investment in incomplete markets.
Ann. Appl. Probab., 13(4):1504–1516, 2003.

Part III

Utility based valuation and risk-tolerance wealth processes

Outline

Goal: study the **dependence** of prices for non-traded securities on **trading volume** due to market incompleteness (inability to replicate).

Model of a financial market

There are $d+1$ *traded* or *liquid* assets:

1. a *savings account* with zero interest rate.
2. d *stocks*. The price process S of the stocks is a semimartingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P})$.

Let $\mathcal{Q}$ denote the family of **martingale measures** for S, that is,

$$\mathcal{Q} = \{\mathbb{Q} \sim \mathbb{P} : \quad S \text{ is a local martingale under } \mathbb{Q}\}$$

Assumption (No Arbitrage)

$$\mathcal{Q} \neq \emptyset$$

Contingent claims

Consider a family of m **non-traded** or **illiquid** European contingent claims with

1. maturity T
2. payment functions $f = (f_i)_{1 \leq i \leq m}$.

Assumption

No nonzero portfolio of f is replicable:

$$\langle q, f \rangle = \sum_{i=1}^{m} q_i f_i \text{ is } \textbf{replicable} \ \Leftrightarrow q = 0$$

Pricing problem

Question

What is the (marginal) price $p = (p_i)_{1 \leq i \leq m}$ of the contingent claims f?

Definition (Intuitive)

The marginal price p for the (one-dimensional) contingent claim f is the **threshold** such that given a chance to buy or sell at a price p^{trade} an investor will

$$\text{buy at } p^{\text{trade}} < p \;\&\; \text{sell at } p^{\text{trade}} > p$$
$$\Updownarrow$$
$$\text{do nothing at } p^{\text{trade}} = p$$

Economic agent or investor

Consider an investor with a portfolio (x, q), where

x: *liquid* capital

$q = (q_i)$: quantities of the *illiquid* contingent claims.

His preferences with respect to consumption at maturity are modeled by

1. **subjective probability measure** $\mathbb{P}$
2. **utility function** $U = U(x)$:
   2.1 $U : (0, \infty) \to \mathbf{R}$, strictly increasing and strictly concave
   2.2 The Inada conditions hold true:

$$U'(0) = \infty \quad U'(\infty) = 0$$

Problem of optimal investment

The goal of the investor is to maximize **the expected utility of terminal wealth**:

$$u(x,q) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T + \langle q, f \rangle)],$$

where $\mathcal{X}(x)$ is the set of strategies with initial wealth x.

Order structure:

$$(x,q) \text{ is ``better'' than } (x',q')$$
$$\Updownarrow$$
$$u(x,q) \geq u(x',q').$$

Marginal utility based price

Definition

A **marginal utility based price** for the claims f given the portfolio (x, q) is a vector $p(x, q)$ such that

$$u(x, q) \geq u(x', q')$$

for any pair (x', q') satisfying

$$x + \langle q, p(x, q) \rangle = x' + \langle q', p(x, q) \rangle.$$

In other words, given the portfolio (x, q) the investor **will not trade** the options at $p(x, q)$.

Computation of $p(x) = p(x, 0)$

Define the conjugate function

$$V(y) = \max_{x>0} [U(x) - xy], \quad y > 0.$$

and consider the following **dual** optimization problem:

$$v(y) = \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}\left[V\left(y(\frac{d\mathbb{Q}}{d\mathbb{P}})\right)\right], \quad y > 0$$

Denote by $\mathbb{Q}(y)$ the **minimizer above** for y (if exists).
Mark Davis gave heuristic arguments to show that if y corresponds to x in the sense that

$$x = -v'(y)$$

then

$$p(x) = \mathbb{E}_{\mathbb{Q}(y)}[f].$$

Computation of $p(x) = p(x, 0)$

Precise mathematical statement is given in a joint paper with Julien Hugonnier and Walter Schachermayer [HKS05].

Theorem (Bounded contingent claims)

Let $x > 0$ and $y = u'(x)$. The following conditions are equivalent:

1. $p(x)$ *is unique for* **any bounded** f.
2. $\mathbb{Q}(y)$ *exists*

Moreover, in this case

$$p(x) = \mathbb{E}_{\mathbb{Q}(y)}[f].$$

Computation of $p(x) = p(x, 0)$

Theorem (General case)

Let $x > 0$, $y = u'(x)$ *and* X *be a non-negative wealth process. The following conditions are equivalent:*

1. $p(x)$ *is unique for* **any** f *such that*

$$|f| \leq K(1 + X_T) \text{ for some } K > 0$$

2. $\mathbb{Q}(y)$ *exists and* X *is a martingale under* $\mathbb{Q}(y)$*.*

Moreover, in this case

$$p(x) = \mathbb{E}_{\mathbb{Q}(y)}[f].$$

Trading problem

Suppose that we can trade f at p^{trade} and our initial position is given by the cash amount $x > 0$.

Direction of trade:

$$p(x) > p^{\text{trade}} \Rightarrow \text{buy}$$

$$p(x) < p^{\text{trade}} \Rightarrow \text{sell}$$

Question

What quantity $q = q(p^{\text{trade}})$ the investor should trade (buy or sell) at the price p^{trade}?

Answer

Using the field marginal utility based prices $p(x, q)$ we can compute the optimal quantity from the "equilibrium" condition:

$$p^{\text{trade}} = p(x - qp^{\text{trade}}, q)$$

Sensitivity analysis of utility based prices

Main difficulty: the marginal prices $p(x, q)$ are hard to compute except for the case $q = 0$.

Linear approximation: for "small" Δx and q

$$p(x + \Delta x, q) \approx p(x) + p'(x)\Delta x + D(x)q,$$

where $p'(x)$ is the derivative of $p(x)$ and

$$D^{ij}(x) = \frac{\partial p^i}{\partial q^j}(x, 0), \quad 1 \leq i, j \leq m.$$

The vector $p'(x)$ and the matrix $D(x)$ measure the **sensitivity** of $p(x, q)$ with respect to x and q at $(x, 0)$. Hereafter, we present results of a joint paper with Mihai Sîrbu, [KS06].

Technical assumptions

Assumption

The financial model can be **completed** by an addition of a finite number of securities.

Assumption

There are strictly positive constants c_1 and c_2 such that

$$c_1 < A(x) = -\frac{xU''(x)}{U'(x)} < c_2, \quad x > 0.$$

Assumption

There is a wealth process $X \geq 0$ such that

1. $\|f\| \leq X_T$
2. $X^{X(x)} = X \frac{x}{X(x)}$ is a square integrable martingale under $\mathbb{Q}^{X(x)}$

Quantitative and qualitative questions

Question (Quantitative)

How to compute $p'(x)$ and $D(x)$?

Question (Qualitative)

When the following (desirable) properties hold true for **any** family of contingent claims f?

1. The marginal utility based price $p(x) = p(x, 0)$ **does not depend** (locally) on x, that is,

$$p'(x) = 0$$

2. The sensitivity matrix $D(x)$ has **full rank**, that is,

$$D(x)q = 0 \Leftrightarrow q = 0.$$

Qualitative questions

3. The sensitivity matrix $D(x)$ is **symmetric**, that is,

$$D^{ij}(x) = D^{ji}(x) \quad \text{for all } i, j.$$

4. The sensitivity matrix $D(x)$ is **negative semi-definite**, that is,

$$\langle q, D(x)q \rangle \leq 0.$$

5. **Stability** of the linear approximation: for any p^{trade} the linear approximation to the "equilibrium" equation:

$$p^{\text{trade}} = p(x - qp^{\text{trade}}, q)$$

that is,

$$p^{\text{trade}} \approx p(x) - p'(x)qp^{\text{trade}} + D(x)q$$

has a qualitatively *"correct"* solution.

Main qualitative result

Theorem

Assume that the technical assumptions hold. The following assertions are equivalent:

1. $p'(x) = 0$ *for any* f.
2. $D(x)$ *is symmetric for any* f
3. $D(x)$ *has full rank for any (non-replicable)* f.
4. $D(x)$ *is negative semidefinite for any* f.
5. *There exists the risk-tolerance wealth process* $R(x)$.

If either of the above conditions hold true, then $D(x)$ is symmetric, negative definite (for non replicable f) and

$$p(x + \Delta x, q) \approx p(x) + D(x)q$$

Risk-tolerance wealth process

Recall that the ratio $-U'(x)/U''(x)$ is called the **risk-tolerance** coefficient of U at x.

Definition

A maximal wealth process $R(x)$ is called the **risk-tolerance wealth process** if

$$R_T(x) = -\frac{U'(X_T(x))}{U''(X_T(x))},$$

where $X(x)$ is the solution of

$$u(x) := u(x,0) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)].$$

Risk-tolerance wealth process

Some properties of $R(x)$ (if it exists):

1. Initial value:

$$R_0(x) = -\frac{u'(x)}{u''(x)},$$

where

$$u(x) = \max_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)].$$

2. Derivative of optimal investment strategy:

$$\frac{R(x)}{R_0(x)} = X'(x) := \lim_{\Delta x \to 0} \frac{X(x+\Delta x) - X(x)}{\Delta x}.$$

Shows what the investor does with extra penny.

Existence of $R(x)$

Recall that $\mathbb{Q}(y)$ is the minimal martingale measure (the solution to the dual problem) for y:

$$v(y) = \inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}\left[V\left(y(\frac{d\mathbb{Q}}{d\mathbb{P}}) \right) \right].$$

where V is the convex conjugate of U.

Theorem

The following assertions are equivalent:

1. $R(x)$ *exists.*
2. $\frac{d}{dy} \frac{d\mathbb{Q}(y)}{d\mathbb{P}} = 0$ *at* $y = u'(x)$ *(derivative in probability).*

In particular, $R(x)$ *exists for any* $x > 0$ *if and only if* $\mathbb{Q}(y)$ *is the same for all* y*:*

$$\mathbb{Q} = \widehat{\mathbb{Q}}.$$

Second order stochastic dominance

Definition

If ξ and η are nonnegative random variables, then $\xi \succeq_2 \eta$ if

$$\int_0^t \mathbb{P}(\xi \geq x)dx \geq \int_0^t \mathbb{P}(\eta \geq x)dx, \; t \geq 0.$$

We have that $\xi \succeq_2 \eta$ iff

$$\mathbb{E}[W(\xi)] \leq \mathbb{E}[W(\eta)]$$

for any *convex* and *decreasing* function W.

Existence of $R(x)$

Case 1: a utility function U is arbitrary.

Theorem

The following assertions are equivalent:

1. $R(x)$ *exists for any* $x > 0$ *and any utility function* U.
2. *There exists a unique* $\widehat{\mathbb{Q}} \in \mathcal{Q}$ *such that*

$$\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} \succeq_2 \frac{d\mathbb{Q}}{d\mathbb{P}} \quad \forall \mathbb{Q} \in \mathcal{Q}.$$

Existence of $R(x)$

Case 2: a financial model is arbitrary.

Theorem

The following assertions are equivalent:

1. $R(x)$ *exists for any* $x > 0$ *and any financial model.*
2. *The utility function* U *is*
   2.1 *a power utility:*

$$U(x) = (x^{\alpha} - 1)/\alpha, \quad \alpha < 1, \ \textit{if } x \in (0, \infty);$$

   2.2 *an exponential utility:*

$$U(x) = -\exp(-\gamma x), \quad \gamma > 0, \ \textit{if } x \in (-\infty, \infty).$$

Computation of $D(x)$

We choose

$$R(x)/R_0(x) = X'(x)$$

as a **numéraire** and denote

$f^R = fR_0(x)/R(x)$: discounted contingent claims

$X^R = XR_0(x)/R(x)$: discounted wealth processes

$\mathbb{Q}^R$: the martingale measure for X^R, that is

$$\frac{d\mathbb{Q}^R}{d\widehat{\mathbb{Q}}} = \frac{R_T(x)}{R_0(x)}$$

Computation of $D(x)$

Consider the Kunita-Watanabe decomposition:

$$P_t^R = \mathbb{E}_{\mathbb{Q}^R}\left[f^R|\mathcal{F}_t\right] = M_t + N_t, \quad N_0 = 0,$$

where

1. M is $R(x)/R_0(x)$-discounted wealth process. Interpretation: **hedging process**.
2. N is a martingale under $\mathbb{Q}^R$ which is orthogonal to all $R(x)/R_0(x)$-discounted wealth processes. Interpretation: **risk process**.

Computation of $D(x)$

Denote

$$a(x) := -xu''(x)/u'(x)$$

the relative risk-aversion coefficient of

$$u(x) = \max_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)].$$

Theorem

Assume that the risk-tolerance wealth process $R(x)$ *exists. Then*

$$D(x) = -\frac{a(x)}{x}\mathbb{E}_{\mathbb{Q}^R}\left[N_T N'_T\right]$$

Computation of $D(x)$ in practice

Question

How to compute $D(x)$ in *practice*?

Inputs:

1. $\widehat{\mathbb{Q}}$. *Already implemented!*
2. $R(x)/R_0(x)$. Recall that

$$\frac{R(x)}{R_0(x)} = \lim_{\Delta x \to 0} \frac{X(x+\Delta x) - X(x)}{\Delta x}.$$

   Decide what to do with one penny!

3. Relative risk-aversion coefficient $a(x)$. *Deduce from mean-variance preferences*. In any case, this is just a number!

Model with basis risk

Traded asset:

$$dS_t = S_t \left(\mu dt + \sigma dW_t \right).$$

Non traded asset:

$$d\widetilde{S} = (\widetilde{\mu} dt + \widetilde{\sigma} d\widetilde{W_t}).$$

Denote by

$$\rho = \frac{d\widetilde{W} dW}{dt}$$

the **correlation** coefficient between S and $\widetilde{S}$. In practice, we want to chose S so that

$$\rho \approx 1.$$

Model with basis risk

Consider contingent claims whose payoffs are determined by $\widetilde{S}$ (maybe path dependent):

$$f = f((\widetilde{S}_t)_{0 \le t \le T}).$$

To compute $D(x)$ we make (as an example) the following choices:

1. $\widehat{\mathbb{Q}}$ is a martingale measure for $\widetilde{S}$.
2. $R(x)/R_0(x) = 1$

Then

$$D_{ij}(x) = -\frac{a(x)}{x}(1 - \rho^2)\mathrm{Cov}_{\widehat{\mathbb{Q}}}(f_i, f_j).$$

Outline of the proof

The proof is based on the second order expansion of the value function

$$u(x, q) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T + \langle q, f \rangle)],$$

in the neighborhood of $(x, 0)$:

$$u(x + \Delta x, q) = u(x) + u'(x)\Delta x + \langle u_q(x, 0), q\rangle \\ + \frac{1}{2} \begin{pmatrix} \Delta x & q \end{pmatrix} G(x) \begin{pmatrix} \Delta x \\ q \end{pmatrix} + o((\Delta x)^2 + \|q\|^2).$$

In this approximation $G(x)$ is the matrix of the second order derivatives of $u(x, q)$ at $(x, 0)$ in the sense that

$$\lim_{|\Delta x| + \|q\| \to 0} \sup_{z \in \partial u(x + \Delta x, q)} \frac{\left\| z - \begin{pmatrix} u'(x) \\ u_q(x, 0) \end{pmatrix} - G(x) \begin{pmatrix} \Delta x \\ q \end{pmatrix} \right\|}{|\Delta x| + \|q\|} = 0.$$

Lower bound

It is relatively straightforward to “guess” the expression for $G(x)$ and to prove the **lower bound**:

$$u(x+\Delta x, q) \geq u(x) + u'(x)\Delta x + \langle u_q(x,0), q\rangle + \frac{1}{2}\begin{pmatrix}\Delta x & q\end{pmatrix} G(x) \begin{pmatrix}\Delta x \\ q\end{pmatrix} + o((\Delta x)^2 + \|q\|^2).$$

Remark

The lower bound is “easy” to verify because $u(x,q)$ is the value function of a **maximization** problem.

Key idea of the proof

Instead of proving upper bound for the second order expansion of $u(x,q)$ directly, we prove upper bound for the second order expansion of the **conjugate** function

$$v(y,r) = \max_{(x,q)}(u(x,q) - xy - \langle q, r\rangle).$$

More precisely, we show that at the conjugate point $(y,r) = (u'(x), u_q(x,0))$

$$v(y+\Delta y, r+\Delta r) \leq v(y) + v'(y)\Delta y + \frac{1}{2}\begin{pmatrix}\Delta y & \Delta r\end{pmatrix} H(y) \begin{pmatrix}\Delta y \\ \Delta r\end{pmatrix} + o((\Delta y)^2 + \|\Delta r\|^2),$$

where $H(y)$ is the **inverse** matrix to $-G(x)$. Jointly with dual relationship between $u(x,q)$ and $v(y,r)$ this inequality implies the upper bound for $u(x,q)$.

Dual problem

The proof of the upper bound for the second order expansion of $v(y, r)$ relies on the fact (established in joint paper with Julien Hugonnier) that $v(y, r)$ is the value function of the **minimization** problem:

$$v(y, r) = \min_{\mathcal{Q}(y,r)} \mathbb{E}[V(\frac{d\mathbb{Q}}{d\mathbb{P}})].$$

Here $\mathcal{Q}(y, r)$ is the subset of the family $\mathcal{Q}$ of the martingale measures $\mathbb{Q}$ for S such that

$$\mathbb{E}_{\mathbb{Q}}[X_T + \langle q, f \rangle] \leq xy + qr$$

for all $X \in \mathcal{X}(x, q)$.

References

Julien Hugonnier, Dmitry Kramkov, and Walter Schachermayer.
On utility-based pricing of contingent claims in incomplete markets.
Math. Finance, 15(2):203–212, 2005.

Dmitry Kramkov and Mihai Sîrbu.
Sensitivity analysis of utility-based prices and risk-tolerance wealth processes.
Ann. Appl. Probab., 16(4):2140–2194, 2006.

Part IV

Mean-Variance and Utility Based Hedging

Outline

Mean-Based Hedging

Utility Based Hedging

Risk-Tolerance Wealth Process

References

Hedging in Incomplete Markets

Basic idea of hedging: mitigate risk by offsetting the payoffs of non-traded derivatives by a (dynamic) portfolio of traded assets.

Complete markets: perfect *replication*.

Incomplete markets:

- Risk-elimination or *super-hedging* (El Karoui & Quenez, Cvitanic & Karatzas , K., Föllmer & K.. . .)
- Risk-minimization. Hedging is an *approximation*.
  - Mean-variance hedging (Föllmer & Sondermann, Föllmer & Schweizer, . . .)
  - Quantile hedging (Kulldorff, Föllmer & Leukert, . . .)
  - Coherent risk measures (. . .)
- Trade-off between risk and return. Hedging is *a part of investment strategy*.
  - Utility-based hedging (Hodges & Neuberger, Davis, Kallsen, Hobson, Henderson, Monoyois, . . .)

Black and Scholes model

There are two assets:

1. Bank account with interest rate $r = 0$.
2. Stock:

$$dS_t = S_t(\cdot dt + \sigma dW_t)$$

Wealth evolution of a strategy is a stochastic integral:

$$X_t = x + \int_0^t \Delta_u dS_u,$$

where Δ_t is the *number* of stocks at time t.
Consider a European contingent claim with maturity T and payoff

$$\Psi = g((S_u)_{0 \leq u \leq T})$$

Hedging in Black and Scholes model

1. Evaluate the price of the option:

$$P_t = \mathbb{E}_t^{\mathbb{Q}}[\Psi]$$

   where $\mathbb{Q}$ is the (unique!) martingale measure for S.

2. Determine the number of stocks Δ_t at time t:

$$\Delta_t = \frac{dP_t}{dS_t} = \frac{d\mathbb{E}_t^{\mathbb{Q}}[\Psi]}{dS_t}$$

   (Use Ito's or Malliavin's calculus.)

3. Put the amount $P_t - \Delta_t S_t$ in the bank account.

This strategy is *self-financing*:

$$P_{t+dt} = P_t + \Delta_t dS_t \quad (!)$$

Model with basis risk

"Black and Scholes model" $+$ *"non-traded asset Y"*:

$$dY_t = Y_t \left(\cdot dt + \eta dB_t \right)$$

Consider a European contingent claim with maturity T and payoff

$$\Psi = g((Y_u)_{0 \leq u \leq T})$$

We want to price and hedge this contingent claim on Y using traded asset S.

Remark

In practice, one wants to choose S as closely correlated with Y as possible.

"Practical" hedging

1. Start with the "price" at t

$$P_t = \mathbb{E}_t^{\mathbb{Q}}[\Psi]$$

   where $\mathbb{Q}$ is *some* martingale measure for S. (Usually, we take $\mathbb{Q}$ to be the unique martingale measure for (S, Y).)

2. Evaluate the number of stocks Δ_t as the minimizer

$$\mathbb{E}_t^{\mathbb{Q}}[(dP_t - \Delta_t dS_t)^2] \overset{\Delta_t}{\to} \min$$

   (Kunita-Watanabe decomposition).

3. Put the amount $P_t - \Delta_t S_t$ in the bank account.

This is the *mean-variance hedging* strategy introduced by Föllmer and Sondermann. However, this strategy is not *self-financing*:

$$P_{t+dt} \neq P_t + \Delta_t dS_t \quad (!)$$

Investment of mismatch wealth

Question

Where does the hedging mismatch wealth

$$dP_t - \Delta_t dS_t$$

go?

Answer

This capital is *invested* according to the preferences of an economic agent.

For example, if the mismatch wealth is small, then it is put in an artificial asset X' showing the optimal investment of extra penny.

$\Rightarrow$ Should measure hedging error in terms of X' (not in terms of savings account).

Updated "practical" hedging

1. Start with the price
$$P_t = \mathbb{E}_t^{\mathbb{Q}}[\Psi]$$
where $\mathbb{Q}$ is *some* martingale measure for S.
2. Decide what to do with extra penny: X'. Choose X' as a numeraire: $(\mathbb{Q} \to \mathbb{R})$
$$\frac{d\mathbb{R}}{d\mathbb{Q}} = X'.$$
3. Compute Δ_t and β_t as the minimizers of
$$\mathbb{E}_t^{\mathbb{R}}[(d\frac{P_t}{X'_t} - \Delta_t d\frac{S_t}{X'_t} - \beta_t d\frac{1}{X'_t})^2] \stackrel{(\Delta_t,\beta_t)}{\longrightarrow} \min .$$
4. Put the mismatch amount $dP_t - \Delta_t dS_t$ in X'.

Independence on the choice of X'

It is interesting to note that Δ_t and β_t computed as minimizers of

$$\mathbb{E}_t^{\mathbb{R}}[(d\frac{P_t}{X'_t} - \Delta_t d\frac{S_t}{X'_t} - \beta_t d\frac{1}{X'_t})^2] \overset{(\Delta_t,\beta_t)}{\rightarrow} \min .$$

do not depend on X'!
In particular, we can use the original formulas of Föllmer and Sondermann:

$$\mathbb{E}_t^{\mathbb{Q}}[(dP_t - \Delta_t dS_t)^2] \overset{\Delta_t}{\rightarrow} \min$$
$$\beta_t = P_t - \Delta_t S_t$$

Remark

This independence of hedge on the choice of X' holds true for continuous S but fails for discontinuous.

Hedging and investment

Consider an economic agent, whose portfolio consists of two parts:

1. Liquid or traded securities.
2. Non-traded derivatives.

- **Investment problem:** trade in liquid securities to achieve an optimal (subjectively for investor) trade-off between risk and return.
- **Main difficulty:** presence of non-traded securities.
- **Complete markets:** can split the investment problem into 2 *completely independent* parts:
  1. Hedging (replication) of derivatives.
  2. Investment without derivatives ("pure" investment).
- **Incomplete markets:** hedging and "pure" investment can not be completely separated.

Definition of hedging through investment

To specify hedging strategy we need to solve *one valuation and two (!) investment problems*:

1. Investment of the liquid part of the portfolio with derivatives
2. Cash valuation of the portfolio with derivatives: computation of *CEV* (*certainty equivalent value*)
3. "Pure" investment of *CEV*.

Then, formally,

$$\begin{aligned} \text{Hedging} &= \text{"Pure" investment starting with } CEV \\ &\quad - \text{Investment of the liquid part of the portfolio} \end{aligned}$$

- *Does not look nice!*
- Look at the asymptotic case of *small hedging error*.

Model of a financial market

There are $d+1$ *traded* or *liquid* assets:

1. a *savings account* with zero interest rate.
2. d *stocks*. The price process S of the stocks is a semimartingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$.

$\mathcal{Q}$: the family of equivalent local martingale measures for S.

Assumption (No Arbitrage)

$$\mathcal{Q} \neq \emptyset$$

Consider a family of m *non-traded* or *illiquid* European contingent claims with

1. maturity T
2. payment functions $\Psi = (\Psi_i)_{1 \leq i \leq m}$.

Investment with random endowment

Consider an investor with a portfolio (x, q), where

- x: *liquid* capital
- $q = (q_i)$: quantities of the *illiquid* contingent claims.

His preferences are modeled by a *utility function* U:

1. $U : (0, \infty) \to \mathbf{R}$, strictly increasing and strictly concave
2. The Inada conditions hold true:

$$U'(0) = \infty \quad U'(\infty) = 0$$

Goal: maximize *the expected utility of terminal wealth*

$$u(x, q) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T + \langle q, \Psi \rangle)],$$

where $\mathcal{X}(x)$ is the set of strategies with initial wealth x.

Utility based hedging strategy

- **Certainty equivalence value** $c(x, q)$ of the portfolio (x, q):

$$u(c(x, q), 0) = u(x, q)$$

  (Investor is indifferent between the choice of current portfolio (x, q) and "pure" portfolio (without derivatives) with wealth $c(x, q)$.
- $X(x, q)$: wealth process (for liquid part) of the optimal investment strategy for initial portfolio (x, q))
- **Hedging strategy** $Z(x, q)$ is formally defined as

$$Z(x, q) = X(c(x, q), 0) - X(x, q),$$

  (Hedging is the difference between the "pure" investment starting from $c(x, q)$ and the liquid part of investment with derivatives).

Marginal hedging strategy

- The problem of optimal investment

$$u(x, q) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T + \langle q, \Psi \rangle)],$$

  can almost never be solved explicitly if $q \neq 0$.
- Computational challenge: $Z(x, q)$ is not linear w.r.t. q.
- *Linear approximation for* $Z(x, q)$ *with respect to* q (for small quantities of derivatives):

$$Z(x, q) = \langle L(x), q \rangle + o(\|q\|), \quad L(x) = \frac{\partial Z}{\partial q}(x, 0).$$

  We call $L(x)$ the **marginal hedging strategy**. The computation of $L(x)$ should be feasible due to linearity: (Davis, Hobson, Hendersen, Kallsen, Monoyois, ...).

Marginal utility based price

Linear approximation for certainty equivalence value:

$$c(x, q) = x + \langle p(x), q \rangle + o(\|q\|),$$

where $p(x)$ is the *marginal utility based price* (Davis price) given by

$$p(x) = \mathbb{E}^{\mathbb{Q}(y)}[\Psi].$$

Here $\mathbb{Q}(y)$ (*"the dual minimizer"*) is the solution of

$$v(y) = \inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}\left[V\left(y(\frac{d\mathbb{Q}}{d\mathbb{P}})\right)\right],$$

$V(y)$ is the conjugate function to $U(x)$:

$$V(y) = \max_{x>0}\left[U(x) - xy\right], \quad y > 0,$$

and y corresponds to x in the sense that $x = -v'(y)$.

Optimal investment of extra penny

Hereafter, we denote by $u(x)$ and $X(x)$ the value function and the optimal investment strategy for the "pure" investment case:

$$u(x) = u(x,0) = \max_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)], \quad x > 0.$$

- The derivative wealth process $X'(x)$ is defined as a (maximal) wealth process such that

$$X'_T(x) = \frac{d}{dx} X_T(x) = \lim_{\Delta x \to 0} \frac{X_T(x + \Delta x) - X_T(x)}{\Delta x}.$$

  This process shows what the investor does with extra penny added to his portfolio.

Mean-variance approximation problems

Choose $X(x)/x$ as a *numéraire* and denote

- $\Psi^{X(x)} = \Psi \frac{x}{X_T(x)}$: discounted contingent claims
- $\mathcal{M}^{X(x)}$: the set of discounted wealth processes starting from 0
- $\mathbb{Q}^{X(x)}$: the martingale measure for $\mathcal{M}^{X(x)}$, that is

$$\frac{d\mathbb{Q}^{X(x)}}{d\mathbb{Q}(y)} = \frac{X_T(x)}{x}, \quad y = u'(x)$$

Denote also by $A(x)$ the *relative risk aversion coefficient* of U:

$$A(x) = -\frac{xU''(x)}{U'(x)}, \quad x > 0,$$

Finally, denote by $M(x) = (M_i(x))$ the solutions of

$$a_i(x) = \min_{M \in \mathcal{M}^{X(x)}} \mathbb{E}^{\mathbb{Q}^{X(x)}}[A(X_T(x))(\Psi_i^{X(x)} - M_T)^2].$$

Technical assumptions

Assumption

The financial model can be **completed** by an addition of a finite number of securities.

Assumption

There are strictly positive constants c_1 and c_2 such that

$$c_1 < A(x) = -\frac{xU''(x)}{U'(x)} < c_2, \quad x > 0.$$

Assumption

There is a wealth process $X \geq 0$ such that

1. $\|\Psi\| \leq X_T$
2. $X^{X(x)} = X\frac{x}{X(x)}$ is a square integrable martingale under $\mathbb{Q}^{X(x)}$

Main "quantitative" result

Inputs:

- $p(x) = (p_i(x))$: marginal utility based prices (Davis prices)
- $X(x)$: the optimal investment strategy without options
- $X'(x)$: optimal investment of extra penny
- $M(x) = (M_i(x))$: solutions to the auxiliary mean-variance approximation problems

Theorem

Under the technical assumptions above the marginal hedging strategy $L(x)$ is well-defined and is given by

$$L(x) = p(x)X'(x) + X(x)M(x).$$

Main “qualitative” result

Inputs:

- $\mathbb{Q}(y)$: the dual minimizer (the martingale measure used to compute marginal prices).

Theorem

Assume that S is continuous and other technical conditions hold true. Then the following statements are equivalent:

1. *For any family of contingent claims Ψ (satisfying the technical assumptions) the marginal utility based strategy is, in fact, the Föllmer-Sondermann mean-variance hedging strategy (given $\mathbb{Q}(y)$).*
2. *There exists a risk-tolerance wealth process $R(x)$.*

Risk-tolerance wealth process

Definition

A maximal wealth process $R(x)$ is called the **risk-tolerance wealth process** if

$$R_T(x) = -\frac{U'(X_T(x))}{U''(X_T(x))},$$

where $X(x)$ is the solution of

$$u(x) := u(x,0) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)].$$

Remark

If $R(x)$ exists then

$$\frac{R(x)}{R_0(x)} = X'(x) := \lim_{\Delta x \to 0} \frac{X(x+\Delta x) - X(x)}{\Delta x}.$$

Existence of $R(x)$

Recall that $\mathbb{Q}(y)$ is the minimal martingale measure, that is, the solution of the following dual problem:

$$v(y) = \inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}\left[V\left(y(\frac{d\mathbb{Q}}{d\mathbb{P}}) \right) \right],$$

where V is the convex conjugate of U.

Theorem

The following assertions are equivalent:

1. $R(x)$ *exists.*
2. $\frac{d}{dy}\mathbb{Q}(y) = 0$ *at* $y = u'(x)$.

In particular, $R(x)$ *exists for any* $x > 0$ *if and only if* $\mathbb{Q}(y)$ *is the same for all* y.

Existence of $R(x)$

Theorem

The following assertions are equivalent:

1. $R(x)$ *exists for any* $x > 0$ *and any utility function* U.
2. *There exists a unique* $\widehat{\mathbb{Q}} \in \mathcal{Q}$ *such that*

$$\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} \succ_2 \frac{d\mathbb{Q}}{d\mathbb{P}} \quad \forall \mathbb{Q} \in \mathcal{Q},$$

where $\succ_2$ *is the second order stochastic dominance relation.*

Theorem

The following assertions are equivalent:

1. $R(x)$ *exists for any* $x > 0$ *and any financial model.*
2. *The utility function* U *on* $(0, \infty)$ *is a power utility:* $U(x) = (x^\alpha - 1)/\alpha$, $\alpha < 1$.

References

Hans Föllmer and Dieter Sondermann.
Hedging of non-redundant contingent claims.
Contributions to mathematical economics, North-Holland, 205–223, 1986.

Dmitry Kramkov and Mihai Sîrbu.
Asymptotic analysis of utility-based hedging strategies for small number of contingent claims.
Stochastic Process. Appl., 117(11):1606–1620, 2007.