Essential Uses of Probability in Analysis
Part V
Parabolic Boundary Harnack Principle

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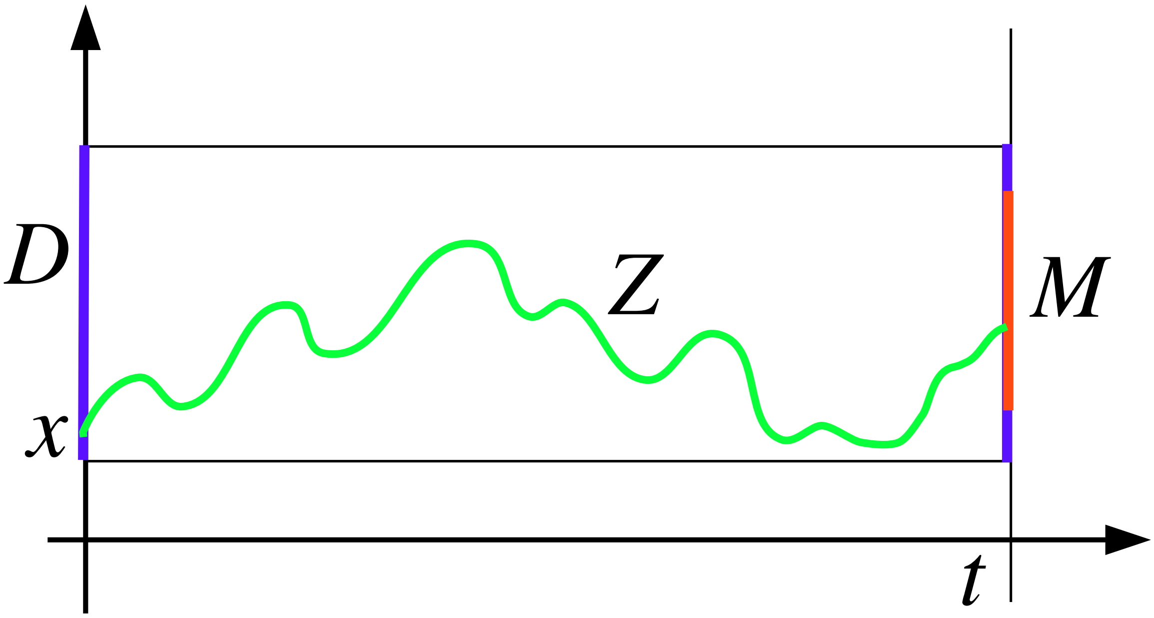
“Technical Lemma”

The subject of this lecture is a “technical lemma” that was proved and used in Bass and B (1992). Variants of the lemma were later used in Adelman, B and Pemantle (1998), B and Kendall (2000) and Atar and B (2004).

Intuition



D — open set,

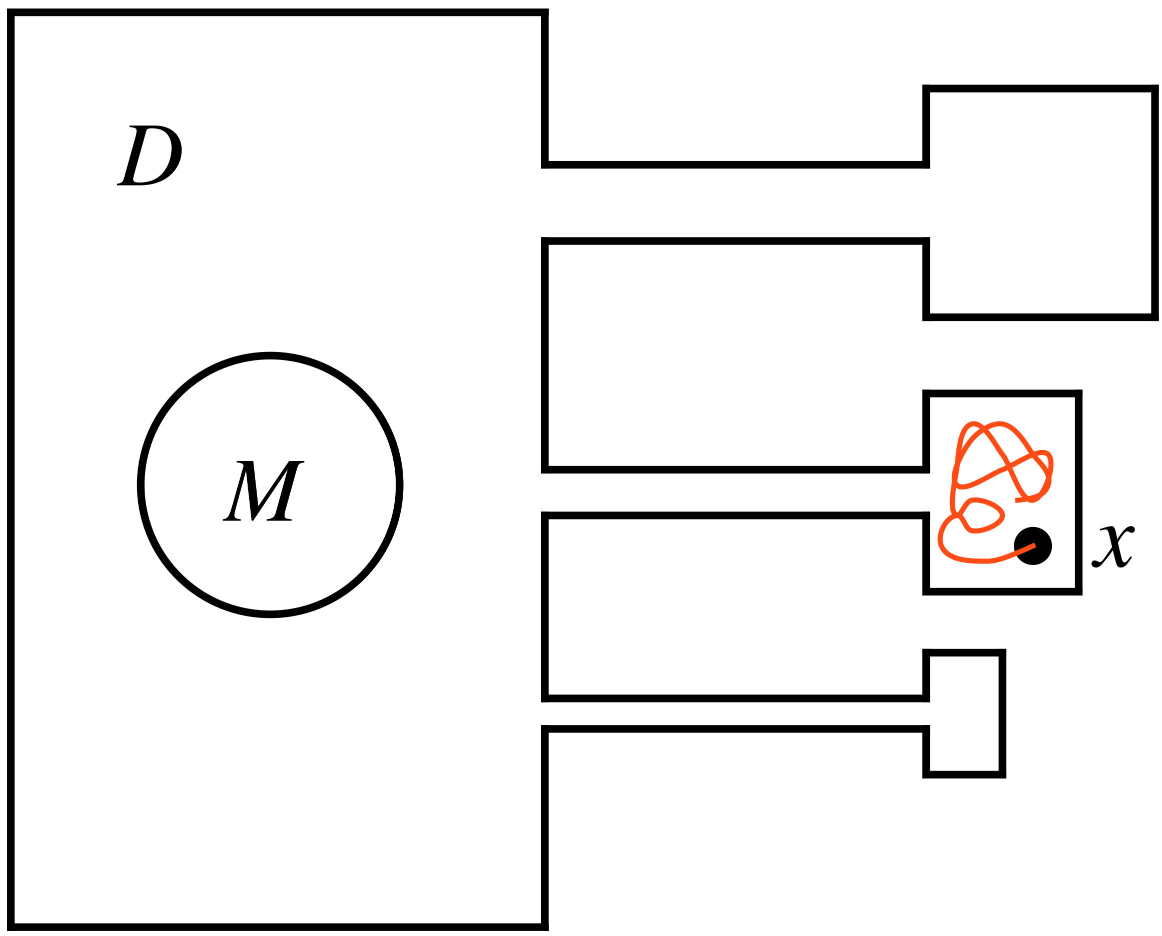
$M \subset D$ — compact set,

$B(0, r) \subset M, r > 0.$

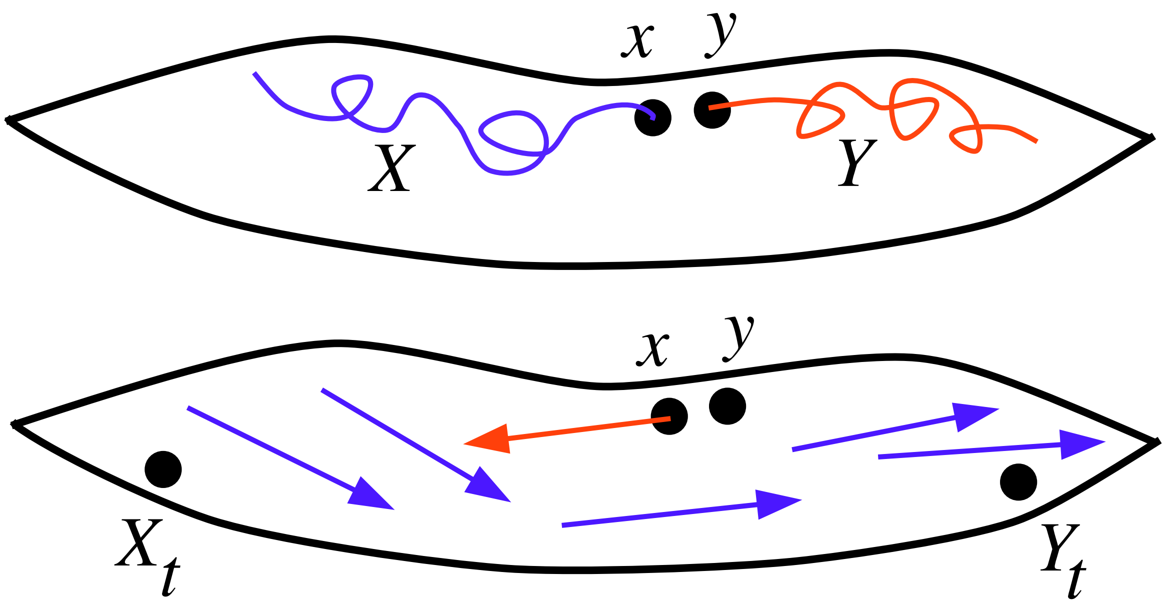
We expect that $\exists c, s > 0 \; \forall x \in D \; \forall t > s,$

$$P^x(Z_t \in M \mid T_{D^c} > t) > c.$$

Counterexample



Application — Atar and B (2004)



$Z_t = (X_t, Y_t)$ — mirror coupling,

$(X_0, Y_0) = (x, y)$,

$T = \inf\{t > 0 : X_t = Y_t\}$.

Lemma: $\exists c, p, s > 0\ \forall t > s\ \forall x, y \in D$

$$P(|X_t - Y_t| > c \mid T > t) > p.$$

$$\varphi(x) - \varphi(y) = e^{\mu_2 t} E^{(x,y)}[\varphi(X_t) - \varphi(Y_t)]$$

Main Theorem — Preliminary Statement

Z — continuous strong Markov process,

D — state space,

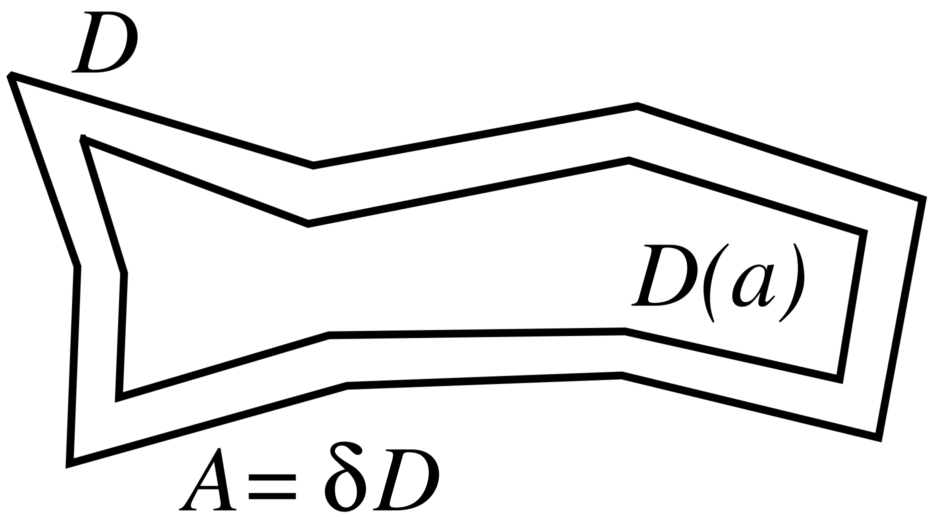
$A \subset D$, $T_A = \inf\{t \geq 0 : Z_t \in A\}$.

We want to prove that under suitable assumptions $\exists c, s, p > 0 \ \forall t > s \ \forall x \in D \setminus A$

$$\boxed{P^x(\mathrm{dist}(Z_t, A) > c \mid T_A > t) > p.}$$

Assumptions

$$D(a) = \{x \in D : \mathrm{dist}(x, A) \geq a\}$$



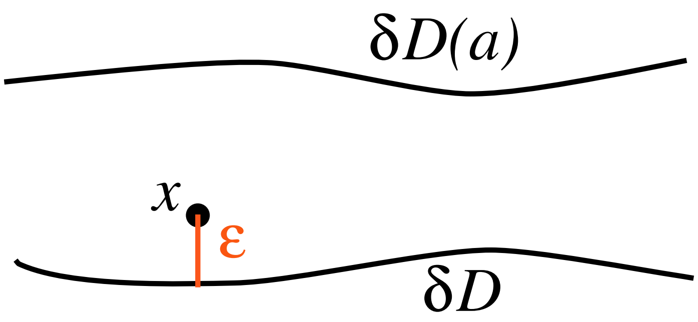
Assumption I:

$\exists c, a_0, \alpha \; \forall a \in (0, a_0) \; \forall \varepsilon \in (0, a) \; \forall x \in D(\varepsilon)$

$$\boxed{P^x(T_{D(a)} < T_A) \geq c\varepsilon^\alpha.}$$

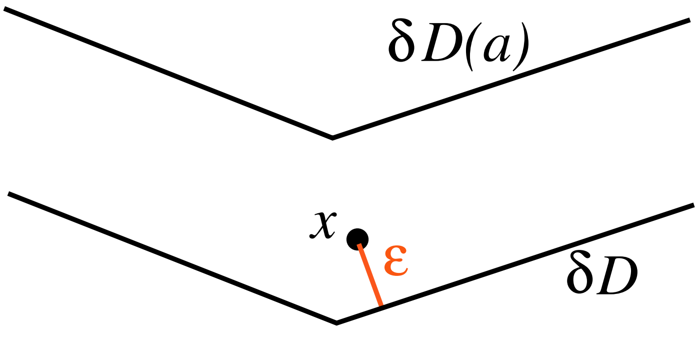
Assumption I: Examples

∂D — smooth



$$P^x(T_{D(a)} < T_A) \approx \varepsilon$$

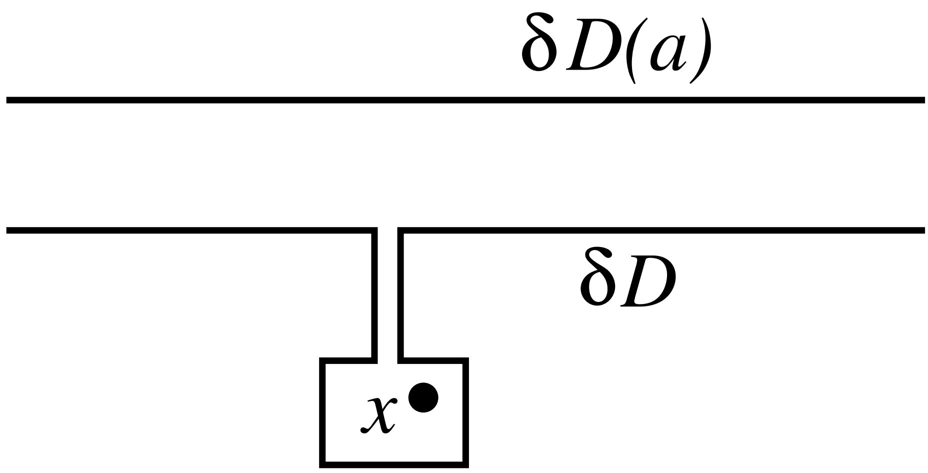
∂D — Lipschitz



$$P^x(T_{D(a)} < T_A) \approx \varepsilon^{\alpha}$$

Assumption I: Examples

Assumption I rules out this picture:



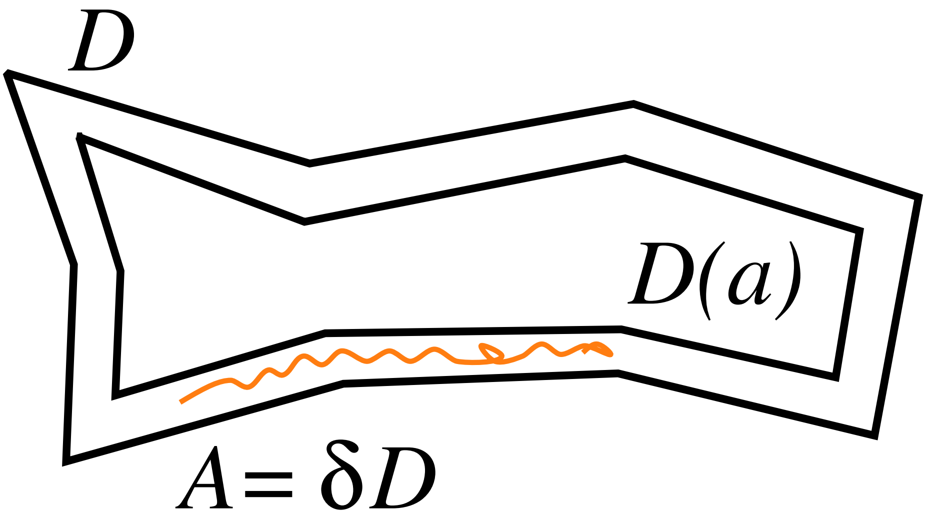
Assumptions (ctnd)

Assumption II:

$\exists \beta > 0 \; \exists c < \infty \; \forall \varepsilon > 0 \; \forall x \in D^c(\varepsilon)$

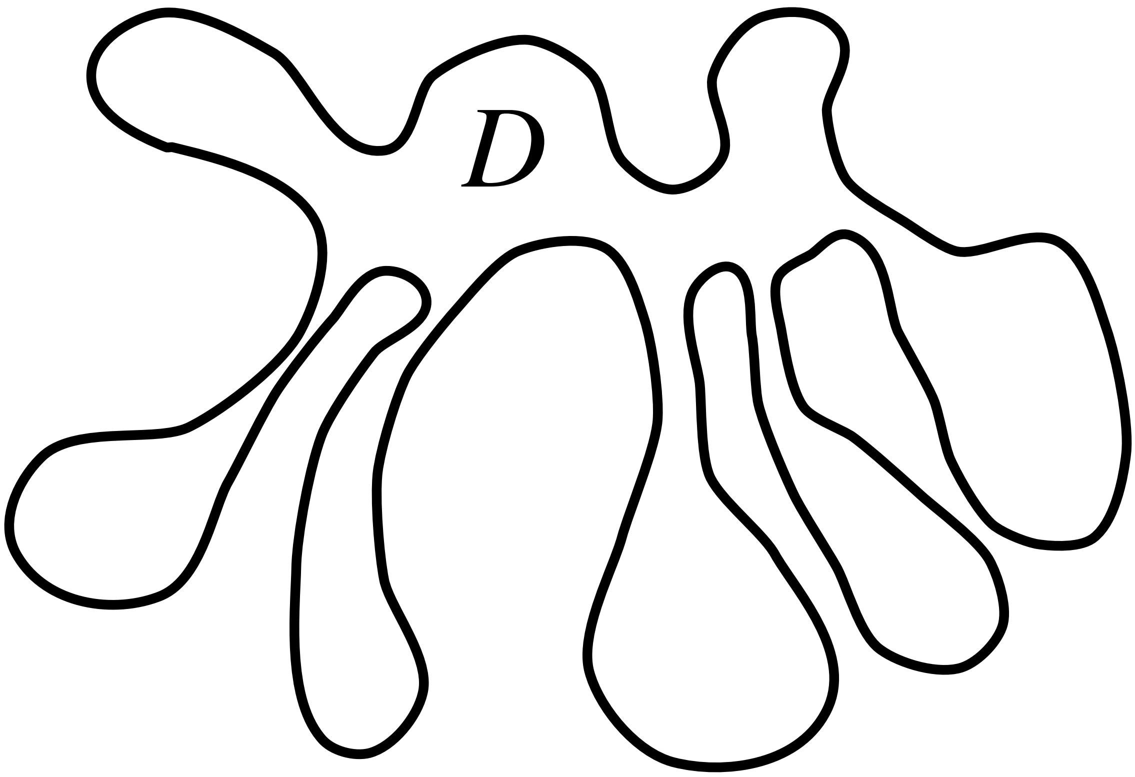
$$E^x T_{A \cup D(\varepsilon)} \le c\varepsilon^\beta .$$

This is ruled out:



Remark

If ∂D is fractal and $A = \partial D$ then replace $D(a) = \{x \in D : \text{dist}(x, A) \geq a\}$ with $\widetilde{D}(a) = \{x \in D : G(x_0, x) \geq a\}$.



Main Theorem

Z — continuous strong Markov process,

D — state space,

$A \subset D$, $T_A = \inf\{t \geq 0 : Z_t \in A\}$.

"Main" Theorem: If assumptions I and II hold then $\exists c, s, p > 0\ \forall t > s\ \forall x \in D \setminus A$

$$\boxed{P^x(\operatorname{dist}(Z_t, A) > c \mid T_A > t) > p.}$$

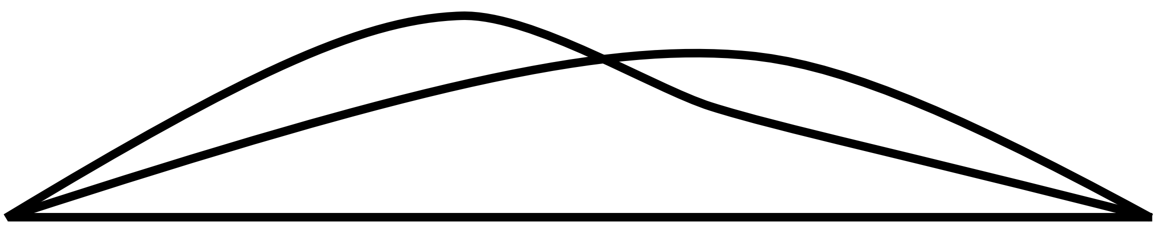
Parabolic Boundary Harnack Principle

$D \subset \mathbf{R}^n$,

$p_t(x, y)$ — transition density for Brownian motion killed on ∂D.

PBHP: $\exists s, c > 0 \ \forall t > s \ \forall x, y, z, v \in D$

$$\frac{p_t(x, y)}{p_t(x, z)} \geq c \, \frac{p_t(v, y)}{p_t(v, z)}.$$



Metatheorem:

"Main Theorem" $\Leftrightarrow$ PBHP.

Parabolic Boundary Harnack Principle

Theorem (Bass and B (1992)) PBHP holds in $D \subset \mathbf{R}^n$ if D belongs to one of the following classes:
(i) uniformly regular twisted L^p domains with $p > n-1$, or
(ii) twisted Hölder domains with $\alpha \in (1/3, 1]$.
There are counterexamples for $p < n-1$ and $\alpha < 1/3$.

PBHP: $\exists s, c > 0 \ \forall t > s \ \forall x, y, z, v \in D$

$$\frac{p_t(x,y)}{p_t(x,z)} \geq c \, \frac{p_t(v,y)}{p_t(v,z)}.$$

Ideas in the Proof of the "Main Theorem" Harmonic case

Chung (1984), Cranston (1985), Cranston and McConnell (1983)

a — fixed,

$h(z) = P^z(T_{D(a)} < T_A)$,

$U_k = \{z : h(z) \in [2^{k-1}, 2^k]\}$.

Assumption I $\Rightarrow$

$$U_k \subset D^c(c_1 2^{k/\alpha}). \qquad (*)$$

(*) and Assumption II $\Rightarrow$

$$\forall z \in U_k \quad E^z T_{U_k^c} \leq c_2 2^{\beta k/\alpha}$$

$$\Downarrow$$

$$\sum_{-\infty}^{0} \sup_{z \in U_k} E^z T_{U_k^c} < \infty.$$

Chung's Argument

P_h^z — distribution of Z conditioned by $\{T_{D(a)} < T_A\}$.

Chung (1984): The number of crossings of $[2^{k-1}, 2^k]$ made by $h(Z_t)$ under P_h^z is stochastically majorized by a geometric random variable with mean independent of k so $E_h^z T_{D(a)}$ is majorized by the sum of expected times to escape from U_k.

$$\sum_{-\infty}^{0} \sup_{z \in U_k} E^z T_{U_k^c} < \infty \Rightarrow \sup_z E_h^z T_{D(a)} < \infty$$

$$\Rightarrow \exists t_1, p_1 > 0 \quad \inf_z P_h^z(T_{D(a)} < t_1) > p_1.$$

Ideas in the Proof of the "Main Theorem"
Parabolic case

The argument discussed above was concerned with the process Z conditioned by the event $\{T_{D(a)} < T_A\}$ ("harmonic conditioning"). The "main theorem" is concerned with the process conditioned by the event $\{T_A > t\}$ ("parabolic conditioning"). The rest of the proof is based on a complicated and technical argument translating the "harmonic" estimate to the "parabolic" estimate.

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