Quasiconvexity at the boundary and weak lower semicontinuity of integral functionals

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It is well-known that Morrey's quasiconvexity is closely related to gradient Young measures, i.e., Young measures generated by sequences of gradients in $L^p(\Omega; I\!R^{m\times n})$. Concentration effects, however, cannot be treated by Young measures. One way how to describe both oscillation and concentration effects in a fair generality are the so-called DiPerna-Majda measures.

DiPerna and Majda showed that having a sequence $\{y_k\}$ bounded in $L^p(\Omega; I\!R^{m\times n})$, $1 \le p < +\infty$, and a complete separable subring $\mathcal{R}$ of continuous bounded functions on $I\!R^{m\times n}$ then there exists a subsequence of $\{y_k\}$ (not relabeled), a positive Radon measure σ on $\bar\Omega$, and a family of probability measures on $\beta_{\mathcal{R}} I\!R^{m\times n}$ (the metrizable compactification of $I\!R^{m\times n}$ corresponding to $\mathcal{R}$), $\{\hat\nu_x\}_{x\in\bar\Omega}$, such that for all $g \in C(\bar\Omega)$ and all $v_0 \in \mathcal{R}$

$$\lim_{k\to\infty} \int_\Omega g(x) v(y_k(x)) dx \ = \int_{\bar\Omega} \int_{\beta_{\mathcal{R}} I\!R^{m\times n}} g(x) v_0(s) \hat\nu_x(ds) \sigma(dx) \ ,$$

where $v(s) = v_0(s)(1 + |s|^p)$. Our talk will address the question: *What conditions must $(\sigma, \hat\nu)$ satisfy, so that $y_k = \nabla u_k$ for $\{u_k\} \subset W^{1,p}(\Omega; I\!R^m)$* We are going to state necessary and sufficient conditions. The notion of *quasiconvexity at the boundary* due to Ball and Marsden plays a crucial role in this characterization.

Based on this result, we then find sufficient and necessary conditions ensuring sequential weak lower semicontinuity of $I : W^{1,p}(\Omega; I\!R^m) \to I\!R$,

$$I(u) = \int_\Omega v(\nabla u(x))\, \mathrm{d}x \ ,$$

where $v : I\!R^{m\times n} \to I\!R$ satisfies $|v| \le C(1 + |\cdot|^p)$, $C > 0$.