Well-posedness of 2-phase Stefan equation with surface tension in $\mathbb{R}^3$, under radial symmetry

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2-phase Stefan equation w Gibbs-Thomson law (e.g. Visintin 1998)

Given an initial set of ice $D(0) \subset \mathbb{R}^d$ and initial temperature distribution $u(0,\cdot)\colon \mathbb{R}^d \to \mathbb{R}$, find $\{D(t)\}_{t>0}$ and $\{u(t,\cdot)\colon \mathbb{R}^d \to \mathbb{R}\}_{t>0}$ such that

$$\partial_t u(t,z) = \frac{1}{2}\Delta_z u(t,z), \quad z \in \mathbb{R}^d \backslash \partial D(t), \quad t > 0,$$
$$u(t,z) = \gamma H_t(z), \quad z \in \partial D(t),$$
$$-V_t(z) = \frac{1}{2}\nabla_z u(t,z)\cdot n_t^+(z) + \frac{1}{2}\nabla_z u(t,z)\cdot n_t^-(z), \; z \in \partial D(t),$$

where H_t is mean curvature of interface $\partial D(t)$, V_t is the "velocity" of interface, n_t^+ and n_t^- are the unit normals along $\partial D(t)$ pointing outside and inside of $D(t)$.

- **Applications** of Stefan-type equations: melting/solidification, crystal growth, aging of alloys, interaction of nonmixing fluids, dynamics of neurons' membrane potentials, tumor growth, cholesterol plug growth, etc.
- **Well-posedness**: *Luckhaus (1990)* shows existence of a weak (variational) solution, but shows **non-uniqueness**. *N.-Shkolnikov (2021)* show existence of a (stronger) probabilistic solution under radial symmetry. Local results exist.

2-phase Stefan equation w Gibbs-Thomson law, under radial symmetry in $\mathbb{R}^3$

Given an initial ball (centered at 0) of ice $B_{\Lambda_0} \subset \mathbb{R}^3$ and (radially symmetric) initial temperature distribution $u(0,\cdot)\colon \mathbb{R}^3 \to \mathbb{R}$, find $\{\Lambda_t\}_{t>0}$ and $\{u(t,\cdot)\colon \mathbb{R}^3 \to \mathbb{R}\}_{t>0}$ such that

$$\begin{aligned}
&\partial_t u(t,z) = \frac{1}{2}\Delta_z u(t,z), \quad z \in \mathbb{R}^3 \backslash \partial B_{\Lambda_t}, \;\; t > 0,\\
&u(t,z) = -\gamma/\Lambda_t, \quad z \in \partial B_{\Lambda_t},\\
&-\dot{\Lambda}_t = \frac{1}{2}\nabla_z u(t,z)\cdot n_t^+(z) + \frac{1}{2}\nabla_z u(t,z)\cdot n_t^-(z),\;\; z \in \partial B_{\Lambda_t},
\end{aligned}$$

where n_t^+ and n_t^- are the unit normals along ∂B_{Λ_t} pointing outside and inside of B_{Λ_t}.

- **Goal**: establish existence and uniqueness (of an appropriate notion) of solution to the above.

Transformation of the equation

- Using the fact that $\gamma/|z|$ solves heat equation, we consider $v(t,z) := u(t,z) + \gamma/|z|$ and solve

$$\partial_t v(t,z) = \frac{1}{2}\Delta_z v(t,z), \quad z \in \mathbb{R}^3 \backslash \partial B_{\Lambda_t}(0), \;\; t > 0, \tag{1}$$

$$v(t,z) = 0, \quad z \in \partial B_{\Lambda_t}(0), \tag{2}$$

$$-\dot{\Lambda}_t = \frac{1}{2}\nabla_z v(t,z) \cdot n_t^+(z) + \frac{1}{2}\nabla_z v(t,z) \cdot n_t^-(z), \; z \in \partial B_{\Lambda_t}(0). \tag{3}$$

- Integrating (3) and using (1)–(2), we obtain

$$\frac{1}{3}\left(\Lambda_0^3 - \Lambda_t^3\right) = \alpha \int_{\mathbb{R}^3} v_0(z)\, dz - \alpha \int_{\mathbb{R}^3} v(t,z)\, dz,$$

  where α^{-1} is the area of S^2.

- **Standard solution approach**. Show a contraction-type property (after normalization) of the mapping: from the cubed **bdry** Λ^3 to the **temperature energy** $3\alpha \int_{\mathbb{R}^3} v(t,z)\, dz$, where v solves (1)–(2).

Challenges

$$\partial_t v(t,z) = \frac{1}{2}\Delta_z v(t,z), \quad z \in \mathbb{R}^3 \backslash \partial B_{\Lambda_t}(0), \;\; t > 0,$$
$$v(t,z) = 0, \quad z \in \partial B_{\Lambda_t}(0), \tag{4}$$
$$\frac{1}{3}\left(\Lambda_0^3 - \Lambda_t^3\right) = \alpha \int_{\mathbb{R}^3} v_0(z)\,dz - \alpha \int_{\mathbb{R}^3} v(t,z)\,dz.$$

- If v_0 is sufficiently large, $t \mapsto \Lambda_t$ cannot be smooth (and may **jump**).
- Then, the bdry condition (4) **cannot be satisfied at all** $t > 0$.
- If $\Lambda_s = \tilde{\Lambda}_s$ for $s \in [0,t)$, then, for small $\delta > 0$, one may have

$$3\alpha \sup_{s\in[t,t+\delta]} \left| \int_{\mathbb{R}^3} v(s,z)\,dz - \int_{\mathbb{R}^3} \tilde{v}(s,z)\,dz \right| \approx v(t^-,\cdot)\Big|_{\partial B_{\Lambda_{t-}}} \sup_{s\in[t,t+\delta]} \left|\Lambda_s^3 - \tilde{\Lambda}_s^3\right|$$

- If $v_0 < 1$, the comparison principle yields desired contraction.
- But, if not, the **contraction does not hold**.

Probabilistic solution

- Assume $v_0 \geq 0$ and $\int_{\mathbb{R}^3} v_0(z)\, dz < \infty$.
- **Itô's formula** yields: $v(t,\cdot)$ is the **density of BM** started from v_0 and **killed at hitting** $(\Lambda_s)_{s\in[0,t]}$.
- **Probabilistic growth condition**:

$$\frac{1}{3}(\Lambda_{0-}^3 - \Lambda_t^3) = \alpha \int_{\mathbb{R}^3} v_0(z)\, dz - \alpha \int_{\mathbb{R}^3} v(t,z)\, dz = \alpha \int_{\mathbb{R}^3} \mathbb{P}(\tau^z(\Lambda) \leq t)\, v_0(z)\, dz$$

$$\tau^z(\Lambda) := \inf\{t \geq 0 :\ (|z + W_t| - \Lambda_t)\,(|z| - \Lambda_{0-}) < 0\}.$$

- **Jump condition**. The potential jumps of Λ introduce ambiguity. We define the notion of a "physical" solution by assuming that the jumps are the **smallest possible**:

$$\Lambda_{t-} - \Lambda_t = \inf\left\{ y \in (0, \Lambda_{t-}] :\ \mathtt{Leb}\left(B_{\Lambda_{t-}} \setminus B_{\Lambda_{t-}-y}\right) > \int_{\Lambda_{t-}-y \leq |z| \leq \Lambda_{t-}} v(t-,z)\, dz \right\}$$

Existence and unqueness

(Λ, v) is a **probabilistic solution** if Λ is right-cont., $v(t,z)$ is a function of $|z|$, and:

$v(t,\cdot)$ – density of BM started from v_0 and killed at hitting $(\Lambda_s)_{s\in[0,t]}$,

$$\frac{1}{3}(\Lambda_{0-}^3 - \Lambda_t^3) = \alpha \int_{\mathbb{R}^3} \mathbb{P}(\tau^z(\Lambda) \le t)\, v_0(z)\, dz,$$

and the minimal jump condition is satisfied.

- *N.-Shkolnikov (2021)* shows existence of probabilistic solution (by Euler-type approximation).
- **Theorem** (*Guo-N.-Shkolnikov 2023*). Assume that $v_0 \ge 0$ is bounded and decays sufficiently fast at infinity. Assume that v_0 is **piecewise monotone**. Then, the probabilistic solution is **unique**.

Proof of uniqueness: preliminaries

- Argue by contradiction: assume there exist two non-increasing solutions Λ, $\tilde{\Lambda}$ and $t_0 \geq 0$ s.t. $\Lambda_s = \tilde{\Lambda}_s$ for $s < t_0$, and $\sup_{s\in[t_0,t_0+\varepsilon]} |\Lambda_s - \tilde{\Lambda}_s| > 0$ for all small enough $\varepsilon > 0$. W.l.o.g. $t_0 = 0$.
- **Desired contraction-type property**:

$$\|\Gamma(\Lambda^3) - \Gamma(\tilde{\Lambda}^3)\| \leq F\left(\|\Lambda^3 - \tilde{\Lambda}^3\|\right), \quad F(x) < x \text{ for } x > 0,$$

$$\Gamma_t(\Lambda^3) := \Lambda^3_{0-} - 3\alpha \int_{\mathbb{R}^3} \mathbb{P}(\tau^z(\Lambda) \leq t)\, v_0(z)\, dz.$$

- **Challenge**. Recall that, for small $\delta > 0$, one may have

$$\|\Gamma(\Lambda^3) - \Gamma(\tilde{\Lambda}^3)\|$$

$$= 3\alpha \sup_{s\in[0,\delta]} \left| \int_{\mathbb{R}^3} \mathbb{P}(\tau^z(\Lambda) \leq s)\, v(0^-,z)\, dz - \int_{\mathbb{R}^3} \mathbb{P}(\tau^z(\tilde{\Lambda}) \leq s)\, v(0^-,z)\, dz \right|$$

$$\approx v(0^-,\cdot)\Big|_{\partial B_{\Lambda_{0-}}} \cdot \sup_{s\in[0,\delta]} \left|\Lambda^3_s - \tilde{\Lambda}^3_s\right|, \text{ and } v(0^-,\cdot)\Big|_{\partial B_{\Lambda_{0-}}} \text{ may be large.}$$

Proof of uniqueness: resolution of initial jump

- **Proposition (difficult)**. If $v_0(z)$ is **piecewise monotone** as a function of $|z|$, then so are $v(t,\cdot)$ and $v(t^-,\cdot)$ for **all** $t \geq 0$.
- The above prop. and minimal jump condition imply that $\Lambda_0 = \tilde{\Lambda}_0$, $v(0,z) = \tilde{v}(0,z)$, yielding:

$$\sup_{s\in[0,\delta]} \left|\Lambda^3_s - \widetilde{\Lambda}^3_s\right|$$

$$= 3\alpha \sup_{s\in[0,\delta]} \left| \int_{\mathbb{R}^3} \mathbb{P}(\tau^z(\Lambda) \leq s)\, v(0,z)\, dz - \int_{\mathbb{R}^3} \mathbb{P}(\tau^z(\tilde{\Lambda}) \leq s)\, v(0,z)\, dz \right| .$$

where $v(0,\cdot)\Big|_{\partial B_{\Lambda_0}} \leq 1$.

- Since the case $v(0,\cdot)\Big|_{\partial B_{\Lambda_0}} < 1$ is easy, we focus on $v(0,\cdot)\Big|_{\partial B_{\Lambda_0}} = 1$.
- The minimal jump condition yields, for small enough $\Lambda_0 - |z| > 0$:

$$v(0,z) = 1 - \psi(\Lambda_0 - |z|), \quad \psi \uparrow, \quad \psi(0^+) = 0.$$

Proof of uniqueness: steps 1 and 2

$$v(0,z) = 1 - \psi(\Lambda_0 - |z|), \quad \psi \uparrow, \quad \psi(0^+) = 0.$$

Consider t s.t. $\Lambda^3_t - \tilde{\Lambda}^3_t = \sup_{s \le t} |\Lambda^3_s - \tilde{\Lambda}^3_s|$ and deduce

$$\begin{aligned}
&\frac{1}{3}(\Lambda^3_t - \widetilde{\Lambda}^3_t) = \alpha \int_{\mathbb{R}^3} \mathbb{P}(\tau^z(\Lambda) \le s)\, v(0,z)\, dz - \alpha \int_{\mathbb{R}^3} \mathbb{P}(\tau^z(\tilde{\Lambda}) \le s)\, v(0,z)\, dz \\
&\le \alpha \int_{B_{\Lambda_0}} \mathbb{P}(\tau^z(\tilde{\Lambda}) \le t,\, \tau^z(\Lambda) > t)\, v(0,z)\, dz \\
&+ \alpha \int_{\mathbb{R}^3 \setminus B_{\Lambda_0}} \mathbb{P}(\tau^z(\tilde{\Lambda}) \le t,\, \tau^z(\Lambda) > t)\, v(0,z)\, dz.
\end{aligned}$$

Goal: obtain a contradiction to the above, for small enough $t > 0$.

1. $\alpha \int_{\mathbb{R}^3 \setminus B_{\Lambda_0}} \mathbb{P}(\tau^z(\tilde{\Lambda}) \le t,\, \tau^z(\Lambda) > t)\, v(0,z)\, dz \le C_7\, \mathbb{E} \max_{0 \le s \le t} (B_s + \widetilde{\Lambda}_s - \Lambda_0)$
2. $\alpha \int_{B_{\Lambda_0}} \mathbb{P}(\tau^z(\tilde{\Lambda}) \le t,\, \tau^z(\Lambda) > t)\, v(0,z)\, dz \le$
$\frac{1}{3}(\Lambda^3_t - \tilde{\Lambda}^3_t) - C_8\, \mathbb{E}\, \psi\left(\max_{0 \le s \le t} (B_s - \Lambda_s + \Lambda_0)\right)$

Proof of uniqueness: steps 1 and 2

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$$\le \alpha \int_{B_{\Lambda_0}} \mathbb{P}(\tau^z(\tilde{\Lambda}) \le t,\, \tau^z(\Lambda) > t)\, v(0,z)\, dz$$

$$+ \alpha \int_{\mathbb{R}^3 \setminus B_{\Lambda_0}} \mathbb{P}(\tau^z(\tilde{\Lambda}) \le t,\, \tau^z(\Lambda) > t)\, v(0,z)\, dz.$$

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Proof of uniqueness: step 3

To obtain the desired **contradiction**, it suffices to show that, for any $\bar{C} > 0$,

$$\mathbb{E}\,\psi\left(\max_{0\le s\le t}(B_s - \Lambda_s + \Lambda_0)\right) > \bar{C}\,\mathbb{E}\max_{0\le s\le t}(B_s + \widetilde{\Lambda}_s - \Lambda_0), \text{ for small enough } t > 0.$$

- **Lemma 3 (easy).** There exists a solution $\hat{\Lambda}$ to 1-phase Stefan problem with the same ψ (up to multiplicative constant) and with $\hat{\Lambda}_0 = \Lambda_0$, s.t. $\hat{\Lambda} \ge \Lambda \vee \widetilde{\Lambda}$. Hence, it suffices to show

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- Using growth condition for $\hat{\Lambda}$, obtain:

$$\mathbb{E}\psi(\sup_{s\le t}(B_s - \hat{\Lambda}_s + \Lambda_0)) > \bar{C}\,\mathbb{E}\Psi(\sup_{s\le t}(B_s - \hat{\Lambda}_s + \Lambda_0))$$

$$= \bar{C}\,\mathbb{E}\sup_{s\le t}(B_s - \hat{\Lambda}_{t-s} + \hat{\Lambda}_t), \quad \Psi(x) := \int_0^x \psi(y)\,dy.$$

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Proof of uniqueness: step 3

Goal: show that

$$\mathbb{E} \sup_{s \le t} (B_s - \hat{\Lambda}_{t-s} + \hat{\Lambda}_t) \ge \mathbb{E} \max_{0 \le s \le t} (B_s + \hat{\Lambda}_s - \Lambda_0)$$

- The above follows from "semi-convexity" of $\hat{\Lambda}$:

$$\hat{\Lambda}_t - \hat{\Lambda}_{t-s} \le \Lambda_0 - \hat{\Lambda}_s, \quad s \in [0, t],$$

- which in turn follows from the scaling property:

  Lemma 4 (easy) For any $q \in (0, 1]$:

$$\hat{\Lambda}_{qt} - \Lambda_0 \ge \sqrt{q}\,(\hat{\Lambda}_t - \Lambda_0) \ge q\,(\hat{\Lambda}_t - \Lambda_0)$$