Short course: Convergence of minimizers in local and nonlocal total variation regularization

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We focus on the solution of linear operator equations of the form $Au = f + w$, with f assumed to be an ideal measurement (that we do not have access to), and w some perturbation or instance of noise. Such equations may or may not have a unique solution, and even if they do, they might still be *ill-posed*. That is, small perturbations w could lead to huge variations in the solution u. This is the case in many models of image acquisition and image processing tasks: tomography, deblurring...

A way to try to reduce this ill-posedness is to consider a Tikhonov-type regularization approach: for $A : X \to Y$ between Banach spaces X, Y one considers a minimization problem of the type

$$\min_{u \in X} \|Au - (f + w)\|_Y^\sigma + \alpha \mathcal{R}(u),$$

with $\mathcal{R}$ a convex functional on X that encodes some a priori information about the expected solutions. One of the most basic mathematical questions that one can ask about such a scheme is its behavior in the low-noise regime. That is, if $\|w\|_Y \to 0$, can we choose the regularization parameter α depending on it in such a way that the corresponding minimizers converge to an exact or approximate solution? In which topology? And how fast?

First, we plan to review the standard theory to answer these questions in the case where $\mathcal{R}$ is strongly convex. Then, we turn our attention to a specific choice of non-strongly convex $\mathcal{R}$: the total variation, widely used in situations where discontinuous solutions are desirable, and nonlocal variants thereof. In that case, although convergence rates are complicated to interpret, recent results show that it is possible to have convergence of the level sets of solutions (a proxy for 'objects' in images) in the sense of Hausdorff convergence. This is accomplished through additional regularity, and to prove it one can use tools from the theory of quasi-minimizers of (standard or fractional) perimeter, convex analysis, and some rudimentary geometry of Banach spaces. All of these can be introduced in different levels of detail according to the background of the audience.

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