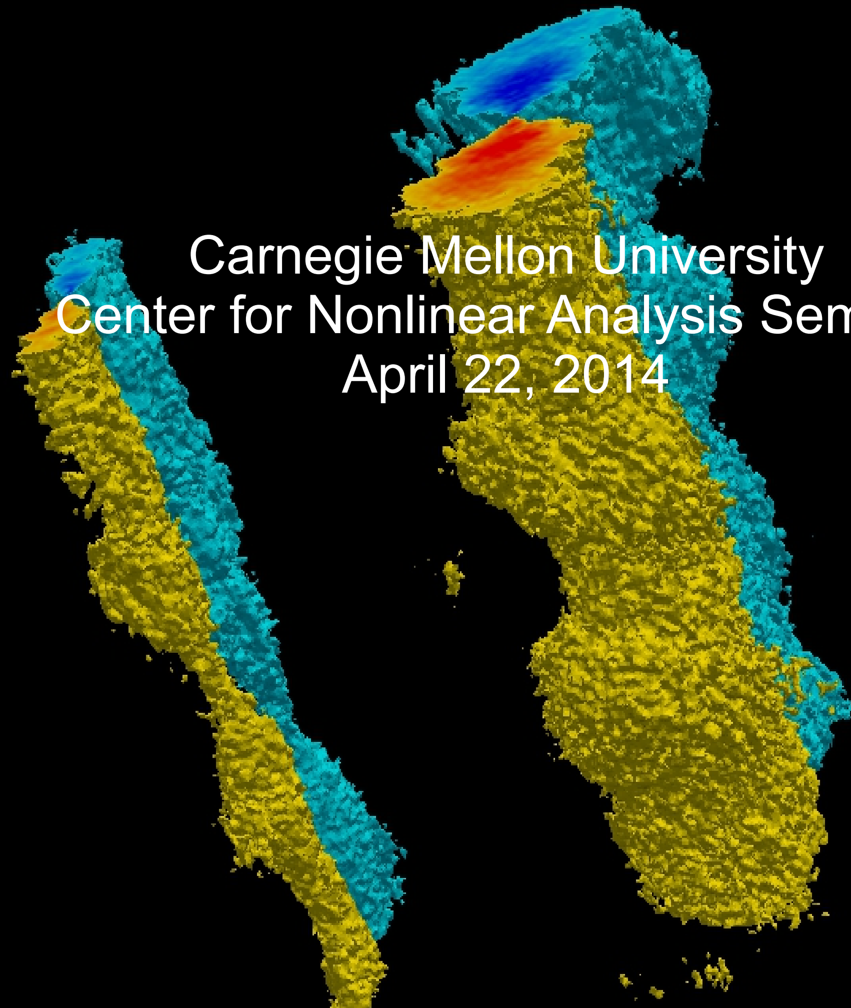
Multiple time scales and geophysical fluid dynamics: reduced equation sets and slow manifolds



Carnegie Mellon University
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Jared P. Whitehead

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- Terry Haut, Center for Nonlinear Studies, Los Alamos National Laboratory.

Los Alamos
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EST. 1943

- Beth A Wingate, formerly at LANL, and now at Department of Mathematics, University of Exeter

UNIVERSITY OF
EXETER

Motivation

Image courtesy of http://personal.maths.surrey.ac.uk/st/T.Bridges/WATERWAVES/

- Waves are ubiquitous in the oceans and atmosphere (we just don't see them in the atmosphere).
- Waves are very fast, and have relatively short spatial scales.

- If these waves are everywhere, why don't we hear the weather man talking about them, i.e.

Motivation

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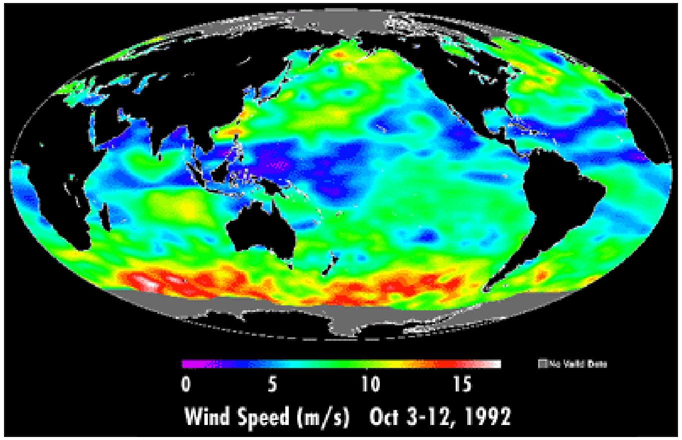

Image courtesy of
http://sealevel.jpl.nasa.gov/education/classactivities/onlinetutorial/tutorial1/windwaves/

- Waves are ubiquitous in the oceans and atmosphere (we just don't see them in the atmosphere).
- Waves are very fast, and have relatively short spatial scales.

- If these waves are everywhere, why don't we hear the weather man talking about them?

- Because most of us don't care what happens on the scale of minutes or meters, and it is a little hard to capture such scales anyway (weather/climate is a global problem).

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- How do we predict those scales of interest, while avoiding these fast, small scales?

Some historical perspective



S V E N S K A G E O F Y S I S K A F Ö R E N I N G E N

Tellus

VOLUME 2, NUMBER 4

A QUARTERLY JOURNAL O


VOL. 10, NO. 2 JOURNAL OF METEOROLOGY APRIL 1953

NUMERICAL INTEGRATION OF THE QUASI-GEOSTROPHIC EQUATIONS FOR BAROTROPIC AND SIMPLE BAROCLINIC FLOWS

By J. G. Charney and N. A. Phillips

Institute for Advanced Study [1]

(Manuscript received 5 January 1953)

ABSTRACT

An n-level generalization of the $2\frac{1}{2}$-dimensional model is derived by specialization of the complete three-dimensional quasi-geostrophic equations. In the case $n = 1$, it reduces to the two-dimensional single-layer barometric model. In the case $n = 2$, it reduces to the double-layer barotropic model, or — what is shown to be mathematically equivalent—the $2\frac{1}{2}$-dimensional model. Methods of numerical integration of the 2- and $2\frac{1}{2}$-dimensional equations, and the machine requirements for such integrations, are discussed.

The results of a series of six two-dimensional and six $2\frac{1}{2}$-dimensional forecasts for 12 and 24 hours are presented. Although the $2\frac{1}{2}$-dimensional forecasts are noticeably superior to the two-dimensional forecasts, it is apparent that considerable improvement will be possible with models in which there are fewer artificial constraints. A method of integration is therefore proposed for the n-level generalization of the $2\frac{1}{2}$-dimensional model, and computation schemes are outlined for the general three-dimensional quasi-geostrophic equations. The semi-Lagrangian coordinate system with potential temperature as vertical coordinate is shown to exhibit favorable properties for machine integration.

Numerical Integration of the Barotr

By J. G. CHARNEY, R. FJÖRTOFT[1],

The Institute for Advanced Study, Prin

(Manuscript received 1 Novem

Abstract

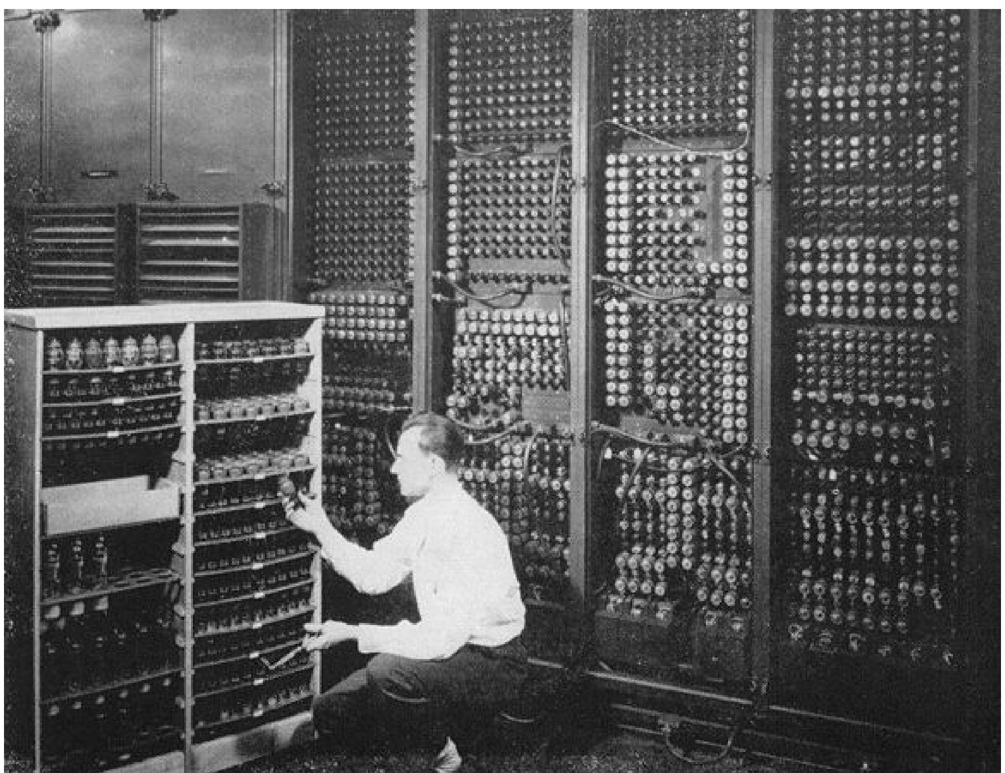
A method is given for the numerical solution of t over a limited area of the earth's surface. The lack o investigation of the appropriate boundary conditions. These are determined by a heuristic argument and are shown to be sufficient in a special case. Approximate conditions necessary to insure the mathematical stability of the difference equation are derived. The results of a series of four 24-hour forecasts computed from actual data at the 500 mb level are presented, together with an interpretation and analysis. An attempt is made to determine the causes of the forecast errors. These are ascribed partly to the use of too large a space increment and partly to the effects of baroclinicity. The rôle of the latter is investigated in some detail by means of a simple baroclinic model.

Some historical perspective

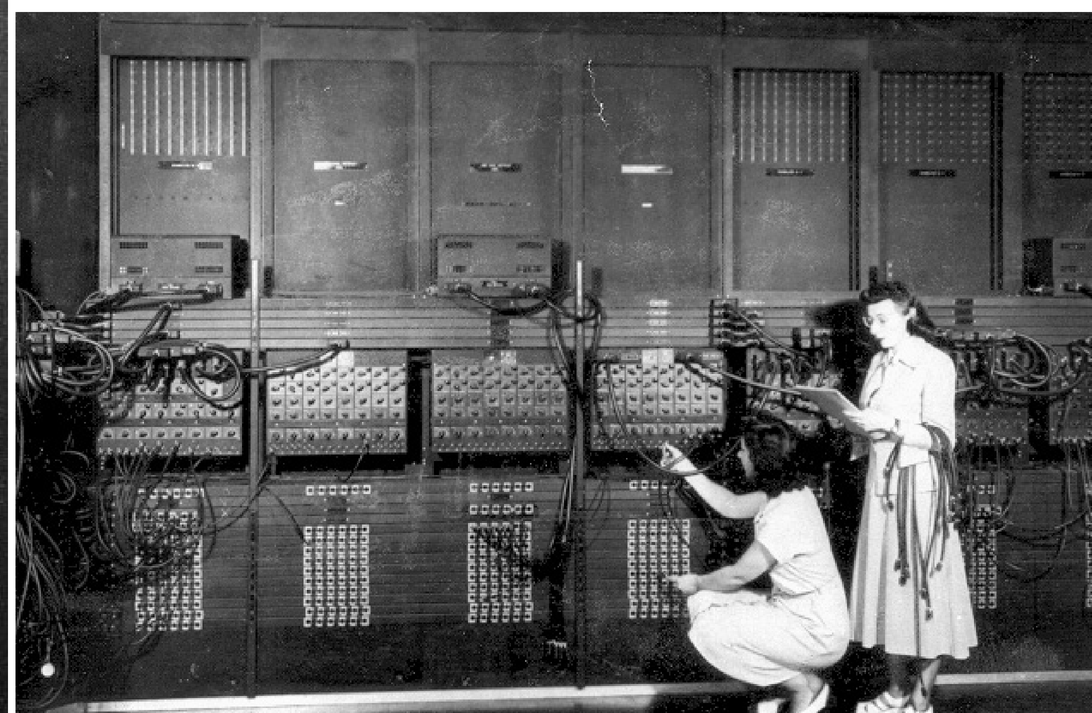
In his baroclinic instability study, Charney had derived a mathematically tractable equation for the unstable waves ‘by eliminating from consideration at the outset the meteorologically unimportant acoustic and shearing-gravitational oscillations’ [6]. The multi-scale nature of atmospheric dynamics, with low-frequency and high-frequency components, is also found in a wide range of other physical contexts. The advantages

From P. Lynch 'The origins of computer weather prediction and climate modeling', JCP 2008.

• Removed the 'fast' gravity waves thus allowing for the 1st accurate numerical weather prediction model to be developed in the early 1950s.



Replacing a bad tube meant checking among ENIAC's 19,000 possibilities.



What did Charney et. al. really do?

Most evolution equations for GFD are of the form:

$$\frac{\partial \vec{u}}{\partial t} + B(\vec{u}, \vec{u}) + \frac{1}{\epsilon} L \vec{u} = D \vec{u}$$

Typically L represents rotation and/or stable stratification, and is skew-Hermitian (purely imaginary, discrete spectrum) meaning it is wave-generating.

What happens if we let $\epsilon \rightarrow 0$, i.e. the waves get faster and faster?

To avoid this unpleasantry, we can consider information (flow) that lives in the kernel of L, so there are no waves. This is equivalent to what Charney & Co. did in the 1950s.

In meteorology this is referred to as an O(1) balance relation, and the resultant set of solutions is called the slow manifold.

Of slow 'manifolds' and 'balance relations'

1236 JOURNAL OF THE ATMOSPHERIC SCIENCES VOLUME 57

Balance and the Slow Quasimanifold: Some Explicit Results

RUPERT FORD,* MICHAEL E. MCINTYRE, AND WARWICK A. NORTON+

Centre for Atmospheric Science,# Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge, United Kingdom

(Manuscript received 19 June 1998, in final form 24 March 1999)

The ideas of balanced flow and slow manifold for stratified, rotating fluid systems (e.g., Charney 1948; Leith 1980; Lorenz 1980) are among the most useful, important, and arguably central ideas in dynamical meteorology and oceanography, for well-known reasons.

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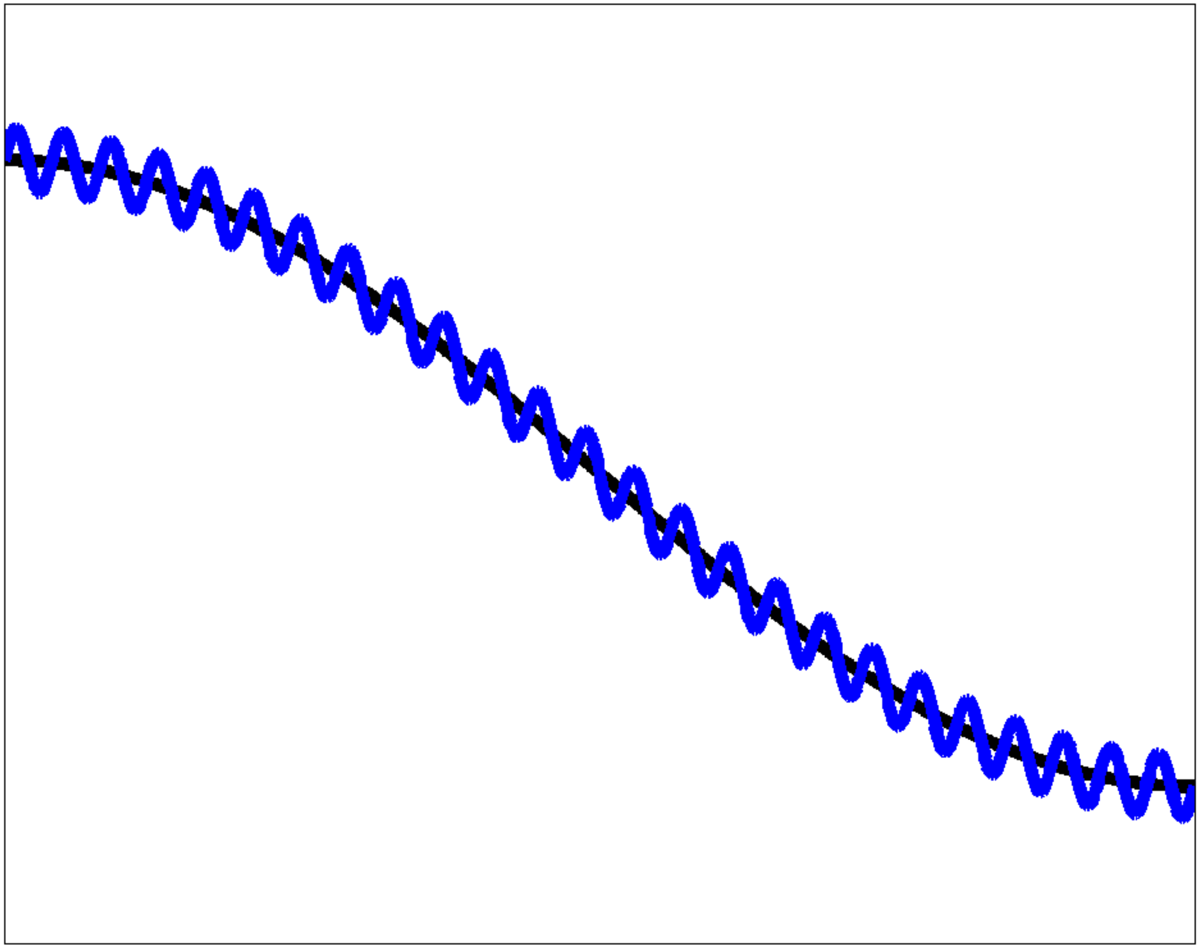

scale.

- If the data lives on the slow manifold, i.e. in the kernel of L (and fast waves don't really matter) this restriction vanishes.
- If the slow manifold is invariant, and the initial data is in the kernel of L, then numerical models in this space are not restricted by the fast waves.

Speaking of manifolds

The meteorological definition of the slow manifold likely does not coincide with the mathematical one.

Basically it is an invariant region of phase space where there are no fast waves.

The biggest difficulty in defining and utilizing the slow manifold is that typically the invariance property does not hold.

This means that either the fast waves can influence the slow mean flow, or the slow mean flow can spontaneously generate fast waves.

Interactions between 'fast' and 'slow'

J. Fluid Mech. (1989), *vol.* 206, *pp.* 433–462

Printed in Great Britain

Wave–vortex dynamics in rotating shallow water

(shallow-water) equations. In the case of small geopotential fluctuations considered here, we find no energy exchange between the inertio–gravitational and the

Pergamon

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0967–0637(95)00040–2

Abstract—This paper shows that the rapid, inertia–gravity oscillations present in all numerical integrations of nonlinear rotating fluid systems have no effect on the slower, quasi-geostrophic, oscillatio **at least to leading order in Rossby number.** rically with reduced-... c frictions. Differences between these findings and those from theoretical turbulence are discussed; these are probably due to the frictional terms, which are active in the numerical calculations here and act to damp out the enstrophy cascade.

Spontaneous generation of 'fast' waves

Fast

ABSTRACT

The spontaneous generation of inertia–gravity waves by balanced motion is investigated in the limit of small Rossby number $\epsilon \ll 1$. Particular (sheared disturbance) solutions of the three-dimensional Boussinesq equations are considered. For these solutions, there is a strict separation between balanced motion and inertia–gravity waves for large times. This makes it possible to estimate the amplitude of the inertia–gravity waves that are generated spontaneously from perfectly balanced initial conditions. It is shown analytically using exponential asymptotics, and confirmed numerically, that this amplitude is proportional to $\epsilon^{-1/2} \exp(-\alpha/\epsilon)$, with a constant $\alpha > 0$ and a proportionality constant that are giv This result demonstrates nonstrates the inevitability of inertia–gravit and hence the nonexistence of an invariant slow manifold also exemplifies the remarkable, of the singularity structure of the balanced motion for complex values of time is emphasized, and some general implications of the results are discussed.

Spontaneous generation of 'fast' waves

The above results seem to mitigate against the existence of a slow manifold, or at the very least, they indicate that if it exists, it is unstable, since, except for steady flows, the trajectories seem to possess a gravity-inertial wave component which persists for all time. If this is indeed the case, it becomes necessary to reconsider the definition of balanced flow for the truncated model under

generated naturally. The sequence of manifolds, $U^{(n)}$, can be regarded as an asymptotic sequence for the G's which can be used to define an open, somewhat "fuzzy" balanced set, B, which has the same dimension as the underlying space (Fig. 11).

Spontaneous generation of 'fast' waves

We conclude that the stochastic-layer hypothesis is strongly supported by our analysis. In other words, it appears that neither a strict slow manifold nor a unique generalized slow manifold exists. As Warn (1997) and others have already argued in other ways, it seems practically certain then that the entity traditionally called the slow manifold—whose practical usefulness is not in question—is not, in fact, a manifold. We therefore suggest, for the sake of continuity with the traditional terminology, that this entity might be referred to as the slow quasimanifold.

Is the slow manifold truly invariant?

$$\frac{\partial \vec{u}}{\partial t} + B(\vec{u}, \vec{u}) + \frac{1}{\epsilon} L \vec{u} = D \vec{u}$$

JOURNAL OF DIFFERENTIAL EQUATIONS **114**, 476–512 (1994)

Singular Limits of Quasilinear Hyperbolic Systems with Large Parameters and the Incompressible Limit of Compressible Fluids

SERGIU KLAINERMAN
Courant Institute

AND

ANDREW MAJDA
University of California at Berkeley

Fast Singular Limits of Hyperbolic PDEs

STEVEN SCHOCHET

School of Mathematical Sciences, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Ramat Aviv 69978, Israel

Received April 13, 1992

The theory of cancellation of oscillations was originally pioneered by Bogliubov & company, and extended to singular limits of hyperbolic problems:

Essentially, a discrete purely imaginary spectrum will generate oscillatory terms, that when averaged across (to get the long time behavior) will vanish...

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} e^{is} \, ds = 0$$

Is the slow manifold truly invariant?

$$\frac{\partial \vec{u}}{\partial t} + B(\vec{u}, \vec{u}) + \frac{1}{\epsilon} L \vec{u} = D \vec{u}$$

So long as the nonlinearity is relatively well-behaved...

The theory of cancellation of oscillations guarantees that in the infinite limit, the fast waves and the part of the flow living in the kernel of L (slow manifold), are completely decoupled.

In reality this never occurs (we live in a 'finite world').

To understand what this means, we need to revisit the cancellation of oscillations argument developed by Schochet and others.

Cancellation of Oscillations was motivated by multiple scales asymptotics

$$\frac{\partial \vec{u}}{\partial t} + \frac{1}{\epsilon} L \vec{u} + B(\vec{u}, \vec{u}) = D \vec{u}$$

To treat the singular limit $\epsilon \to 0$, introduce the 'fast' time scale $\tau = \frac{t}{\epsilon}$.

Using the ansatz $\vec{u}(t, \tau, \vec{x}) = e^{-\tau L} \vec{u}_0(t, \vec{x}) + O(\epsilon)$ leads to a reduced, averaged equation for $\vec{u}_0(t, \vec{x})$ that enforces no secular growth of the lower order terms:

$$\frac{\partial \vec{u}_0}{\partial t} = -\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} e^{sL} B\left(e^{-sL} \vec{u}_{0,} e^{-sL} \vec{u}_0\right) ds + \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} e^{sL} D e^{-sL} ds\, \vec{u}_0$$

Multiple time scales and renormalization

$$\frac{\partial \vec{u}}{\partial t} + \frac{1}{\epsilon} L \vec{u} + B(\vec{u}, \vec{u}) = D \vec{u}$$

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J. Differential Equations 208 (2005) 215–257

Journal of Differential Equations

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Renormalization group method applied to the primitive equations

M. Petcu,[a,c] R. Temam,[a,b,*] and D. Wirosoetisno[b]

[a] *Laboratoire d'Analyse Numérique, Université de Paris-Sud, Orsay, France*
[b] *The Institute for Scientific Computing and Applied Mathematics, Indiana University, Rawles Hall, 831 E Third Street, Bloomington, IN, 47405-7106, USA*
[c] *The Institute of Mathematics of the Romanian Academy, Bucharest, Romania*

Received June 27, 2003

Dedicated to George Sell on the occasion of his 65th birthday

With different notation, this same reduced system can be found via the method of renormalization propagated by Temam et. al..

Limiting dynamics and linear algebra

$$\frac{\partial \vec{u}}{\partial t}+\frac{1}{\epsilon}L\vec{u}+B(\vec{u},\vec{u})=D\vec{u}$$

To treat the singular limit $\epsilon \to 0$, introduce the 'fast' time scale $\tau=\frac{t}{\epsilon}$.

Using the ansatz $\vec{u}(t,\tau,\vec{x})=e^{-\tau L}\vec{u}_0(t,\vec{x})+O(\epsilon)$ leads to a reduced, averaged equation for $\vec{u}_0(t,\vec{x})$ that enforces no secular growth of the lower order terms:

$$\frac{\partial \vec{u}_0}{\partial t}=-\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^{\tau} e^{sL}B\left(e^{-sL}\vec{u}_{0,}\, e^{-sL}\vec{u}_0\right)ds+\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^{\tau} e^{sL}De^{-sL}ds\,\vec{u}_0$$

It turns out that this is exactly the same as projecting onto the kernel of L.

Limiting dynamics and linear algebra

$$\frac{\partial \vec{u}}{\partial t} + \frac{1}{\epsilon} L \vec{u} + B(\vec{u}, \vec{u}) = D \vec{u}$$

Define P as the projection onto the kernel of L. We then decompose $\vec{u} = \vec{u}_0 + \vec{u}\,'$, where $P \vec{u}_0 = \vec{u}_0$ and $P \vec{u}\,' = 0$, where $\vec{u}_0$ is the part of the flow on the slow manifold, and $\vec{u}\,'$ is everything off it (both $O(1)$ fast, and fast *and* slow $O(\epsilon)$ terms).

This allows us to diagnose how the fluctuations (primed variables) affect the O(1) slow manifold and vice versa. The theory of cancellation of fast oscillations guarantees that there is no interaction between these terms in the infinite limit, but what happens when $\epsilon > 0$?

Can we clarify how the invariance of the slow manifold fails, i.e. is it spontanteous generation of 'fast' waves or is it because these 'fast' waves affect the evolution of the mean flow on the slow manifold?

Returning to geophysical applications:

Consider the rotating, stratified Boussinesq equations with equally strong rotation and stratification, written in nonlocal form (the pressure is eliminated by solving the inherent Poisson problem):

$$\frac{\partial \vec{u}}{\partial t}+\frac{1}{Ro}L\vec{u}+B(\vec{u},\vec{u})=D\vec{u} \qquad where \qquad \vec{u}=\begin{pmatrix}\vec{v}\\ \rho\end{pmatrix}, \qquad D\vec{u}=\begin{pmatrix}\dfrac{1}{\mathrm{Re}}\Delta\vec{v}\\ \dfrac{1}{Pr\,\mathrm{Re}}\Delta\rho\end{pmatrix}$$

$$L\vec{u}=\begin{pmatrix}\hat{z}\times\vec{v}+\nabla\Delta^{-1}\omega \quad \hat{z}\rho-\nabla\Delta^{-1}\left(\dfrac{\partial\rho}{\partial z}\right)\\ -w\end{pmatrix}, \qquad B(\vec{u},\vec{u})=\begin{pmatrix}\vec{v}\cdot\nabla\vec{v}-\nabla\Delta^{-1}\left(\nabla\cdot\left(\vec{v}\cdot\nabla\vec{v}\right)\right)\\ \vec{v}\cdot\nabla\rho\end{pmatrix}$$

Returning to geophysical applications:

Geophys. Astrophys. Fluid Dynamics, Vol. 87, pp. 1 - 50
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LOW FROUDE NUMBER LIMITING DYNAMICS FOR STABLY STRATIFIED FLOW WITH SMALL OR FINITE ROSSBY NUMBERS

PEDRO F. EMBID[a] and ANDREW J. MAJDA[b]

[a]*Mathematics Department, The University of New Mexico, Albuquerque, New Mexico 87131,USA;* [b]*Courant Institute of Mathematical Sciences, New York University, New York 10012, USA*

(Received 8 October 1996; In final form 5 June 1997)

Schochet's theory is extended to include this particular quadratic nonlinear interaction, i.e. in the limit as the rotation and stratification simultaneously dominate, the resultant slow dynamics (quasi-geostrophy) is the rigorously justified limiting system.

Returning to geophysical applications:

What if the rotation and stratification were not equally strong (fast)?

$$\frac{\partial \vec{u}}{\partial t} + \frac{1}{Ro} K\vec{u} + \frac{1}{Fr} L\vec{u} + B(\vec{u},\vec{u}) = D\vec{u} \qquad where \qquad \vec{u} = \begin{pmatrix} \vec{v} \\ \rho \end{pmatrix}, \qquad D\vec{u} = \begin{pmatrix} \frac{1}{\mathrm{Re}} \Delta \vec{v} \\ \frac{1}{Pr\,\mathrm{Re}} \Delta \rho \end{pmatrix}$$

$$K\vec{u} = \begin{pmatrix} \hat{z} \times \vec{v} + \nabla \Delta^{-1} \omega \\ 0 \end{pmatrix}, \qquad L\vec{u} = \begin{pmatrix} \hat{z}\rho - \nabla \Delta^{-1} \left(\frac{\partial \rho}{\partial z} \right) \\ -w \end{pmatrix}, \qquad B(\vec{u},\vec{u}) = \begin{pmatrix} \vec{v} \cdot \nabla \vec{v} - \nabla \Delta^{-1} \left(\nabla \cdot (\vec{v} \cdot \nabla \vec{v}) \right) \\ \vec{v} \cdot \nabla \rho \end{pmatrix}$$

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(Received 8 October 1996; In final form 5 June 1997)

J. Fluid Mech. (2011), *vol.* 676, *pp.* 546–571. © Cambridge University Press 2011. The online version of this article is published within an Open Access environment subject to the conditions of the Creative Commons Attribution-NonCommercial-ShareAlike licence <http://creativecommons.org/licenses/by-nc-sa/2.5/>. The written permission of Cambridge University Press must be obtained for commercial re-use.
doi:10.1017/jfm.2011.69

Low Rossby limiting dynamics for stably stratified flow with finite Froude number

BETH A. WINGATE[1]†, PEDRO EMBID[2], MIRANDA HOLMES-CERFON[3] AND MARK A. TAYLOR[4]

[1]MS D413, Los Alamos National Laboratory, Los Alamos, NM 87544, USA
[2]Department of Mathematics and Statistics, the University of New Mexico, Albuquerque, NM 87131, USA
[3]New York University, Courant Institute of Mathematical Sciences, New York, NY 10012, USA
[4]Sandia National Laboratories, Albuquerque, NM 87185, USA

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Limiting dynamics and linear algebra for the Boussinesq equations:

Define P as the projection onto the kernel of L (or K). We decompose $\vec{u}=\vec{u}_0+\vec{u}\,'$, where $P\vec{u}_0=\vec{u}_0$ and $P\vec{u}\,'=0$, where $\vec{u}_0$ is the part of the flow on the slow manifold, and $\vec{u}\,'$ is everything off it (both $O(1)$ fast, and *fast and slow* $O(\epsilon)$ terms).

$$P_{Ro}\vec{u}=\begin{pmatrix}\langle\vec{v}_H\rangle_z-\nabla_H\Delta_H^{-1}(\nabla_H\cdot\langle\vec{v}_H\rangle_z)\\ \langle w\rangle_z\\ \rho\end{pmatrix}\qquad P_{Fr}\vec{u}=\begin{pmatrix}\vec{v}_H-\nabla_H\Delta_H^{-1}(\nabla_H\cdot\vec{v}_H)\\ 0\\ \langle\rho\rangle_H\end{pmatrix}$$

$$P_{QG}\vec{u}=\begin{pmatrix}\vec{v}_H-\frac{Fr^2}{Ro^2}\Delta_{QG}^{-1}\frac{\partial^2\vec{v}_H}{\partial z^2}-\Delta_{QG}^{-1}\left(\nabla_H(\nabla_H\cdot\vec{v}_H)+\frac{Fr}{Ro}\nabla_H\times(\hat{z}\rho)\right)\\ \rho-\frac{Fr}{Ro}\Delta_{QG}^{-1}\left(\partial_z(\nabla_H\times\vec{v}_H)\right)-\Delta_{QG}^{-1}\Delta_H\rho\end{pmatrix}$$

$$where\qquad \Delta_{QG}=\Delta_H+\frac{Fr^2}{Ro^2}\frac{\partial^2}{\partial z^2}.$$

Approaching the Limit: Evolution of the dynamics on the slow manifold

Rapid Rotation

$$\frac{\partial \vec{v}_H^{Ro}}{\partial t} + \vec{v}_H^{Ro} \cdot \nabla_H \vec{v}_H^{Ro} - \nabla_H \Delta_H^{-1} \left(\nabla_H \cdot (\vec{v}_H^{Ro} \cdot \nabla_H \vec{v}_H^{Ro}) \right) - \frac{1}{\mathrm{Re}} \Delta_H \vec{v}_H^{Ro} = - \left(1 - \nabla_H \Delta_H^{-1} \nabla_H \cdot \ \right) \langle \{ \vec{v}\,' \cdot \nabla \vec{v}\,' \}_H \rangle_z ,$$

$$\nabla_H \cdot \vec{v}_H^{Ro} = 0,$$

$$\frac{\partial w^{Ro}}{\partial t} + \vec{v}_H^{Ro} \cdot \nabla_H w^{Ro} + \frac{1}{Fr} \langle \rho \rangle_z - \frac{1}{\mathrm{Re}} w^{Ro} = - \langle \vec{v}\,' \cdot \nabla w' \rangle_z ,$$

$$\frac{\partial \rho}{\partial t} + \vec{v}^{Ro} \cdot \nabla \rho - \frac{1}{Fr} w^{Ro} - \frac{1}{\mathrm{Re}\, Pr} \Delta \rho = - \vec{v}\,' \cdot \nabla \rho + \frac{1}{Fr} w' .$$

$$\frac{1}{2} \frac{d}{dt} \left\| \vec{v}_H^{Ro} \right\|_2^2 = - \int \vec{v}_H^{Ro} \cdot \langle [\vec{v}\,' \cdot \nabla \vec{v}\,']_H \rangle_z \, d\vec{x}$$

$$\frac{1}{2} \frac{d}{dt} \left\| w^{Ro} \right\|_2^2 = - \frac{1}{Fr} \int w^{Ro} \langle \rho \rangle_z \, d\vec{x} - \int w^{Ro} \langle \vec{v}_H\,' \cdot \nabla_H w' \rangle_z \, d\vec{x}$$

$$\frac{1}{2} \frac{d}{dt} \| \rho \|_2^2 = \frac{1}{Fr} \int w^{Ro} \rho \, d\vec{x} + \frac{1}{Fr} \int w' \rho \, d\vec{x}$$

Approaching the Limit: Evolution of the dynamics on the slow manifold

Strong Stratification

$$\frac{\partial \vec{v}_H^{Fr}}{\partial t}+\vec{v}_H^{Fr}\cdot\nabla_H \vec{v}_H^{Fr}+\frac{1}{Ro}\hat{z}\times\vec{v}_H^{Fr}+\nabla_H\Delta_H^{-1}\left(\frac{1}{Ro}\omega^{Fr}-\nabla_H\cdot(\vec{v}_H^{Fr}\cdot\nabla_H \vec{v}_H^{Fr})\right)-\frac{1}{\mathrm{Re}}\Delta\vec{v}_H^{Fr}$$
$$=-\left(1-\nabla_H\Delta_H^{-1}\nabla_H\cdot\ \right)\{\vec{v}\,'\cdot\nabla\vec{v}\,'\}_H,$$
$$\nabla_H\cdot\vec{v}_H^{Fr}=0,$$
$$\frac{\partial\rho^{Fr}}{\partial t}-\frac{1}{\mathrm{Re}\,Pr}\frac{\partial^2\rho^{Fr}}{\partial z^2}-\langle\vec{v}\,'\cdot\nabla\rho\,'\rangle_H$$

$$\frac{1}{2}\frac{d}{dt}\left\|\vec{v}_H^{Fr}\right\|_2^2=-\int\vec{v}_H^{Fr}\cdot[\vec{v}\,'\cdot\nabla\vec{v}\,']_H\,d\vec{x}$$

$$\frac{1}{2}\frac{d}{dt}\left\|\rho^{Fr}\right\|_2^2=-\int\rho^{Fr}\langle\vec{v}\,'\cdot\nabla\rho\,'\rangle_H\,dz$$

Approaching the Limit: Evolution of the dynamics on the slow manifold

Quasi-Geostrophy

$$\frac{\partial \vec{v}_H^{QG}}{\partial t} + \vec{v}_H^{QG}\cdot\nabla_H \vec{v}_H^{QG} - Bu^2 \Delta_{QG}^{-1} \partial^2 \partial z^2 \left(\vec{v}_H^{QG}\cdot\nabla_H \vec{v}_H^{QG}\right)$$

$$-\Delta_{QG}^{-1}\left(\nabla_H\left(\nabla_H\cdot \vec{v}_H^{QG}\cdot\nabla_H \vec{v}_H^{QG}\right) - Bu\,\nabla_H\times\left(\hat{z}\frac{\partial}{\partial z}\left[\vec{v}_H^{QG}\cdot\nabla\rho^{QG}\right]\right)\right) - \frac{1}{\mathrm{Re}}\Delta \vec{v}_H^{QG}$$

$$= -\{\vec{v}\,'\cdot\nabla\vec{v}\,'\} + \Delta_{QG}^{-1}\left(\nabla_H\left(\nabla_H\cdot\{\vec{v}\,'\cdot\nabla\vec{v}\,'\}_H\right) - Bu\,\nabla_H\times\left(\hat{z}\frac{\partial}{\partial z}\{\vec{v}\,'\cdot\nabla\rho\,'\}\right)\right) + Bu^2\Delta_{QG}^{-1}\frac{\partial^2}{\partial z^2}\{\vec{v}\,'\cdot\nabla\vec{v}\,'\}_H$$

$$\frac{\partial \rho^{QG}}{\partial t} + \vec{v}_H^{QG}\cdot\nabla\rho^{QG} - Bu\,\Delta_{QG}^{-1}\frac{\partial}{\partial z}\left(\vec{v}_H^{QG}\cdot\nabla_H\omega^{QG}\right) - \Delta_{QG}^{-1}\Delta_H\left(\vec{v}_H^{QG}\cdot\nabla\rho^{QG}\right) - \frac{1}{\mathrm{Re}\,Pr}\Delta\rho^{QG}$$

$$= -\vec{v}\,'\cdot\nabla\rho\,' + Bu\,\Delta_{QG}^{-1}\frac{\partial}{\partial z}\left(\nabla_H\times\{\vec{v}\,'\cdot\nabla\vec{v}\,'\}_H\right) + \Delta_{QG}^{-1}\Delta_H\left(\vec{v}\,'\cdot\nabla\rho\,'\right)$$

$$\frac{1}{2}\frac{d}{dt}\left\|\vec{v}_H^{QG}\right\|_2^2 = -\frac{Fr}{Ro}\int\left[\vec{v}_H^{QG}\cdot\Delta_{QG}^{-1}\left[\nabla_H\times\left(\hat{z}\frac{\partial}{\partial z}\left[\vec{v}_H^{QG}\cdot\nabla_H\rho^{QG}\right]\right)\right]\right]d\vec{x}$$

$$-\int\left[\vec{v}_H^{QG}\cdot\left(1-\frac{Fr^2}{Ro^2}\Delta_{QG}^{-1}\frac{\partial^2}{\partial z^2}\right)\left[\vec{v}\,'\cdot\nabla\vec{v}\,'\right]_H\right]d\vec{x} - \frac{Fr}{Ro}\int\left[\vec{v}_H^{QG}\cdot\Delta_{QG}^{-1}\left[\nabla_H\times\left(\hat{z}\frac{\partial}{\partial z}\left[\vec{v}\,'\cdot\nabla\rho\,'\right]\right)\right]\right]d\vec{x}$$

$$\frac{1}{2}\frac{d}{dt}\left\|\rho^{QG}\right\|_2^2 = \frac{Fr}{Ro}\int\left[\rho^{QG}\Delta_{QG}^{-1}\frac{\partial}{\partial z}\left(\vec{v}_H^{QG}\cdot\nabla_H\omega^{QG}\right)\right]d\vec{x}$$

$$-\int\left[\rho^{QG}\left(1-\Delta_{QG}^{-1}\Delta_H\right)\vec{v}\,'\cdot\nabla\rho\,'\right]d\vec{x} + \frac{Fr}{Ro}\int\left[\rho^{QG}\Delta_{QG}^{-1}\left(\nabla_H\times\left[\vec{v}\,'\cdot\nabla\vec{v}\,'\right]_H\right)\right]d\vec{x}$$

What to do with this information?

This is the 21st Century, and I used to work for a DOE Lab...so

Direct Numerical Simulations (DNS) of the rotating, stratified Boussinesq system.

What to do with this information?

Direct Numerical Simulations (DNS) of the rotating, stratified Boussinesq system.

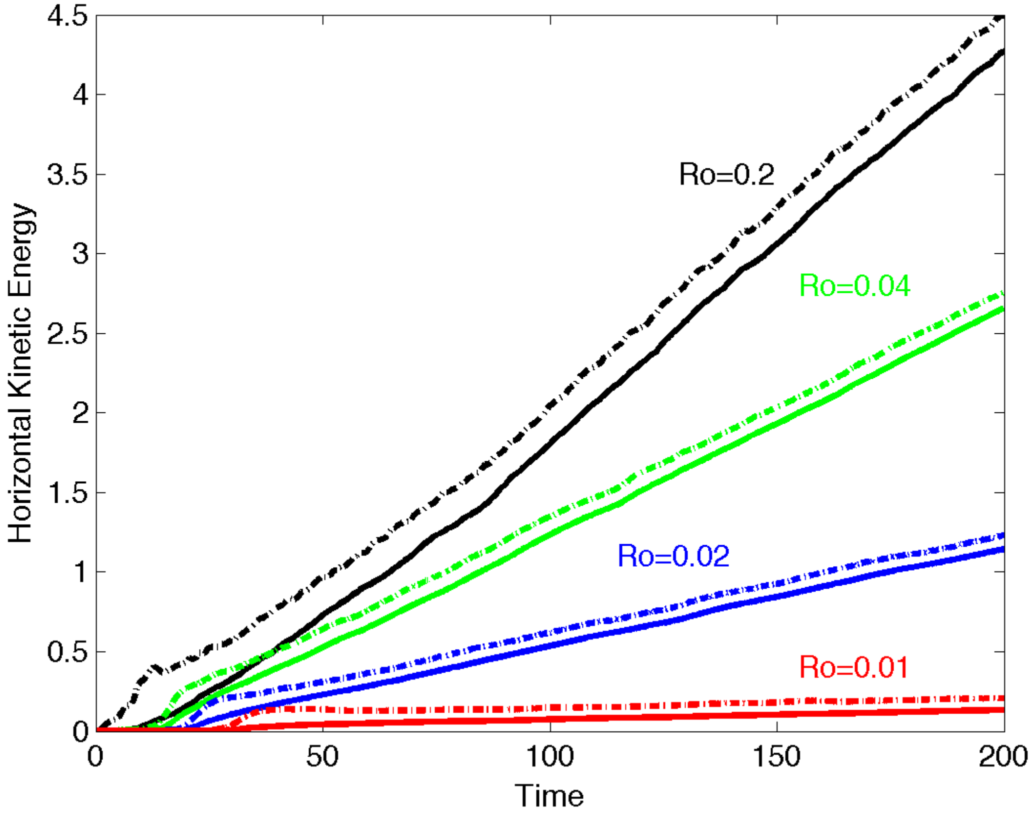
- 512^3 simulations for a variety of Ro and Fr #s.
- Force the density (density and horizontal momentum are decoupled on the slow manifold for Ro=0, or Fr=0).
- Large scale forcing for Ro->0 limit, and small scale forcing for Fr->0 and QG limits.
- Modeled dissipation to ensure longer range of attainable scales.

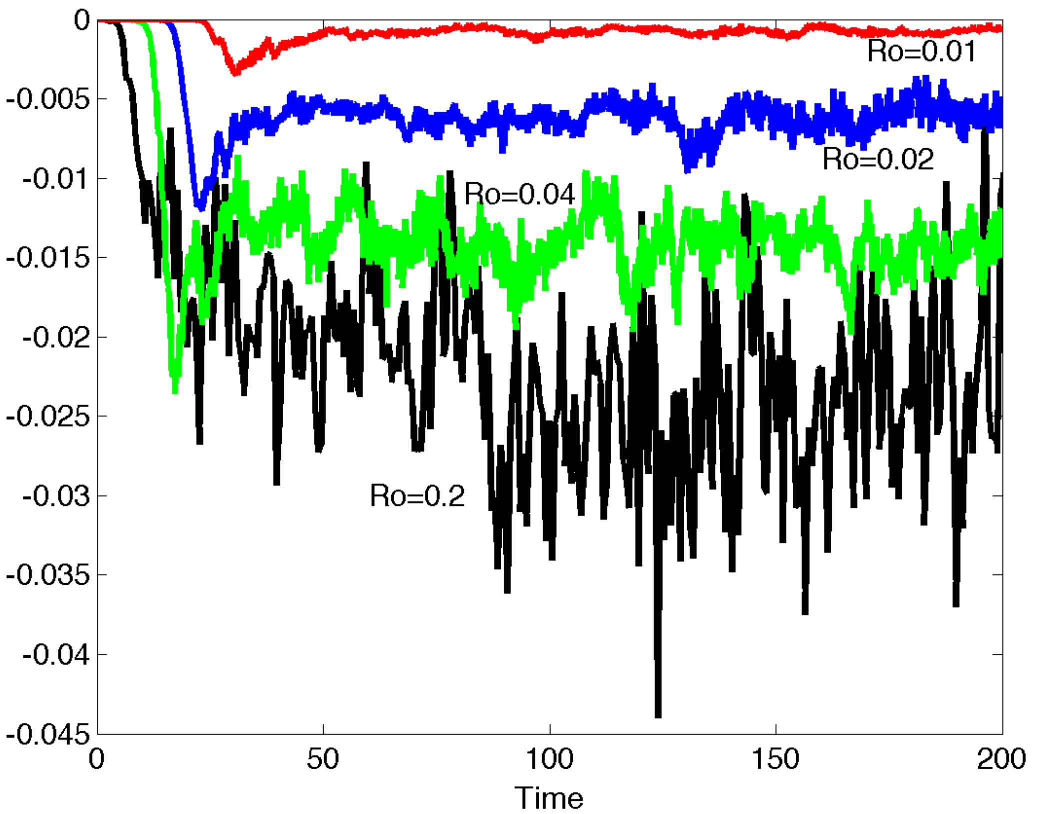
Rapidly rotating: exchanges in energy

$$\frac{1}{2}\frac{d}{dt}\left\|\vec{v}_H^S\right\|_2^2 = -\int \vec{v}_H^S\cdot\langle[\vec{v}\,'\cdot\nabla\vec{v}\,']_H\rangle_z\, d\vec{x}$$

$$\frac{1}{2}\frac{d}{dt}\left\|w^S\right\|_2^2 = -\frac{1}{Fr}\int w^S\langle\rho\rangle_z\, d\vec{x} - \int w^S\langle\vec{v}_H\,'\cdot\nabla_H w'\rangle_z\, d\vec{x}$$

$$\frac{1}{2}\frac{d}{dt}\left\|\rho\right\|_2^2 = \frac{1}{Fr}\int w^S\rho\, d\vec{x} + \frac{1}{Fr}\int w'\rho\, d\vec{x}$$



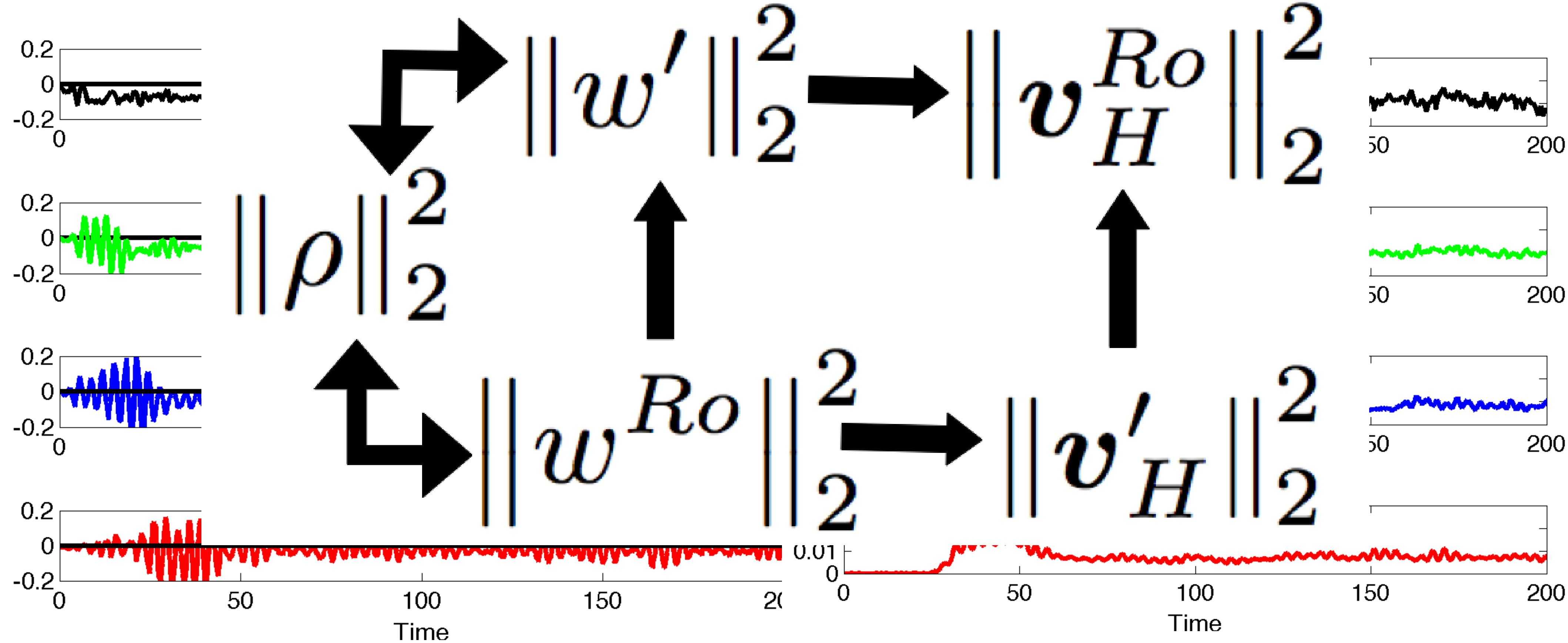


Rapidly rotating: Exchanges in Energy

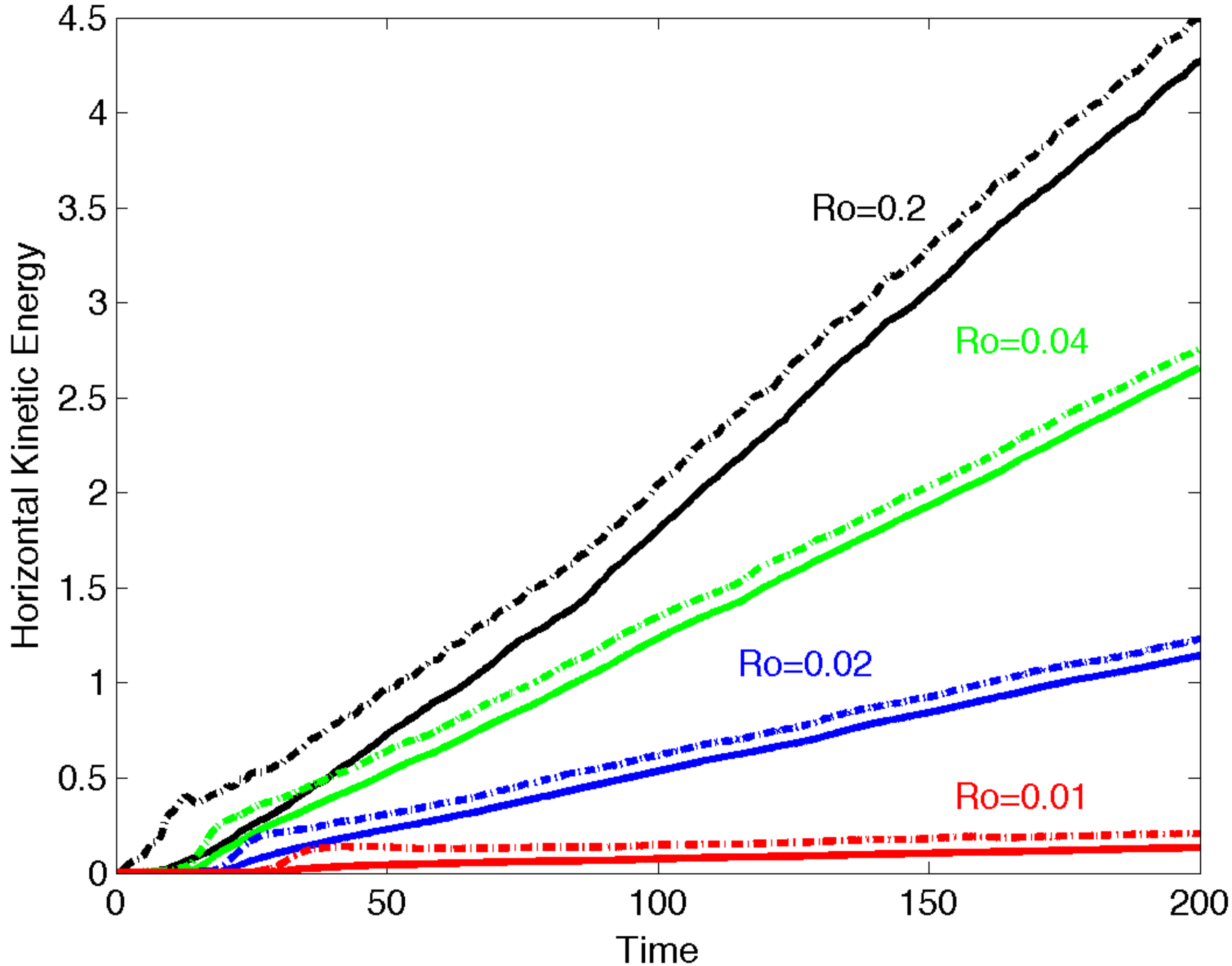
$$\frac{1}{2}\frac{d}{dt}\left\|\vec{v}_H^{Ro}\right\|_2^2 = -\int \vec{v}_H^{Ro}\cdot\langle[\vec{v}\,'\cdot\nabla\vec{v}\,']_H\rangle_z\, d\vec{x}$$

$$\frac{1}{2}\frac{d}{dt}\left\|w^{Ro}\right\|_2^2 = -\frac{1}{Fr}\int w^{Ro}\langle\rho\rangle_z\, d\vec{x} - \int w^{Ro}\langle\vec{v}_H\,'\cdot\nabla_H w'\rangle_z\, d\vec{x}$$

$$\frac{1}{2}\frac{d}{dt}\|\rho\|_2^2 = \frac{1}{Fr}\int w^{Ro}\rho\, d\vec{x} + \frac{1}{Fr}\int w'\rho\, d\vec{x}$$

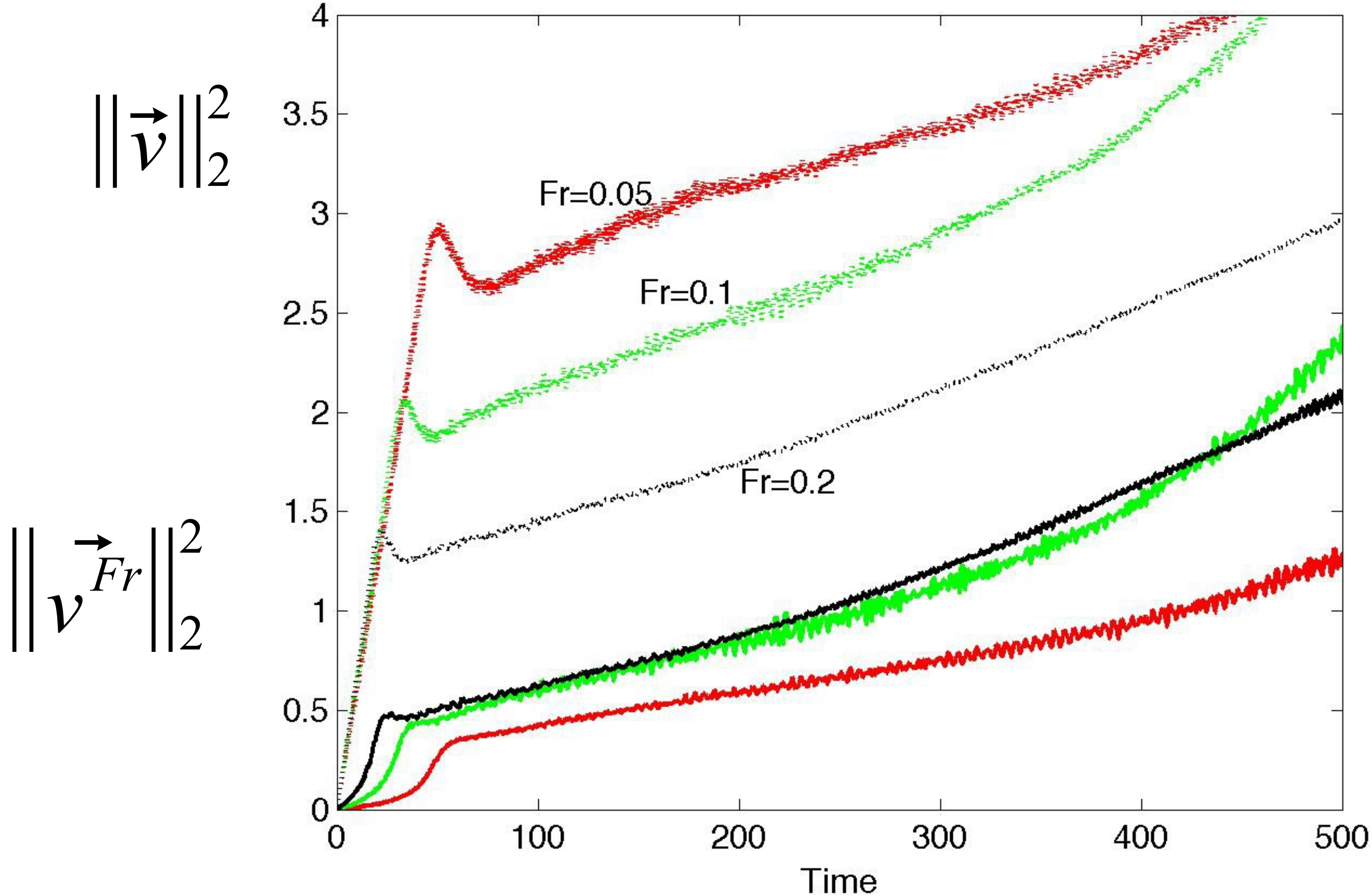


Approaching the limit of rapid rotation: where does all the energy go?



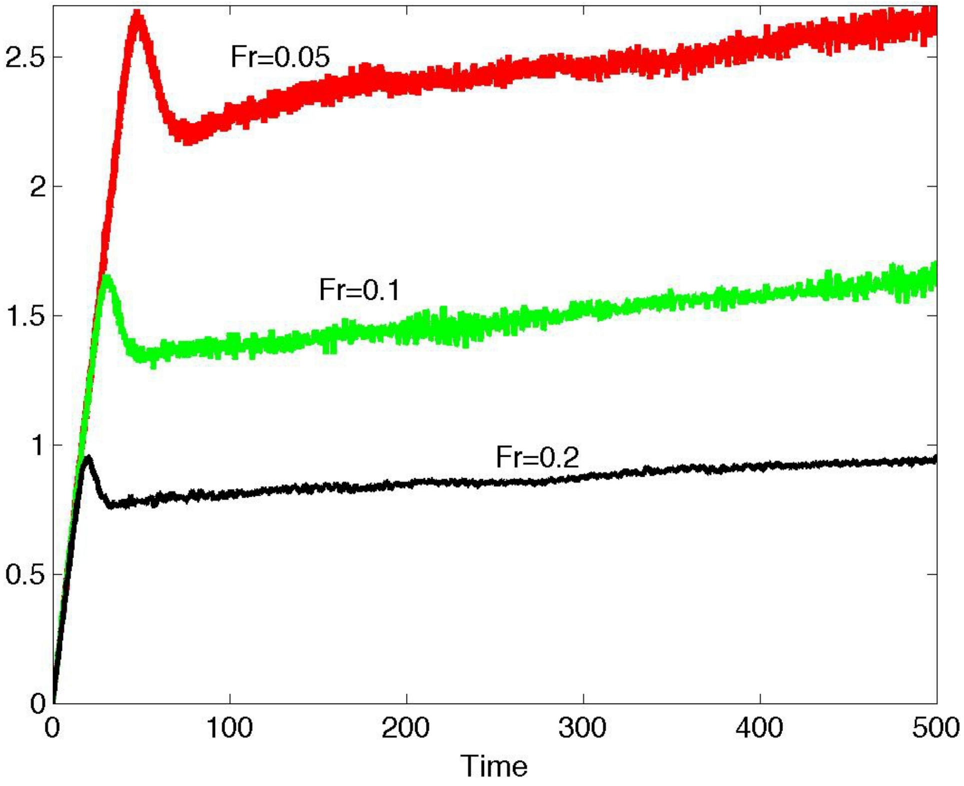
- Recall that the energy is inserted into the potential for these simulations, and then sent to both slow and fast parts of the flow.
- Yet, everything moves onto the slow manifold, even for moderately small Ro (Ro<0.5).
- Does this indicate that the slow manifold is attracting?

Strongly stratified: where does the energy go?

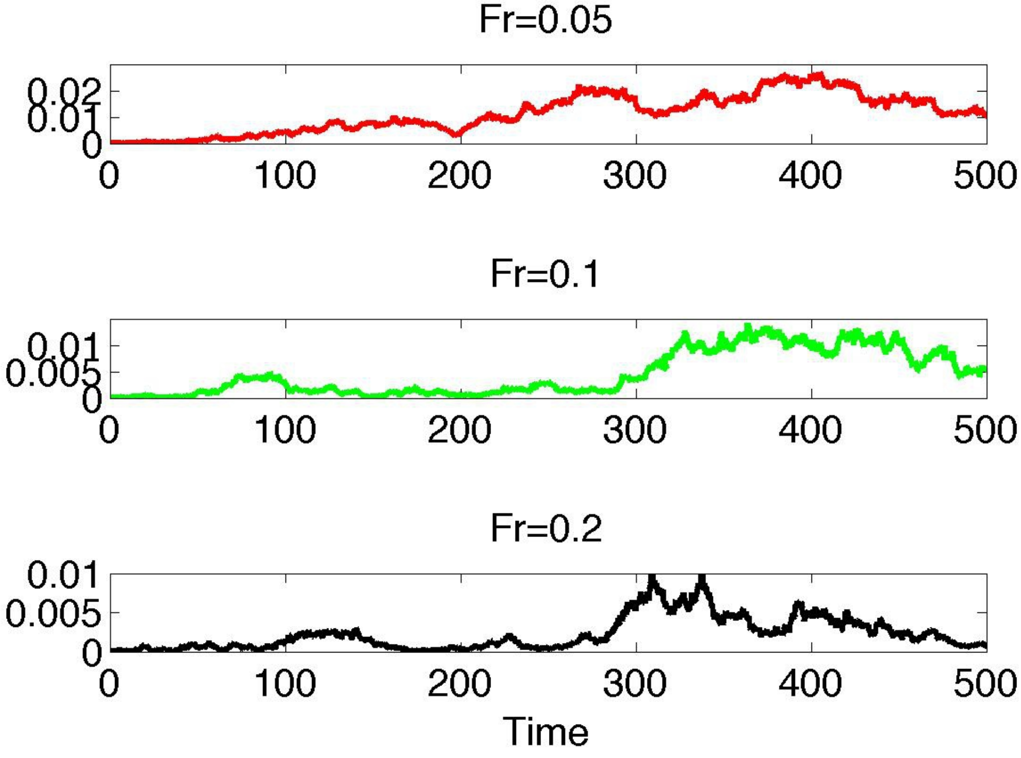


Strongly stratified: where does the energy go?

$$\|\rho\|_2^2$$



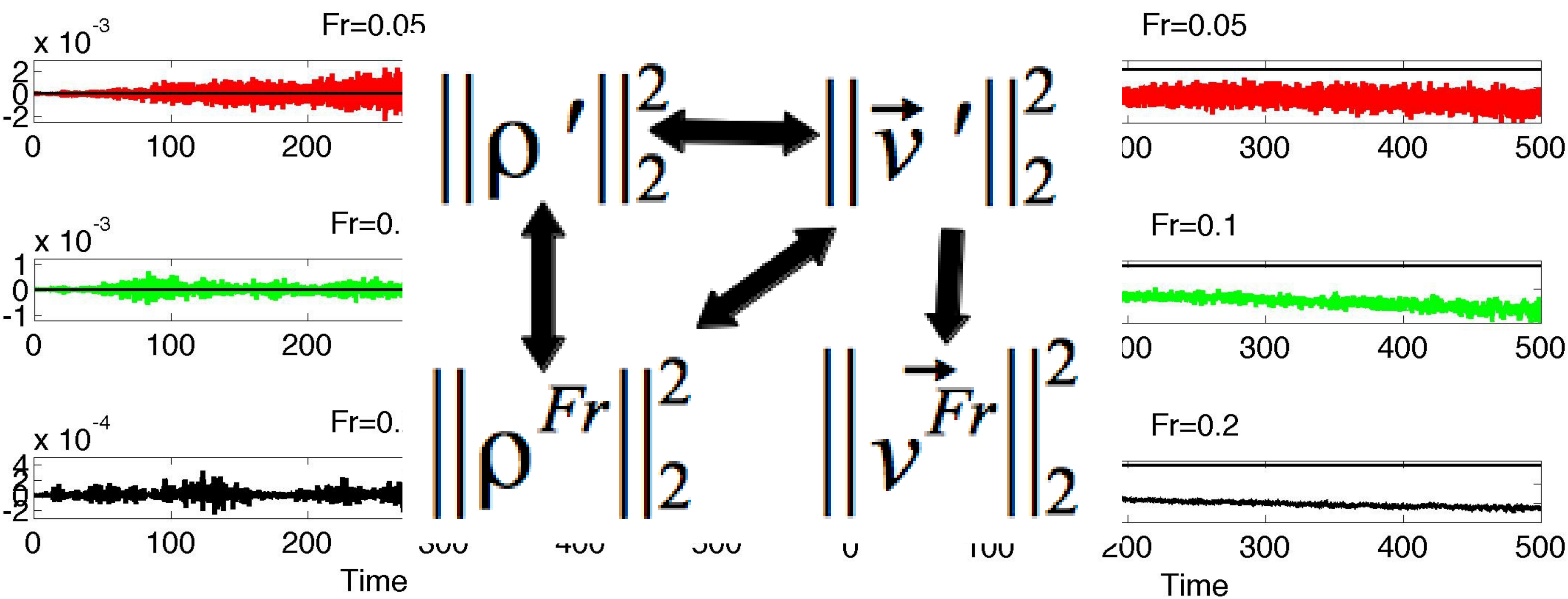
$$\left\|\rho^{Fr}\right\|_2^2$$



Strongly stratified: where does the energy go?

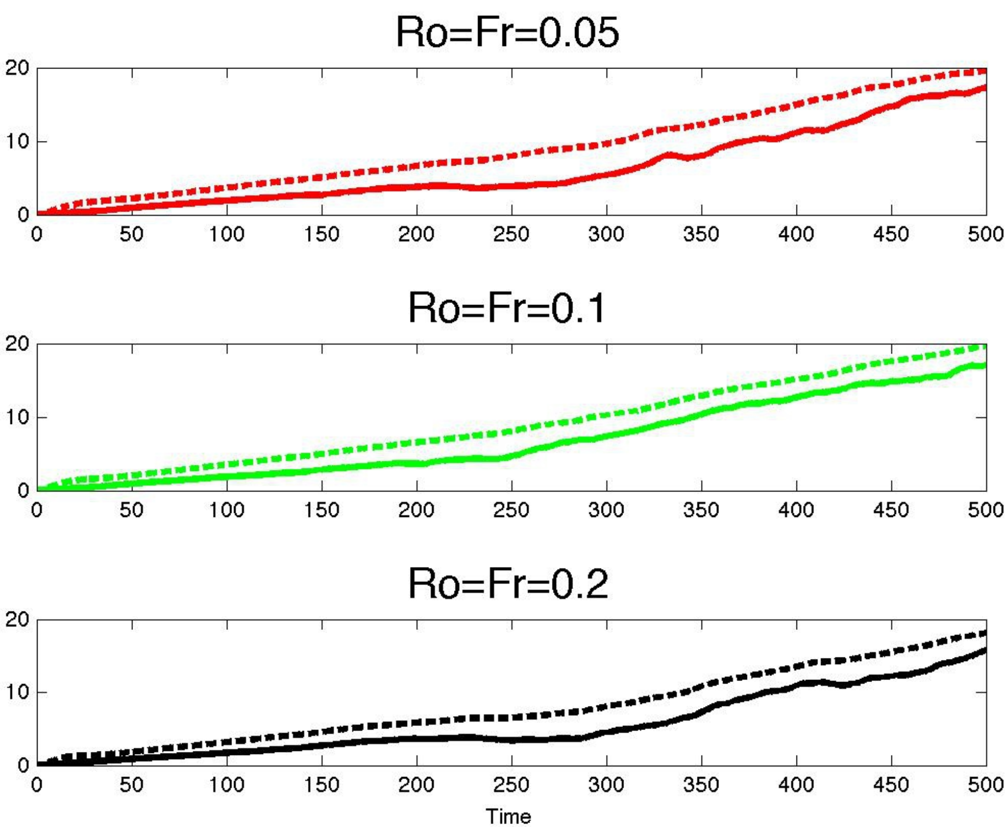
$$\frac{1}{2}\frac{d}{dt}\left\|\vec{v}_H^{Fr}\right\|_2^2 = -\int \vec{v}_H^{Fr}\cdot[\vec{v}\,'\cdot\nabla\vec{v}\,']_H\, d\vec{x}$$

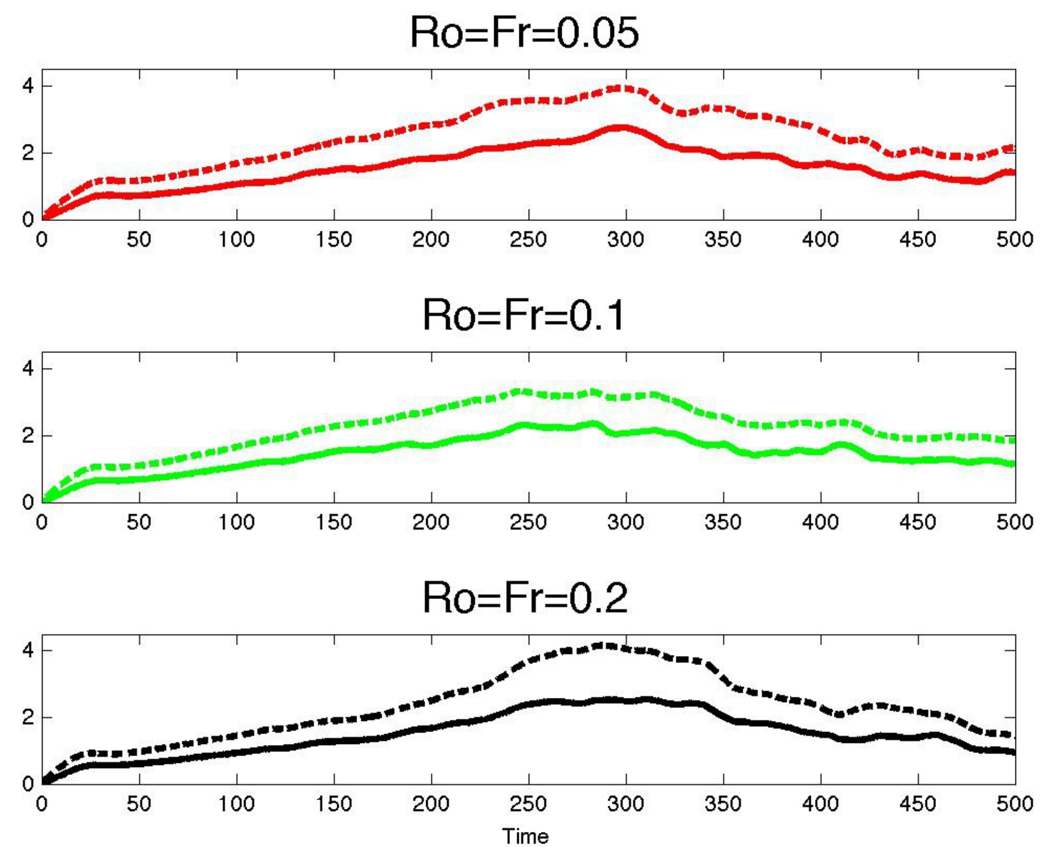
$$\frac{1}{2}\frac{d}{dt}\left\|\rho^{Fr}\right\|_2^2 = -\int \rho^{Fr}\langle\vec{v}\,'\cdot\nabla\rho\,'\rangle_H\, dz$$



Simultaneously strong stratification and rapid rotation: what about the energy?

$$\left\|\vec{v}^{QG}\right\|_2^2 \qquad \left\|\vec{v}\right\|_2^2 \qquad \left\|\rho^{QG}\right\|_2^2 \qquad \left\|\rho\right\|_2^2$$





Simultaneously strong stratification and rapid rotation: what about the energy?

This

$$\frac{1}{2}\frac{d}{dt}\left\|v_H^{\vec{Q}G}\right\|_2^2 = -\frac{Fr}{Ro}\int\left[v_H^{\vec{Q}G}\cdot\Delta_{QG}^{-1}\left[\nabla_H\times\left(\hat{z}\frac{\partial}{\partial z}\left[v_H^{\vec{Q}G}\cdot\nabla_H\rho^{QG}\right]\right)\right]\right]d\vec{x}$$

$$-\int\left[v_H^{\vec{Q}G}\cdot\left(1-\frac{Fr^2}{Ro^2}\Delta_{QG}^{-1}\frac{\partial^2}{\partial z^2}\right)\left[\vec{v}\,'\cdot\nabla\vec{v}\,'\right]_H\right]d\vec{x} - \frac{Fr}{Ro}\int\left[v_H^{\vec{Q}G}\cdot\Delta_{QG}^{-1}\left[\nabla_H\times\left(\hat{z}\frac{\partial}{\partial z}\left[\vec{v}\,'\cdot\nabla\rho\,'\right]\right)\right]\right]d\vec{x}$$

$$\frac{1}{2}\frac{d}{dt}\left\|\rho^{QG}\right\|_2^2 = \frac{Fr}{Ro}\int\left[\rho^{QG}\Delta_{QG}^{-1}\frac{\partial}{\partial z}\left(v_H^{\vec{Q}G}\cdot\nabla_H\omega^{QG}\right)\right]d\vec{x}$$

$$-\int\left[\rho^{QG}\left(1-\Delta_{QG}^{-1}\Delta_H\right)\vec{v}\,'\cdot\nabla\rho\,'\right]d\vec{x} + \frac{Fr}{Ro}\int\left[\rho^{QG}\Delta_{QG}^{-1}\left(\nabla_H\times\left[\vec{v}\,'\cdot\nabla\vec{v}\,'\right]_H\right)\right]d\vec{x}$$

is a bit of a mess, so the only message we really get is that the energy off the slow manifold does not grow in time, apparently remaining bounded.

SO WHAT?

- All 3 limits appear to have a bounded amount of energy in the 'fast' component of the flow.
- Although the forcing is applied to the density, the potential energy is bounded.
- The limit of rapid rotation and weak stratification has a very distinctive movement of energy.
- Apparently the fluctuations act as a conduit to move energy onto the 'slow manifold'.
  - Is there some connection here to the attractor?

On 'fast waves' and the slow manifold: 3 different 'slow manifolds' for 1 system?

This brings up another question:

While the O(1) fast waves cannot influence the O(1) slow dynamics, these simulations indicate that the higher order 'fast' part of the flow acts as a conduit to move energy onto the O(1) slow manifold as kinetic energy.

In his baroclinic instability study, Charney had derived a mathematically tractable equation for the unstable waves 'by eliminating from consideration at the outset the meteorologically unimportant acoustic and shearing-gravitational oscillations' [6]. The multi-scale nature of atmospheric dynamics, with low-frequency and high-frequency components, is also found in a wide range of other physical contexts. The advantages

From P. Lynch 'The origins of computer weather prediction and climate modeling', JCP 2008.

Are such waves truly meteorologically unimportant?

A reduced model that considers only the O(1) slow manifold will 'miss' the influence of the higher order 'fast' variables, which may be an important forcing of the slow system.

In other words, the part of the flow living off the O(1) slow manifold is important and provides more than just a dissipative effect.

Were we asking the wrong question before? It is true that the slow manifold can spontaneously generate fast waves, but it appears that these fast waves also have a nontrivial influence on the dynamics on the slow manifold, i.e. their influence should not be ignored.

If the fast part of the flow really matters, are these 3 limits all we need to worry about?

- We need further testing of the assumptions we used in the DNS, however beyond that...
- Ro and Fr are dependent on time and space.
- We need to understand the transitions between these limits.
- How does the slow manifold change with Ro and Fr?

Returning to the Boussinesq equations:

What if the rotation and stratification were not equally strong (fast)?

$$\frac{\partial \vec{u}}{\partial t}+\frac{1}{\delta}K\vec{u}+\frac{1}{\epsilon}L\vec{u}+B(\vec{u},\vec{u})=D\vec{u} \qquad \textit{where} \qquad \vec{u}=\begin{pmatrix}\vec{v}\\ \rho\end{pmatrix}, \qquad D\vec{u}=\begin{pmatrix}\frac{1}{\mathrm{Re}}\Delta\vec{v}\\ \frac{1}{Pr\,\mathrm{Re}}\Delta\rho\end{pmatrix}$$

$$K\vec{u}=\begin{pmatrix}\hat{z}\times\vec{v}+\nabla\Delta^{-1}\omega\\ 0\end{pmatrix}, \qquad L\vec{u}=\begin{pmatrix}\hat{z}\rho-\nabla\Delta^{-1}\left(\frac{\partial\rho}{\partial z}\right)\\ -w\end{pmatrix}, \qquad B(\vec{u},\vec{u})=\begin{pmatrix}\vec{v}\cdot\nabla\vec{v}-\nabla\Delta^{-1}(\nabla\cdot(\vec{v}\cdot\nabla\vec{v}))\\ \vec{v}\cdot\nabla\rho\end{pmatrix}$$

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LOW FROUDE NUMBER LIMITING DYNAMICS FOR STABLY STRATIFIED FLOW WITH SMALL OR FINITE ROSSBY NUMBERS

PEDRO F. EMBID[a] and ANDREW J. MAJDA[b]

[a] *Mathematics Department, The University of New Mexico, Albuquerque, New Mexico 87131, USA;* [b] *Courant Institute of Mathematical Sciences, New York University, New York 10012, USA*

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Low Rossby limiting dynamics for stably stratified flow with finite Froude number

BETH A. WINGATE[1]†, PEDRO EMBID[2], MIRANDA HOLMES-CERFON[3] AND MARK A. TAYLOR[4]

[1]MS D413, Los Alamos National Laboratory, Los Alamos, NM 87544, USA
[2]Department of Mathematics and Statistics, the University of New Mexico, Albuquerque, NM 87131, USA
[3]New York University, Courant Institute of Mathematical Sciences, New York, NY 10012, USA
[4]Sandia National Laboratories, Albuquerque, NM 87185, USA

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Singular limits of a single 'fast' time scale

Considers distinguished limits when either $\delta = O(1)$ or $\epsilon = O(1)$.

But what happens if $\epsilon \to 0$, $\delta \to 0$ and $\frac{\delta}{\epsilon} \neq O(1)$?

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Singular limits of two distinct 'fast' time scales

$$\frac{\partial \vec{u}}{\partial t}+\frac{1}{\delta}K\vec{u}+\frac{1}{\epsilon}L\vec{u}+B(\vec{u},\vec{u})=D\vec{u}$$

Allow $\delta=\epsilon^{p}$ for some integer p.

Introduce the 2 fast time scales $\tau=\frac{t}{\epsilon},\alpha=\frac{t}{\delta}$ and consider the ansatz

$$\vec{u}(t,\vec{x})=\vec{u}_0(t,\tau,\alpha,\vec{x})+\epsilon\vec{u}_1(t,\tau,\alpha,\vec{x})+O(\epsilon^2).$$

Using this ansatz, and matching terms of each order implies that:

$$O\left(\frac{1}{\epsilon^{p-q}}\right):\ \frac{\partial \vec{u}_q}{\partial \alpha}+L\vec{u}_q=0 \quad\Rightarrow\quad \vec{u}_q(t,\tau,\alpha,\vec{x})=e^{-\alpha L}\hat{\vec{u}}_q(t,\tau,\vec{x})$$

for all $q=0,\ldots,p-2$.

The final 2 terms (up to O(1)) yield:

$$O\left(\frac{1}{\epsilon}\right):\ \frac{\partial \vec{u}_{p-1}}{\partial \alpha}+L\vec{u}_{p-1}+\frac{\partial \vec{u}_0}{\partial \tau}+K\vec{u}_0=0$$

$$O(1):\ \frac{\partial \vec{u}_p}{\partial \alpha}+L\vec{u}_p+\frac{\partial \vec{u}_1}{\partial \tau}+K\vec{u}_1+\frac{\partial \vec{u}_0}{\partial t}+B(\vec{u}_{0,}\vec{u}_0)-D\vec{u}_0=0.$$

Singular limits of two distinct 'fast' time scales

$$\frac{\partial \vec{u}}{\partial t}+\frac{1}{\delta}K\vec{u}+\frac{1}{\epsilon}L\vec{u}+B(\vec{u},\vec{u})=D\vec{u}$$

$$\vec{u}(t,\vec{x})=\vec{u}_0(t,\tau,\alpha,\vec{x})+\epsilon\vec{u}_1(t,\tau,\alpha,\vec{x})+O(\epsilon^2).$$

$$\vec{u}_q(t,\tau,\alpha,\vec{x})=e^{-\alpha L}\hat{\vec{u}}_q(t,\tau,\vec{x})\text{ for all } q=0,\ldots,p-2.$$

$$O\left(\frac{1}{\epsilon}\right):\ \ \frac{\partial \vec{u}_{p-1}}{\partial \alpha}+L\vec{u}_{p-1}+\frac{\partial \vec{u}_0}{\partial \tau}+K\vec{u}_0=0$$

$$\Rightarrow \quad e^{\alpha L}\vec{u}_{p-1}=\vec{u}_{p-1}|_{\alpha=0}-\alpha\frac{\partial \hat{\vec{u}}_0}{\partial \tau}-\left(\int_0^{\alpha} e^{sL}Ke^{-sL}\,ds\right)\hat{\vec{u}}_0.$$

To ensure the ansatz remains valid, we must force $\vec{u}_{p-1}$ to not have secular growth in α.

This leads to the requirement that $\frac{\partial \hat{\vec{u}}_0}{\partial \tau}=-\left(\frac{1}{\alpha}\int_0^{\alpha} e^{sL}Ke^{-sL}\,ds\right)\hat{\vec{u}}_0.$

Singular limits of two distinct 'fast' time scales

$$\frac{\partial \vec{u}}{\partial t}+\frac{1}{\delta}K\vec{u}+\frac{1}{\epsilon}L\vec{u}+B(\vec{u},\vec{u})=D\vec{u}$$

$$\vec{u}(t,\vec{x})=\vec{u}_0(t,\tau,\alpha,\vec{x})+\epsilon\vec{u}_1(t,\tau,\alpha,\vec{x})+O(\epsilon^2).$$

$$\vec{u}_q(t,\tau,\alpha,\vec{x})=e^{-\alpha L}\hat{\vec{u}}_q(t,\tau,\vec{x})\text{ for all } q=0,\ldots,p-2.$$

$\displaystyle\frac{\partial \hat{\vec{u}}_0}{\partial \tau}=-\left(\frac{1}{\alpha}\int_0^{\alpha} e^{sL}Ke^{-sL}ds\right)\hat{\vec{u}}_0$ needs to be true to avoid secular growth, particularly for large α, i.e.

$$\hat{\vec{u}}_0(t,\tau,\vec{x})=e^{-\tau\overline{M}}\overline{\vec{u}_0}(t,\vec{x})\text{ where }\overline{M}=\lim_{\alpha\to\infty}M(\alpha)$$

$$\text{and } M(\alpha)=\frac{1}{\alpha}\int_0^{\alpha} e^{sL}K e^{-sL}ds.$$

Inserting this back into the $O\left(\frac{1}{\epsilon}\right)$ equation leads to

$$\vec{u}_{p-1}(t,\tau,\alpha,\vec{x})=e^{-\alpha L}\hat{\vec{u}}_{p-1}(t,\tau,\vec{x})+\alpha e^{-\alpha L}\left(\overline{M}-M(\alpha)\right)e^{-\tau\overline{M}}\overline{\vec{u}_0}(t,\vec{x})$$

Singular limits of two distinct 'fast' time scales

$$\frac{\partial \vec{u}}{\partial t}+\frac{1}{\delta}K\vec{u}+\frac{1}{\epsilon}L\vec{u}+B(\vec{u},\vec{u})=D\vec{u}$$

$$\vec{u}(t,\vec{x})=\vec{u}_0(t,\tau,\alpha,\vec{x})+\epsilon\vec{u}_1(t,\tau,\alpha,\vec{x})+O(\epsilon^2).$$

$$\vec{u}_q(t,\tau,\alpha,\vec{x})=e^{-\alpha L}\hat{\vec{u}}_q(t,\tau,\vec{x})\text{ for all } q=0,\ldots,p-2.$$

$$\vec{u}_0(t,\tau,\alpha,\vec{x})=e^{-\alpha L}e^{-\tau\overline{M}}\overline{\vec{u}_0}(t,\vec{x})$$

$$\vec{u}_{p-1}(t,\tau,\alpha,\vec{x})=e^{-\alpha L}\hat{\vec{u}}_{p-1}(t,\tau,\vec{x})+\alpha e^{-\alpha L}\left(\overline{M}-M(\alpha)\right)e^{-\tau\overline{M}}\overline{\vec{u}_0}(t,\vec{x})$$

$$O(1)\text{ leads to :}$$

$$e^{\alpha L}\vec{u}_p=\hat{\vec{u}}_p-\alpha\frac{\partial\hat{\vec{u}}_1}{\partial\tau}-\alpha M(\alpha)\hat{\vec{u}}_1-\left(\int_0^\alpha e^{sL}e^{-\tau\overline{M}}e^{-sL}ds\right)\frac{\partial\overline{\vec{u}_0}}{\partial t}$$

$$-\int_0^\alpha e^{sL}B\left(e^{-\tau\overline{M}}e^{-sL}\overline{\vec{u}_{0,}}e^{-\tau\overline{M}}e^{-sL}\overline{\vec{u}_0}\right)ds+\left(\int_0^\alpha e^{sL}De^{-\tau\overline{M}}e^{-sL}ds\right)\overline{\vec{u}_0}$$

Singular limits of two distinct 'fast' time scales

$O(1)$ leads to :

$$e^{\alpha L}\vec{u}_p=\hat{\vec{u}}_p-\alpha\frac{\partial\hat{\vec{u}}_1}{\partial\tau}-\alpha M(\alpha)\hat{\vec{u}}_1-\left(\int_0^\alpha e^{sL}e^{-\tau\overline{M}}e^{-sL}ds\right)\frac{\partial\overline{\vec{u}_0}}{\partial t}$$

$$-\int_0^\alpha e^{sL}B\left(e^{-\tau\overline{M}}e^{-sL}\overline{\vec{u}_0},e^{-\tau\overline{M}}e^{-sL}\overline{\vec{u}_0}\right)ds+\left(\int_0^\alpha e^{sL}De^{-\tau\overline{M}}e^{-sL}ds\right)\overline{\vec{u}_0}$$

To avoid secular growth of $\vec{u}_p$ in α :

$$e^{\tau\overline{M}}\hat{\vec{u}}_1=\overline{\vec{u}_1}-\tau\frac{\partial\overline{\vec{u}_0}}{\partial t}-\int_0^\tau e^{\beta\overline{M}}\left[\lim_{\alpha\to\infty}\frac{1}{\alpha}\int_0^\alpha e^{sL}B\left(e^{-\beta\overline{M}}e^{-sL}\overline{\vec{u}_0},e^{-\beta\overline{M}}e^{-sL}\overline{\vec{u}_0}\right)ds\right]d\beta$$

$$+\left(\int_0^\tau e^{\beta\overline{M}}\left[\lim_{\alpha\to\infty}\frac{1}{\alpha}\int_0^\alpha e^{sL}De^{-\beta\overline{M}}e^{-sL}ds\right]d\beta\right)\overline{\vec{u}_0}.$$

Where this is highly dependent on the fact that $\overline{M}$ and L commute.

Singular limits of two distinct 'fast' time scales

Avoiding secular growth of $\vec{u}_p$ in α

and $\hat{\vec{u}}_1$ in τ leads to the O(1) slow evolution equation:

$$\frac{\partial \overline{\vec{u}_0}}{\partial t} = -\lim_{\tau\to\infty} \lim_{\alpha\to\infty} \frac{1}{\tau\alpha} \int_0^\tau \int_0^\alpha e^{\beta \overline{M}} e^{sL} B\left(e^{-\beta\overline{M}} e^{-sL} \overline{\vec{u}_0}, e^{-\beta\overline{M}} e^{-sL} \overline{\vec{u}_0}\right) ds\, d\beta + \overline{D}\, \overline{\vec{u}_0}$$

$$\text{where } \overline{D} = \lim_{\tau\to\infty} \lim_{\alpha\to\infty} \frac{1}{\tau\alpha} \int_0^\tau \int_0^\alpha e^{\beta\overline{M}} e^{sL} D\, e^{-\beta\overline{M}} e^{-sL} ds\, d\beta .$$

The same result can be achieved via the renormalization method but the algebra is far more complicated.

This is a double averaging over both fast time scales, but the order is important.

This works for $\delta = \epsilon^{\frac{p}{q}}$

where $p > q$ by a simple re-definition of the parameters, i.e.

this yields the O(1) slow equations whenever $\delta \to 0$ at a faster rate than $\epsilon \to 0$.

Singular limits of two distinct 'fast' time scales

$$\frac{\partial \vec{u}}{\partial t} + \frac{1}{\delta} K \vec{u} + \frac{1}{\epsilon} L \vec{u} + B(\vec{u}, \vec{u}) = D \vec{u}$$

$$\frac{\partial \overline{\vec{u}_0}}{\partial t} = -\lim_{\tau \to \infty} \lim_{\alpha \to \infty} \frac{1}{\tau \alpha} \int_0^{\tau} \int_0^{\alpha} e^{\beta \overline{M}} e^{sL} B\left(e^{-\beta \overline{M}} e^{-sL} \overline{\vec{u}_{0,}}\, e^{-\beta \overline{M}} e^{-sL} \overline{\vec{u}_0} \right) ds\, d\beta + \overline{D} \overline{\vec{u}_0}$$

$$\text{where } \overline{D} = \lim_{\tau \to \infty} \lim_{\alpha \to \infty} \frac{1}{\tau \alpha} \int_0^{\tau} \int_0^{\alpha} e^{\beta \overline{M}} e^{sL} D e^{-\beta \overline{M}} e^{-sL} ds\, d\beta .$$

Using a change of variables motivated by this multiple time scale approach, we can use Schochet's theory of cancellation of oscillations to show that the solution to this system (under suitable restrictions on the nonlinearity and regularity of the solution) satisfies:

$$\vec{u}(t, \vec{x}) = e^{\frac{-t}{\epsilon} \overline{M}} e^{\frac{-t}{\delta} L} \overline{\vec{u}_0}(t, \vec{x}) + o(1)$$

so long as the limits $\delta \to 0$ and $\epsilon \to 0$ can be taken consecutively, where $\delta \to 0$ prior to $\epsilon \to 0$.

Returning to the rotating, stratified Boussinesq system: the limiting system, i.e. SO WHAT?

Rapid rotation with weak stratification

$$\frac{\partial \vec{v}_H^{Ro}}{\partial t} + \vec{v}_H^{Ro} \cdot \nabla_H \vec{v}_H^{Ro} + \nabla_H p^{Ro} = \frac{1}{\mathrm{Re}} \Delta_H \vec{v}_H^{Ro},$$

$$\nabla_H \cdot \vec{v}_H^{Ro} = 0,$$

$$\frac{\partial w^{Ro}}{\partial t} + \vec{v}_H^{Ro} \cdot \nabla_H w^{Ro} + \frac{1}{Fr} \langle \rho \rangle_z = \frac{1}{\mathrm{Re}} w^{Ro},$$

$$\frac{\partial \rho}{\partial t} + \vec{v}^{Ro} \cdot \nabla \rho - \frac{1}{Fr} w^{Ro} = \frac{1}{\mathrm{Re}\, Pr} \Delta \rho,$$

$$\vec{v}_H^{Ro} = \langle \vec{v}_H \rangle_z, \qquad w^{Ro} = \langle w \rangle_z.$$

Rapid rotation dominating strong stratification

$$\frac{\partial \vec{v}_H^{S}}{\partial t} + \vec{v}_H^{S} \cdot \nabla \vec{v}_H^{S} + \nabla_H p^{S} = \frac{1}{\mathrm{Re}} \Delta \vec{v}_H^{S},$$

$$\nabla_H \cdot \vec{v}_H^{S} = 0,$$

$$\frac{\partial w^{S}}{\partial t} + \vec{v}_H^{S} \cdot \nabla w^{S} = \frac{1}{\mathrm{Re}} \Delta_H w^{S},$$

$$\frac{\partial \rho^{S}}{\partial t} + \vec{v}^{S} \cdot \nabla \rho^{S} = \frac{1}{\mathrm{Re}\, Pr} \Delta \rho^{S},$$

$$\vec{v}_H^{S} = \langle \vec{v}_H \rangle_z, \qquad w^{S} = \langle w \rangle_z - \langle w \rangle, \qquad \rho^{S} = \rho - \langle \rho \rangle.$$

Returning to the rotating, stratified Boussinesq system: the limiting system, i.e. SO WHAT?

Strong stratification with weak rotation

$$\frac{\partial \vec{v}_H^{Fr}}{\partial t}+\vec{v}_H^{Fr}\cdot\nabla_H \vec{v}_H^{Fr}+\frac{1}{Ro}\hat{z}\times\vec{v}_H^{Fr}+\nabla_H p^{Fr}=\frac{1}{\mathrm{Re}}\Delta\vec{v}_H^{Fr},$$

$$\nabla_H\cdot\vec{v}_H^{Fr}=0,$$

$$\frac{\partial \rho^{Fr}}{\partial t}=\frac{1}{\mathrm{Re}\,Pr}\frac{\partial^2\rho^{Fr}}{\partial z^2},$$

$$\rho^{Fr}=\langle\rho\rangle_H.$$

Strong stratification dominating rapid rotation

$$\frac{\partial \vec{v}_H^{S}}{\partial t}+\vec{v}_H^{S}\cdot\nabla_H \vec{v}_H^{S}+\nabla_H p^{S}=\frac{1}{\mathrm{Re}}\Delta\vec{v}_H^{S},$$

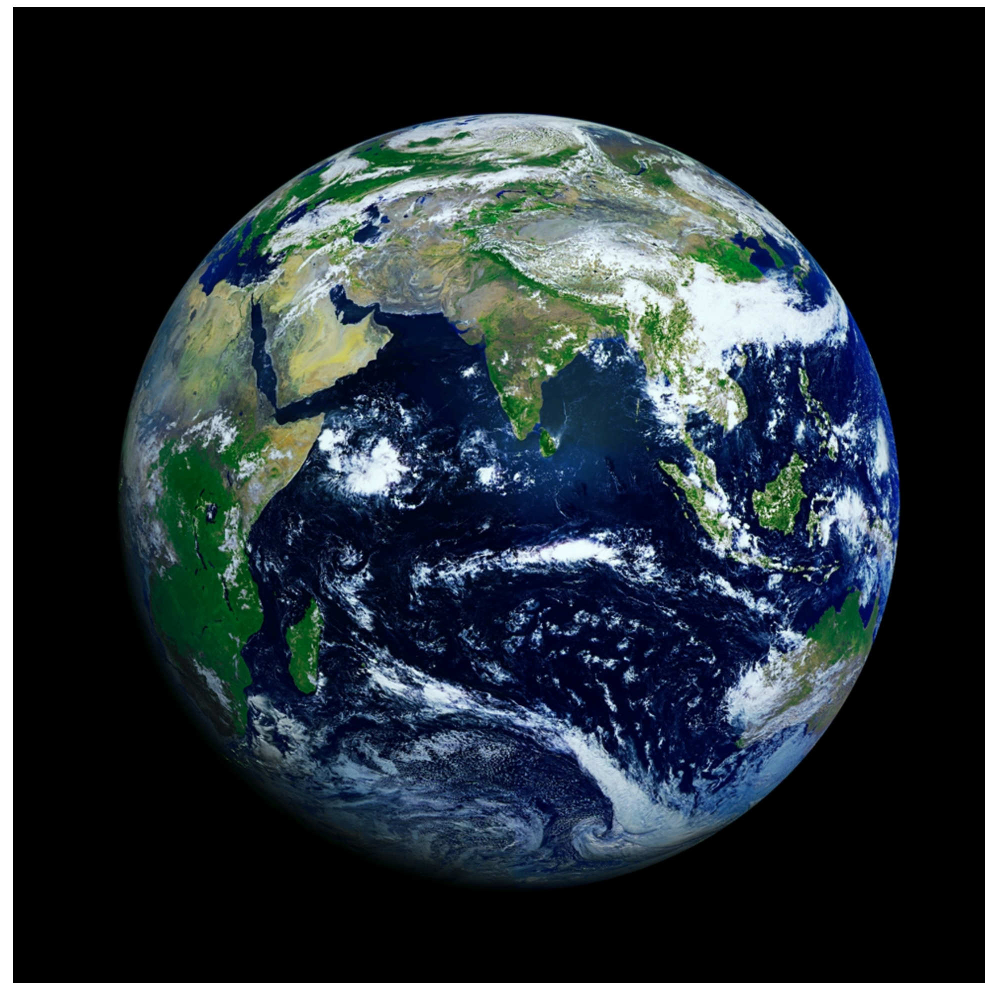
$$\nabla_H\cdot\vec{v}_H^{S}=0,$$

$$\frac{\partial \rho^{S}}{\partial t}=\frac{1}{\mathrm{Re}\,Pr}\frac{\partial^2\rho^{S}}{\partial z^2},$$

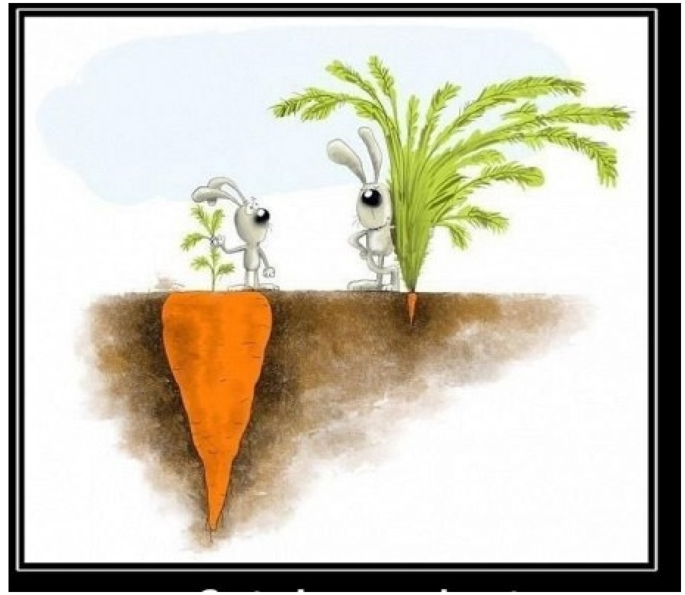
$$\vec{v}_H^{S}=\vec{v}_H-\langle v_H\rangle-\langle\vec{v}_H\rangle_H,\qquad \rho^{S}=\langle\rho\rangle_H.$$

Singular limits of two distinct 'fast' time scales

In any case, whether rotation dominates stratification or stratification dominates rotation, **the O(1) slow limiting system is NOT quasi-geostrophy**. Geophysically there is more going on, particularly when dealing with the entire globe.



At the end of the day



- The invariance (or lack thereof) of the slow manifold goes 2 ways:
  - The slow manifold can spontaneously generate fast waves off of it.
  - Fast waves may act as a source for the dynamics on the slow manifold, and should not be ignored!
- When more than one fast time scale is present, there are several limiting systems (slow manifolds). Not everything is QG!

Thank You!

Approaching the limit of Fast Rotation: what does it look like?