On the size of the Navier – Stokes singular set

Walter Craig
McMaster University

Center for Nonlinear Analysis Seminar
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Joint work with:

Maxim Arnold	University of Illinois Urbana – Champaign
Andrei Biryuk	Kuban State University, Krasnodar

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Outline

- Initial value problem for the Navier – Stokes equations
- The singular sets $S(u)$ and the energy concentration set $S^{L^2}(u)$
- Three estimates on Leray weak solutions
- Phase space volume: lower bounds on $WF(u)$ and $WF^{L^2}(u)$
- Ideas of the proof:
  - new a priori estimate on the Fourier transform
  - microlocal analysis
  - defect measures

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- Navier – Stokes equations for incompressible viscous fluids

$$\begin{aligned} &\partial_t u + (u \cdot \nabla)u = -\nabla p + \nu \Delta u \\ &\nabla \cdot u = 0 \end{aligned} \tag{1}$$

- For $t = 0$ specify an initial velocity field $u_0(x)$, $\nabla \cdot u_0 = 0$
Finite energy

$$\begin{aligned} e(u_0) &:= \frac{1}{2} \int |u_0(x)|^2 \, dx \\ &= \frac{1}{2} \|u_0\|_{L^2}^2 < +\infty \end{aligned}$$

- Space-time domain

$$D = \mathbb{R}^3 \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R}^+ := Q$$

Alternatively

$$D = \mathbb{T}^3 \qquad (x,t) \in \mathbb{T}^3 \times \mathbb{R}^+$$

A domain $D \subseteq \mathbb{R}^3$ with smooth boundary – we leave this open.

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- Question: do solutions exist?

- Answer, *yes* in some sense.

  If no singularities are formed, then *yes*.

  Solutions exist, they are unique, and the mathematical theory is satisfactory

  If singularities form, then weak solutions exist. However they may not be unique, they exhibit infinite velocities, and the theory is less than satisfactory

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Leray weak solutions

The usual definition of a weak solution over $t \in [0, T]$:

1. The pair $(u(x,t), p(x,t))$ is a solution of (1) in the sense of distributions

2. Integrability conditions Initial energy $e(u_0) := \frac{1}{2}R^2 < +\infty$

$$\frac{1}{2}\int |u(x,t)|^2\,dx < +\infty \tag{2}$$

$$\nu \int_0^T \int_D |\nabla u(x,t)|^2\,dxdt < +\infty$$

$$\iint_{loc} |p|^{5/3}\,dxdt < +\infty \qquad (Sohr\ \&\ von Wahl\ (1986))$$

3. The energy inequality is satisfied

$$\frac{1}{2}\int_D |u(x,T)|^2\,dx + \nu \int_0^T \int_D |\nabla u(x,t)|^2\,dxdt \le \frac{1}{2}\int_D |u_0(x)|^2\,dx$$

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Given $u_0 \in L^2(D)$ *divergence free, then there exists at least one weak solution to* (1) *globally in time. Weak solutions satisfy*

$$u \in L_t^\infty(L_x^2) \cap L_t^2(\dot{H}_x^1) \quad p \in L_{loc}^{5/3}(Q) \tag{3}$$

A lot is known about such solutions, including weak continuity

$$u \in C_t(L_x^2 : \text{weak} \;\; \text{topology})$$

as well as

$$u \in L_t^s(L_x^p)\,, \qquad \frac{3}{p} + \frac{2}{s} = \frac{3}{2} \qquad 2 \le p \le 6$$

Uniqueness and global regularity are unknown

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Given a weak solution (u, p) of (1), the singular set $S(u)$ is the set of space-time points at which $u(x, t)$ is not locally bounded.

That is, $(x_0, t_0) \notin S(u)$ if there is a neighborhood $Q_r := Q_r(x_0, t_0)$ such that $u(x, t)$ is bounded in Q_r

Hence $S(u)$ is a closed set

This makes sense due to a theorem of Serrin (1962) which states that if $(x_0, t_0) \notin S(u)$, then for all k (and with some $0 < \alpha < 1$)

$$\partial_x^k u(x, t) \in C^\alpha(Q_{r/2}(x_0, t_0)) \tag{4}$$

Serrin's condition is actually $u \in L_t^s(L_x^p)(Q_r(x_0, t_0))$ for $\frac{3}{p} + \frac{2}{s} < 1$

- Improved by Struwe (1995) to equality, with $s < \infty$
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Upper bounds on the singular set

- Singular times

Theorem (Leray, Foiaş & Temam)

The set of *singular times* $\tau(u) = \pi_t S(u) \in \mathbb{R}^+$ *has zero* $1/2$*-Hausdorff dimensional measure*

$$\mathcal{H}^{1/2}(\tau(u)) = 0 \tag{5}$$

- Partial regularity

Theorem (Caffarelli, Kohn & Nirenberg (1982))

If (u, p) *is a* *suitable* *weak solution of* (1) *then the parabolic one-dimensional Hausdorff measure of* $S(u)$ *is zero;*

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Hausdorff dimension

- Definition (Hausdorff dimension)

  Cover a set S with balls B_{r_j} of radii $r_j < \delta$. The β-dimensional Hausdorff measure of S is

  $$\mathcal{H}^\beta(S) := \liminf_{\delta \to 0} \sum_j r_j^\beta$$

  The Hausdorff dimension of S is the infimum of β such that $\mathcal{H}^\beta(S) = 0$

  - The parabolic Hausdorff dimension is the same, however using parabolic cylinders Q_r for space-time

    $$Q_r(x_0, t_0) := \{(x, t) \,:\, |x_0 - x| < r\,, 0 < t_0 - t < r^2\}$$

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Homogeneous (or box counting) dimension

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  Given a closed set S, consider C_0^∞ cutoff functions $0 \leq \varphi_\varepsilon \leq 1$ such that on an ε-tubular neighborhood $o_\varepsilon(S)$ of S

  $$\varphi_\varepsilon(x) = 1$$

  Then the homogeneous dimension of S is

  $$D(S) := d - \liminf_{\varepsilon \to 0} \frac{\log(\int \varphi_\varepsilon)}{\log(\varepsilon)}$$

  - This is to say that

    $$\int \varphi_\varepsilon \, dx \sim \varepsilon^{d - D(S)}$$

  - The main lemma of the Caffarelli Kohn Nirenberg theorem also implies that $D(S) \leq 1$

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Restrict to a time slice

- For t_0 fixed, the singular set $S_{t_0} := S(u) \cap \{t = t_0\}$ in each time slice is at most one-dimensional, and

$$\mathcal{H}^1(S_{t_0}) = 0 \tag{7}$$

- Suitable weak solutions are those satisfying a local energy inequality

$$\begin{aligned}\int_D \tfrac{1}{2}|u(\cdot,t)|^2\varphi\,dx\Big|_{t=0}^{T} + \nu\int_0^T\int_D |\nabla u(\cdot,t)|^2\varphi\,dxdt \\ \le \tfrac{1}{2}\int_0^T\int_D |u(\cdot,t)|^2\Big(\partial_t\varphi + \nu\Delta\varphi\Big)\,dxdt \\ + \int_0^T\int_D\Big(p + \tfrac{1}{2}|u(\cdot,t)|^2\Big)u\cdot\nabla\varphi\,dxdt\end{aligned} \tag{8}$$

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Theorem 1: lower bounds in phase space

Theorem (2011)

If $t_0 \in \tau(u)$ *is a singular time for* u *then*

$$\mathrm{Dim}\,(WF(u)) \geq \tfrac{1}{2} \tag{9}$$

- dimension comparison: $S(u) \cap \{t = t_0\} := S_{t_0}(u)$ is a subset of $\mathbb{R}^3$ while the wave front set $WF(u) \subseteq T^*(\mathbb{R}^3)$ can be considered as a subset of $S^*(\mathbb{R}^3)$, which is 5 dimensional
- This lower bound is essentially valid fiber-wise $WF_{x_0} \subseteq T^*_{x_0}(D)$, for each fiber for which $(x_0, t_0) \in S_{t_0}(u)$

Phase space dimension of $WF(u)$

- Consider cutoff symbols $0 \leq a(x,\xi) \leq 1$ in $S^0_{\rho\delta}$ such that and $a(x,\xi) = 1$ on $WF(u) \cap B_r(x_0)$ and

$$(1 - a(x,D))u \in C_t(C_x^\infty)(Q_r(x_0,t_0))$$

- The volume growth of $\mathrm{supp}\,(a)$ gives an upper bound on the phase space neighborhood of $WF(u)_{x_0} \subseteq T^*_{x_0}(D)$ supporting the singularity

$$\mathrm{vol}\left(\pi_\xi \mathrm{supp}\,(a) \cap B_R(0)\right) \sim R^{1+\beta}$$

Definition ($\mathrm{Dim} WF(u)_{x_0}$)

$$\bar{\beta}_{x_0}(u) := \liminf_{r,a}(\beta) \tag{10}$$

Three inequalities

1. The energy inequality

$$\frac{1}{2}\int_D |u(x,T)|^2\,dx + \nu \int_0^T \int_D |\nabla u(x,t)|^2\,dxdt \le \frac{1}{2}\int_D |u_0(x)|^2\,dx$$

   and its consequences under interpolation

2. Foiaş, Guillopé & Temam (1981), Chemin (2004)

$$\int_0^T \|u(\cdot,t)\|_{L^\infty}\,dt < +\infty\,, \qquad \forall T \in \mathbb{R}^+ \tag{11}$$

3. Theorem (Biryuk & C. (2009))

Let $B_R(0) \subseteq L^2(D)$ and define

$$A_{R_1} := \{(\hat{u}(\xi))_{\xi\in\mathbb{R}^d} \,:\, |\xi||\hat{u}(\xi)| < R_1\} \tag{12}$$

If $R^2/\sqrt{2\pi}^3 < \nu R_1$ then $A_{R_1} \cap B_R(0)$ is a (future) invariant set for Navier – Stokes flow

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- That is to say, if initial data satisfies $|\hat{u}_0(\xi)| < \frac{R_1}{|\xi|}$ then for all time $t > 0$

$$|\hat{u}(\xi, t)| < \frac{R_1}{|\xi|}\,, \qquad \forall \xi \tag{13}$$

- Corollary (Biryuk & C (2009))

If the initial data satisfies $|\hat{u}_0(\xi)| < \frac{R_1}{|\xi|}$, then additionally for all $T \geq 0$

$$\nu \int_0^T |\hat{u}(\xi, t)|^2 \, dt \leq \frac{R_2^2}{|\xi|^4} \tag{14}$$

- Proof of theorem and corollary given at end of talk (if there is time)

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- Corollary (Biryuk & C (2009))

If the initial data satisfies $|\hat{u}_0(\xi)| < \frac{R_1}{|\xi|}$, then additionally for all $T \geq 0$

$$\nu \int_0^T |\hat{u}(\xi, t)|^2 \, dt \leq \frac{R_2^2}{|\xi|^4} \tag{14}$$

- Proof of theorem and corollary given at end of talk (if there is time)

  The quantity $\sup_t \| |\xi| \hat{u}(\xi, t) \|_{L^\infty}$ scales like the BV norm $\sup_t \|\partial_x u(\cdot, t)\|_{L^1}$ (for which there are no known bounds).

Energy discontinuities

- The energy $e(u(\cdot,t)) = \frac{1}{2}\|u(\cdot,t)\|_{L^2}^2$ could be discontinuous at $t_0 \in \tau(u)$.

  Or else it may be continuous (but nonetheless $\|\nabla u(\cdot,t)\|_{L^2}^2$ is necessarily unbounded on $[t_0 - \delta, t_0]$ for any δ)

- Decompose the set of singular times $\tau(u)$ into

$$\tau(u) := \tau_1 \cup \tau_2$$

  where τ_1 are the energy discontinuities

$$\tau_1 := \{t_0 : \limsup_{t \to t_0^-} \|u(\cdot,t)\|_{L^2}^2 > \|u(\cdot,t_0)\|_{L^2}^2\}$$

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The energy concentration set S^{L^2}

- Suppose that $t_0 \in \tau_1$; that is, $u(x,t)$ is not strong L^2 continuous for $t \mapsto t_0^-$.

Definition (L^2 concentration set)

The point $x_0 \notin S_{t_0}^{L^2}$ if there exists $r > 0$ such that

$$\lim_{t \to t_0^-} \|u(\cdot,t)\|^2_{L^2(B_r(x_0))} = \|u(\cdot,t_0)\|^2_{L^2(B_r(x_0))} \tag{15}$$

Thus L^2 concentration is associated with a point set $S_{t_0}^{L^2} \in \mathbb{R}^3$

- The set $S_{t_0}^{L^2}$ is closed
  Indeed norm convergence plus weak convergence implies strong convergence, and any $B_{r_1}(x_1) \subseteq B_r(x_0)$ shares this strong convergence
- Since $u(\cdot,t)$ is smooth outside the singular set $S_{t_0}(u)$,

$$S_{t_0}^{L^2} \subseteq S_{t_0}(u) \tag{16}$$

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Theorem 2: lower bounds on the energy concentration set

- A lower bound on the size of the energy concentration wave front set $WF^{L^2}(u)$ when $t_0 \in \tau_1$

Theorem (Arnold & C. (2010))

If $t_0 \in \tau_1$ is an energy discontinuity for u then

$$\mathrm{Dim}\left(WF^{L^2}(u)\right) \geq 1 \tag{17}$$

- Remark: $S_{t_0}^{L^2} \subseteq S_{t_0}$, and as well $WF^{L^2} \subseteq WF$, so that whenever $x_0 \in S_{t_0}^{L^2}$ then Theorem 2 implies Theorem 1 with

$$\mathrm{Dim}\left(WF(u)\right) \geq 1$$

However Theorem 1 applies in the case that $x_0 \in S_{t_0} \backslash S_{t_0}^{L^2}$

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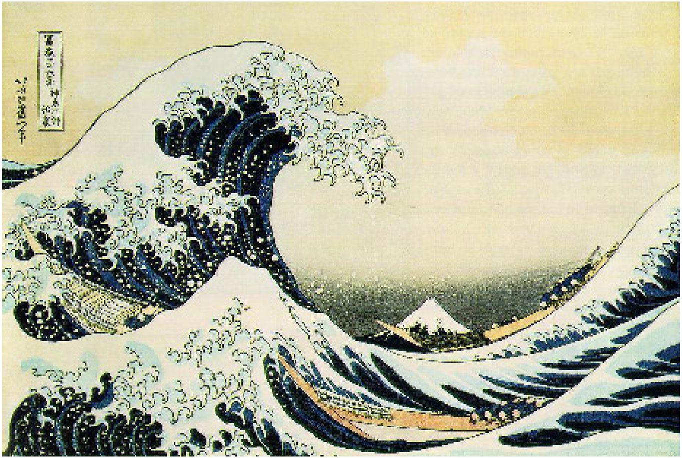
However Theorem 1 applies in the case that $x_0 \in S_{t_0} \backslash S_{t_0}^{L^2}$

Outline of the proof of Theorem 2

- Probe the solution with Weyl calculus pseudodifferential operators
- Identify the L^2 discontinuities with defect measures, and $WF^{L^2}(u) \subseteq WF(u)$ with their support.
  This identifies the microlocal L^2 concentration set $WF^{L^2}(u)$ as a geometric subset of $T^*(\mathbb{R}^3)$
- Define the dimension of the sets $WF(u)$ and $WF^{L^2}(u)$ using symbol classes $S^0_{\rho\delta}$, for $0 < \rho \leq 1$
- If the solution concentrates onto $WF^{L^2}(u)$ on a set which is too small, argue by contradiction, using the third estimate (13) and the Lebesgue dominated convergence theorem



Thank you

proof of the theorem that $A \cap B_R(0)$ is invariant

- For fixed ξ the field $\hat{u}(\xi) \in \mathbb{C}^2_\xi \subseteq \mathbb{C}^3$
  Because of incompressibility $\xi \cdot \hat{u}(\xi) = 0$

- Proposition

  The function $\hat{u}(\xi, t)$ is Lipschitz continuous as a function of t for every ξ

  - The Fourier transform satisfies

$$\begin{aligned} \partial_t \hat{u}(\xi) &= -\nu|\xi|^2 \hat{u}(\xi) - \frac{i\xi}{\sqrt{(2\pi)^3}} \Pi_\xi \int \hat{u}(\xi - \xi_1) \cdot \hat{u}(\xi_1)\, d\xi_1 \\ &:= X(u)_\xi \end{aligned} \tag{18}$$

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- Suppose that $\|u(\cdot)\|_{L^2} \leq R$ and consider the vector field $X(u)_\xi$ when $|\hat{u}(\xi)| = R_1/|\xi|$. The radial component is

$$\mathrm{re}(\hat{u}(\xi) \cdot X(u)_\xi) < -\nu|\xi|^2(R_1/|\xi|)^2 + (R_1/|\xi|)|\xi|\frac{R^2}{\sqrt{(2\pi)^3}} \quad (19)$$

  and therefore the LHS is negative when $R^2/\sqrt{(2\pi)^3} < \nu R_1$

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proof of corollary

- A fact about the vector field $X(\hat{u})$ is that solutions obey

$$|\hat{u}(\xi, T)|^2 - |\hat{u}_0(\xi)|^2 + 2\nu \int_0^T |\xi|^2 |\hat{u}(\xi, t)|^2 \, dt$$
$$= \frac{2}{\sqrt{(2\pi)^3}} \mathrm{im}\Big[\int_0^T \bar{\hat{u}}(\xi) \cdot \int \hat{u}(\xi - \xi_1) \cdot \xi_1 \, \hat{u}(\xi_1) \, d\xi_1 dt \Big] \quad (20)$$

(again setting $f = 0$ for simplicity)

- Writing $I^2(\xi) = \nu \int_0^T |\xi|^4 |\hat{u}(\xi, t)|^2 \, dt$
this gives an inequality

$$I^2(\xi) - \frac{R^2}{2\nu\sqrt{(2\pi)^3}} I(\xi) - \frac{R_1^2}{2} \leq 0 \quad (21)$$

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Microlocal behavior in x and ξ

- The Fourier transform $\hat{u}(\xi, t) \in Lip$ as a function of time
- Take $x_0 \in S_T^{L^2}$, and localize the convergence question

$$v(x,t) := (u(x,t) - u(x,T))\varphi(x) \,, \qquad 0 \le \varphi \in C_0^\infty(B_r(x_0))$$

Proposition

As $t \to T^-$ then $\hat{v}(\xi, t) \to 0$ pointwise in $\xi \in \mathbb{R}^3$

- What causes a lack of strong convergence is loss of L^2 mass at $|\xi| \to \infty$

Weyl calculus

- Given a point $x_0 \in S_T^{L^2}$ we test for energy concentration using the Weyl pseudodifferential calculus

$$a^w(x,D)v(x,t) = \iint e^{i\xi\cdot(x-y)} a\left(\tfrac{x+y}{2},\xi\right)v(y,t)\,dyd\xi \tag{22}$$

for $a(x,\xi) \in S^0_{\rho\delta}$.

- A microlocal test of the energy is

$$\langle v \,|\, a^w(x,D)v\rangle = \iint a(x,\xi)W[v](x,\xi)\,dxd\xi \tag{23}$$

where $W[v]$ is the Wigner transform of v

$$W[v](x,\xi) := \frac{1}{(2\pi)^d}\int e^{i\xi\cdot y}v(x+y/2)\bar{v}(x-y/2)\,dy \tag{24}$$

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microlocal L^2 defects

- For $a(x,\xi) \in S^0_{\rho\delta}$ the operators $a^w(x,D)$ are continuous on L^2 Whenever $u(\cdot,t)$ converges strongly to $u(\cdot,T)$ on $B_r(x_0)$, for $v(x,t) := (u(x,t) - u(x,T))\varphi(x)$ then

$$\lim_{t\to T^-} \langle a \,|\, W[v(t)]\rangle = 0 \tag{25}$$

- However if $u(\cdot,t)$ converges weakly but not strongly to $u(\cdot,T)$, it is detected by a microlocal defect measure $\mu \in \mathcal{M}(S^*(\mathbb{R}^3))$.

Theorem (L. Tartar (1990), P. Gérard (1991))

Let μ_{x_0} be a microlocal defect measure of $\lim_{t_j\to T^-} v(x,t_j)$. Suppose that for all symbols $a \in S^0_{10}$ homogeneous degree zero

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then $\mu_{x_0} = 0$ and $u(\cdot,t_j)$ converges strongly in $L^2(B_r(x_0))$ to $u(\cdot,T)$.

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the L^2 wave front set WF^{L^2}

Definition

Let $\mathcal{A}_r$ be the set of all microlocal defect measures $\mu_{x_0,r}$ as $t \to T^-$ of $\varphi_r(\cdot)(u(\cdot,t) - u(\cdot,T) := v_r$, with $\operatorname{supp}\varphi_r \subseteq B_r(x_0)$. The L^2 wave front set is

$$WF^{L^2}_{x_0 T} := \bigcap_{r>0} \overline{\bigcup_{\mathcal{A}_r} \operatorname{supp}(\mu_{x_0,r})} \tag{27}$$

- The sets $\operatorname{supp}\mu_{x_0,r}$ are monotone decreasing in $r \to 0$
- Our definition of "dimension" gives an upper bound on the Hausdorff dimension of $WF^{L^2}_T \cap S^*(D)$, which is geometric. But it is not necessarily an equality.
- Our resulting lower bounds contain more analytic information, in non-conic neighborhoods of $WF^{L^2}_T \subseteq T^*(D)$.

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Hörmander symbol classes $S^0_{\rho\delta}$ and WF^{L^2}

- One probes $WF^{L^2}_{x_0 T}$ more finely with symbols $a \in S^0_{\rho\delta}$, with $0 \leq \delta < \rho \leq 1$

  For example $a(x,\xi) = \varphi(x)\chi(\frac{\xi'}{\langle\xi_1\rangle^\rho}) \in S^0_{\rho 0}$ for $0 < \rho \leq 1$

  By standard theory, the operators $a^w(x,D)$ are bounded on L^2.

  These symbol classes include ones of quasi-homogeneous type *Beals & Fefferman* (1974), *Boutet de Monvel* (1975), *Lascar* (1977)

- Suppose that $0 \leq a(x,\xi) \leq 1$ is such that

$$\lim_{t\to T^-} \langle v|(1-a)^w(x,D)v\rangle = 0 \tag{28}$$

  then

$$WF^{L^2}_{x_0 T} \subseteq \operatorname{supp}(a)$$

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L^2 concentration at infinity

- Given test symbols $a(x, \xi)$, the volume growth of $\operatorname{supp}(a)$ gives an upper bound on the neighborhood of $WF^{L^2}_{x_0 T} \subseteq T^*(D)$ with L^2 mass concentration
- Consider $a(x, \xi) \in S^0_{\rho\delta}$ such that $0 \leq a \leq 1$,

$$\lim_{t \to T^-} \langle (1-a) \,|\, W[v(t)] \rangle = 0 \tag{29}$$

and

$$\operatorname{vol}\left(\pi_\xi \operatorname{supp}(a) \cap B_R(0)\right) \sim R^{1+\beta} \tag{30}$$

Definition (size of $WF^{L^2}_{x_0 T}$)

$$\bar{\beta}_{x_0}(v) := \inf(\beta) \tag{31}$$

Lower bounds for the L^2 wave front set

Theorem (Arnold & C. (2010))

The set $WF^{L^2}_{x_0T} \subseteq WF_{x_0T}(u)$ *is not too small (if it is nonempty), in that*

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Thank you

Proof of Theorem 2

Taking into account the local character of the problem, the real result states.

Conjecture (A & C)(work in progress)

Suppose that $\bar{\alpha}$ is the Hausdorff dimension of the set $S_T^{L^2}$, then

$$\bar{\beta}_{x_0}(v) \geq \bar{\alpha} + 1 \tag{33}$$

- Assume that $\bar{\beta}_{x_0}(v) < 1$. Choose $a \in S^0_{\rho\delta}$ with above support properties and volume growth $\bar{\beta}_{x_0}(v) < \beta < 1$, and test $v(\cdot, t)$

$$\lim_{t \to T^-} \langle a | W[v] \rangle = \lim_{t \to T^-} \int v a^w(x, D) v \, dx \tag{34}$$
$$= \lim_{t \to T^-} \int d\eta \Big[\int \tilde{a}(\eta, \xi) \hat{v}(\xi + \eta/2, t) \overline{\hat{v}(\xi - \eta/2, t)} \, d\xi \Big]$$

where $\tilde{a}(\eta, \xi)$ is the Fourier transform of a with respect to x.
- For each η, the volume growth of $\operatorname{supp} \tilde{a}(\eta, \xi)$ is bounded by $R^{1+\beta}$
- There is now a majorant: for each η

$$|\tilde{a}(\eta,\xi)||\hat{v}(\xi+\eta/2,t)||\hat{v}(\xi-\eta/2,t)| \leq |\tilde{a}(\eta,\xi)|\langle \xi+\eta/2\rangle^{-1}\langle \xi-\eta/2\rangle^{-1}$$

- This is integrable over ξ. Indeed

$$\int_\xi (*) \, d\xi \leq \int \langle r \rangle^{-2} r^\beta \, dr < +\infty \tag{35}$$

The Lebesgue dominated convergence theorem implies that as $t \to T^-$ the limit vanishes.

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  $|\tilde{a}(\eta, \xi)||\hat{v}(\xi+\eta/2, t)||\hat{v}(\xi-\eta/2, t)| \leq |\tilde{a}(\eta, \xi)|\langle\xi+\eta/2\rangle^{-1}\langle\xi-\eta/2\rangle^{-1}$

- This is integrable over ξ. Indeed

$$\int_\xi (*)\, d\xi \leq \int \langle r\rangle^{-2} r^\beta\, dr < +\infty \tag{35}$$

  The Lebesgue dominated convergence theorem implies that as $t \to T^-$ the limit vanishes.

- Assume that $\bar{\beta}_{x_0}(v) < 1$. Choose $a \in S^0_{\rho\delta}$ with above support properties and volume growth $\bar{\beta}_{x_0}(v) < \beta < 1$, and test $v(\cdot, t)$

$$\lim_{t \to T^-} \langle a | W[v] \rangle = \lim_{t \to T^-} \int v a^w(x, D) v \, dx \tag{34}$$
$$= \lim_{t \to T^-} \int d\eta \Big[\int \tilde{a}(\eta, \xi) \hat{v}(\xi + \eta/2, t) \overline{\hat{v}(\xi - \eta/2, t)} \, d\xi \Big]$$

where $\tilde{a}(\eta, \xi)$ is the Fourier transform of a with respect to x.
- For each η, the volume growth of $\operatorname{supp} \tilde{a}(\eta, \xi)$ is bounded by $R^{1+\beta}$
- There is now a majorant: for each η

$$|\tilde{a}(\eta, \xi)||\hat{v}(\xi+\eta/2, t)||\hat{v}(\xi-\eta/2, t)| \leq |\tilde{a}(\eta, \xi)|\langle \xi+\eta/2 \rangle^{-1} \langle \xi-\eta/2 \rangle^{-1}$$

- This is integrable over ξ. Indeed

$$\int_\xi (*) \, d\xi \leq \int \langle r \rangle^{-2} r^\beta \, dr < +\infty \tag{35}$$

The Lebesgue dominated convergence theorem implies that as $t \to T^-$ the limit vanishes.

Thank you again