# Existence of Gravity-Capillary Waves on Water of Finite Depth 

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## The Physical Problem


$\mathrm{F}=\mathrm{c} /(\mathrm{gh})^{1 / 2}$, Froude number, nondimensional wave speed. $\tau=\mathrm{T} /\left(\rho \mathrm{h}^{2} \mathrm{~g}\right)$, Bond number, nondimensional surface tension.

We are interested in the traveling waves moving on the free surface. Assume the fluid is inviscid and incompressible and the flow is irrotational.

## Introduction

Scott Russell (1844) first observed a single hump wave moving with a constant speed in a canal and subsequently studied such waves by experiments. He found that $F=1$ is a critical value. Later, Boussinesq (1871) and Korteweg and deVries (1895) formally derived a first order model equation for the free surface $\eta(t, x)=1+\epsilon \xi\left(\epsilon^{3 / 2} t, \epsilon^{1 / 2} x\right)(\mathrm{h}=1)$ with $\xi(\mathrm{t}, \mathrm{x})$ satisfying

$$
\xi_{t}-\lambda_{1} \xi_{x}-3 \xi \xi_{x}+(\tau-1 / 3) \xi_{x x x}=0
$$

under the long wave assumption from the exact Euler equations, where $\mathrm{F}^{-2}=1+\lambda_{1} \epsilon$ for small $\epsilon>0$.

They also found the traveling wave solution $\xi(x, t)$
$=-\left(c_{1}+\lambda_{1}\right) \operatorname{sech}^{2}\left(\left(\left(c_{1}+\lambda_{1}\right) /(\tau-1 / 3)\right)^{1 / 2}\left(x-c_{1} t\right) / 2\right)$
for the model equation (KdV equation), where $\mathrm{c}_{1}$ is a free constant.

Question: Is this traveling wave solution an approximation of a solution of the exact Euler equations?

## Steady-State Solution

Since we can always choose a coordinate system moving with the same speed as the traveling wave so that the wave is steady in this coordinate system, in the following we only consider the time-independent case. Therefore the K-dV equation becomes

$$
-\lambda_{1} \xi_{x}-3 \xi \xi_{x}+(\tau-1 / 3) \xi_{x x x}=0
$$

and

$$
\xi(x)=-\lambda_{1} \operatorname{sech}^{2}\left(\left(\lambda_{1} /(\tau-1 / 3)\right)^{1 / 2} x / 2\right)
$$

is the solitary wave solution. Besides the Froude number F , the Bond number $\tau$ is also a very important parameter for the solitary wave solution and $\tau=1 / 3$ is another critical value
Question: Is this solitary wave solution an approximation of a solution of the exact equations?

## Formal Derivation

## Fluid Domain:



## Governing equations:

Assume that the fluid with constant density $\rho$ is inviscid and the flow is irrotational. A coordinate system moving with the wave is chosen so that the governing equations are the following exact Euler equations;

$$
\begin{gathered}
\mathrm{u}_{x^{*}}^{*}+\mathrm{v}^{*}{ }_{y^{*}}=0, \quad \mathrm{u}^{*}{ }_{y^{*}}-\mathrm{v}^{*}{ }_{x^{*}}=0, \\
\rho\left(\mathrm{u}^{*} \mathrm{u}^{*}{ }_{x^{*}}+\mathrm{v}^{*} \mathrm{u}^{*}{y^{*}}^{*}=-\mathrm{p}^{*} x^{*},\right. \\
\rho\left(\mathrm{u}^{*} \mathrm{v}_{x^{*}}+\mathrm{v}^{*} \mathrm{v}_{y^{*}}\right)=-\mathrm{p}^{*}{ }_{y^{*}}-\rho \mathrm{g}
\end{gathered}
$$

at $\mathrm{y}^{\star}=\eta^{\star}(\mathrm{x})$,

$$
\begin{gathered}
\mathrm{u}^{*} \eta^{*}{ }_{\mathrm{x}^{*}}-\mathrm{v}^{*}=0, \\
\mathrm{p}^{*}=-\mathrm{T} \eta_{x^{*} x^{*}}\left(1+\left(\eta_{x^{*}}^{*}\right)^{2}\right)^{-3 / 2}
\end{gathered}
$$

at $\mathrm{y}^{*}=0, \mathrm{v}^{*}=0$, where the variables with * are dimensional variables. Then

$$
u^{*}=\psi_{y^{*}}^{*}, \mathrm{v}^{*}=-\psi_{x^{*}}^{*}
$$

Use $\left(x^{*}, \psi^{*}\right)$ as independent variables and $y^{*}=f^{*}\left(x^{*}, \psi^{*}\right)$ as the dependent variable where

$$
\psi^{\star}\left(x^{\star}, \mathrm{f}^{\star}\right)=\text { a constant }
$$

is a stream line and $f^{*}$ is the stream-line function. Then

$$
u^{*}=1 / f^{*}{ }_{\psi^{*}}, \quad v^{*}=-f^{*}{ }_{x^{*}} / f^{*}{ }_{\psi^{*}}
$$

Make these variables nondimensional to obtain

$$
\begin{gathered}
\left(1+\left(\mathrm{f}_{x}\right)^{2}\right) \mathrm{f}_{\psi \psi}-2 \mathrm{f}_{x} \mathrm{f}_{\psi} \mathrm{f}_{x \psi}+\left(\mathrm{f}_{\psi}\right)^{2} \mathrm{f}_{x x}=0, \\
\text { for }|\mathrm{x}|<\infty, 0<\psi<1 ; \\
\text { at } \psi=1,\left(1+\left(\mathrm{f}_{x}\right)^{2}\right) /\left(2\left(\mathrm{f}_{\psi}\right)^{2}\right)-\mathrm{F}^{-2} \tau \mathrm{f}_{x x}\left(1+\left(\mathrm{f}_{x}\right)^{2}\right)^{-3 / 2}+\mathrm{F}^{-2 \mathrm{f}} \\
=1 / 2+\mathrm{F}^{-1} ;
\end{gathered}
$$

at $\psi=0, \quad f=0$.
where $\psi=1$ is the free surface, $\mathrm{F}^{-2}=\mathrm{gh} / \mathrm{c}^{2}, \tau=\mathrm{T} /($
$\left.\rho h^{2} g\right), f=f^{*} / h, x=x^{*} / h$. The nondimensional constants $\tau$ and $F$ will determine the solutions of these equations. We shall call these as exact equations, which are equivalent to the exact Euler equations.

## Derivation:

Assume that $\mathrm{f}(\mathrm{x}, \psi)$ and $\mathrm{F}^{-1}$ have asymptotic expansions of the following form

$$
\begin{aligned}
& \mathrm{f}=\psi+\epsilon \mathrm{W}_{1}+\epsilon^{2} \mathrm{~W}_{2}+\ldots \ldots \\
& \mathrm{F}^{-2}=\lambda=\lambda_{0}+\epsilon \lambda_{1},\left(\lambda_{1}=1 \quad \text { or }-1\right)
\end{aligned}
$$

and $x_{1}=\epsilon^{1 / 2} x$, which is so called long wave assumption. The expansion of $f$ is called long wave expansion and $\lambda_{0}$ is the critical value of $\mathrm{F}^{-1}$. After substituting these into the exact equations, the equations for the first order approximation are

$$
\begin{array}{ll}
\mathrm{w}_{1 \psi \psi}=0 & \text { in } 0<\psi<1, \\
\mathrm{w}_{1 \psi}-\lambda_{0} \mathrm{w}_{1}=0 & \text { at } \psi=1, \\
\mathrm{w}_{1}=0 & \text { at } \psi=0
\end{array}
$$

From this equation it is easy to obtain (for simplicity, $\mathrm{x}_{1}=\mathrm{x}$ )

$$
\mathrm{w}_{1}=\xi(\mathrm{x}) \psi, \quad \lambda_{0}=1 .
$$

To obtain $\xi(\mathrm{x})$, we need equations for the second approximation.

The equations for the second approximation are

$$
\begin{array}{ll}
\mathrm{w}_{2 \psi \psi}=-\psi \xi_{x x}(\mathrm{x}) & \text { in } 0<\psi<1, \\
\mathrm{w}_{2 \psi}-\lambda_{0} \mathrm{w}_{2}=\lambda_{1} \xi(\mathrm{x})-\tau \xi_{\mathrm{xx}}(\mathrm{x})+(3 / 2) \xi^{2}(\mathrm{x}) \quad \text { at } \psi=1, \\
\mathrm{w}_{2}=0 & \text { at } \psi=0 .
\end{array}
$$

This is a nonhomogeneous boundary value problem. By Fredholm alternative Theorem for this ODE, the nonhomogeneous terms must satisfy a solvability condition to have the solution $\mathrm{w}_{2}$. From this condition we can obtain the following equation for $\xi(\mathrm{x})$

$$
-\lambda_{1} \xi+(3 / 2) \xi^{2}+(\tau-(1 / 3)) \xi_{x x}=0
$$

The solution of this equation, which decays at infinity, is

$$
\xi(x)=-\lambda_{1} \operatorname{sech}^{2}\left(\left(\lambda_{1} /(\tau-1 / 3)\right)^{1 / 2} x / 2\right),
$$

and $\mathrm{f}(\mathrm{x}, \psi)=\psi+\epsilon \xi(\mathrm{x}) \psi+\mathrm{O}\left(\epsilon^{2}\right)$ and $\eta(\mathrm{x})=\mathrm{f}(\mathrm{x}, 1)$.

## Previous work on the exact equations

Without Surface Tension:
(1) M.A. Lavrent'ev (1943), K.O. Friedrichs and D. Hyers (1954), J.T. Beale (1977)

With Surface Tension:
(1) J.K. Hunter and J.M. Vanden-Broeck (1983, and more)
(2) C.J. Amick and K. Kirchgassner (1987), R. Sachs (1991), Buffoni (2002). (large surface tension)
(3) J.T. Beale (1989), S.M. Sun (1989), G. loose and K.

Kirchgassner (1990) (small surface tension)
(4) S.M. Sun and M.C. Shen (1991), E. Lombardi (1997) (exponential estimates on oscillatory tails)
(5) S. M. Sun (1999) (nonexistence proof)

For Other Fluids:
Amick, Benjamin, Bona, Bose, Lankers, Kirchgassner, Sachs, Ter-Krikorov, Turner, many many more.

## Exact Existence results

1. $\tau=0$ :


This solitary wave of elevation decays exponentially at infinity.
2. $\tau>1 / 3$ :
solitary wave of depression decays exponentially at infinity.
3. $0<\tau<1 / 3$ :


This "solitary wave" of elevation has very small oscillations at infinity.
4. $(1 / 3)-\delta<\tau<1 / 3$ ( $\delta>0$ small):

There exist no solitary waves that decay exponentially at infinity.

## Three Dimensional (3D) Surface Waves


$F=c /(g h)^{1 / 2}$, Froude number, nondimensional wave speed.
$\tau=T /\left(\rho h^{2} g\right)$, Bond number, nondimensional surface tension.

## The governing equations

After making variables non-dimensional, we have

$$
\begin{aligned}
& \phi_{\mathrm{xx}}+\phi_{\mathrm{yy}}+\phi_{\mathrm{zz}}=0, \quad 0<\mathrm{y}<\eta(\mathrm{x}, \mathrm{z}, \mathrm{t}) \\
& \phi_{\mathrm{y}}=0 \quad \text { at } \mathrm{y}=0 \\
& \eta_{\mathrm{t}}^{-} \eta_{\mathrm{x}}+\eta_{\mathrm{x}} \phi_{\mathrm{x}}+\eta_{\mathrm{z}} \phi_{\mathrm{z}}-\phi_{\mathrm{y}}=0 \quad \text { at } \mathrm{y}=\eta(\mathrm{x}, \mathrm{z}, \mathrm{t}) \\
& \phi_{\mathrm{t}^{-}} \phi_{\mathrm{x}}+(1 / 2)\left(\left(\phi_{\mathrm{x}}\right)^{2}+\left(\phi_{\mathrm{y}}\right)^{2}+\left(\phi_{\mathrm{z}}\right)^{2}\right) \\
& \quad+\mathrm{F}^{-2}(\eta-1)-\tau\left(\eta_{\mathrm{x}}\left(1+\left(\eta_{\mathrm{x}}\right)^{2}+\left(\eta_{\mathrm{z}}\right)^{2}\right)^{-1 / 2}\right)_{\mathrm{x}} \\
& -\tau\left(\eta_{\mathrm{z}}\left(1+\left(\eta_{\mathrm{x}}\right)^{2}+\left(\eta_{\mathrm{z}}\right)^{2}\right)^{-1 / 2}\right)_{\mathrm{z}}=0 \quad \text { at } \mathrm{y}=\eta(\mathrm{x}, \mathrm{z}, \mathrm{t})
\end{aligned}
$$

with $\eta \longrightarrow 1,\left(\phi_{x}, \phi_{y}, \phi_{z}\right) \longrightarrow 0$ as $|x| \longrightarrow \infty$.

## Transformation

Let $Y=y / \eta(x, z, t)$, which transforms the fluid region to $0<Y<1$, denote

$$
\Phi(x, Y, z, t)=\phi(x, \eta Y, z, t)
$$

The equations with new variables are

$$
\begin{aligned}
& \Phi_{X x}+\Phi_{Y Y^{+}} \Phi_{z z}=\mathrm{N}_{1}(\Phi, \eta), \quad 0<Y<1 \\
& \Phi_{Y}=0 \text { at } Y=0 \\
& \eta_{-} \eta_{X^{-}} \Phi_{Y}=\mathrm{N}_{2}(\Phi, \eta) \quad \text { at } Y=1 \\
& \phi_{\mathrm{t}}-\phi_{\mathrm{x}}+\mathrm{F}^{-2}(\eta-1)-\tau \eta_{x x}-\tau \eta_{z z}=\mathrm{N}_{3}(\Phi, \eta) \\
& \text { at } Y=1
\end{aligned}
$$

where $\mathrm{N}_{1}, \mathrm{~N}_{2}, \mathrm{~N}_{3}$ are nonlinear in $\Phi, \eta$.

## Kadomtsev-Petviashvili (KP) equation

 If $\mathrm{F}^{-2}=1+\lambda_{1} \epsilon$ and$$
\eta(x, z, t)=1+\epsilon \xi\left(\epsilon^{1 / 2} x, \epsilon z, \epsilon^{3 / 2} t\right)
$$

with $\epsilon>0$ small and $\mathrm{h}=1$, then $\xi(\mathrm{x}, \mathrm{z}, \mathrm{t})$ satisfies

$$
\left(\xi_{t}-\lambda_{1} \xi_{x}-3 \xi \xi_{x}+(\tau-1 / 3) \xi_{x x x}\right)_{x}-\xi_{z z}=0
$$

called KP equation. The equation is an approximate equation derived from the exact governing equations under long wave assumption using asymptotic method.
If $\xi(x, z, t)$ is independent of $z$, the equation is reduced to Korteweg-de Vries (KdV) equation

$$
\xi_{t}-\lambda_{1} \xi_{x}-3 \xi \xi_{x}+(\tau-1 / 3) \xi_{x x x}=0
$$

## Kadomtsev-Petviashvili (KP) equation

If $\tau>1 / 3$, it is called KP-I equation.
If $\tau<1 / 3$, it is called KP-II equation.
Here, we mainly consider the case with $\tau>1 / 3$, i.e., KP-I equation. KP-I equation has a family of traveling wave solutions:

$$
\xi_{\omega}(x-c t, z)=-\frac{4\left(\lambda_{1}+c\right)\left(1-\omega^{2}\right)\left(2-\omega^{2}\right)\left(1-\omega^{2} \cosh \left(a_{\omega} x_{1}\right) \cos \left(b_{\omega} z\right)\right)}{\left(4-\omega^{2}\right)\left(\cosh \left(a_{\omega} x_{1}\right)-\omega \cos \left(b_{\omega} z\right)\right)^{2}}
$$

where $\omega \in[0,1), x_{1}=x-c t$

$$
a_{\omega}=\left(\frac{\lambda_{1}+c}{\tau-1 / 3}\right)^{1 / 2} \sqrt{\frac{1-\omega^{2}}{4-\omega^{2}}}, \quad b_{\omega}=\frac{\left|\lambda_{1}+c\right|}{(\tau-1 / 3)^{1 / 2}} \frac{\sqrt{3\left(1-\omega^{2}\right)}}{4-\omega^{2}},
$$

(Tajiri \&Murakami, 1990)

## Three dimensional steady-state solution:

If we choose a coordinate system moving with the wave, the wave is steady and the KP-I equation is

$$
\left(-\lambda_{1} \xi-(3 / 2) \xi^{2}+(\tau-1 / 3) \xi_{\mathrm{xx}}\right)_{\mathrm{xx}}-\xi_{\mathrm{zz}}=0
$$

which has one family of solutions

$$
\xi_{\omega}(x, z)=-\frac{4 \lambda_{1}\left(1-\omega^{2}\right)\left(2-\omega^{2}\right)\left(1-\omega^{2} \cosh \left(a_{\omega} x_{1}\right) \cos \left(b_{\omega} z\right)\right)}{\left(4-\omega^{2}\right)\left(\cosh \left(a_{\omega} x_{1}\right)-\omega \cos \left(b_{\omega} z\right)\right)^{2}}
$$

with $\omega \in[0,1)$,

$$
a_{\omega}=\left(\frac{\lambda_{1}}{\tau-1 / 3}\right)^{1 / 2} \sqrt{\frac{1-\omega^{2}}{4-\omega^{2}}}, \quad b_{\omega}=\frac{\left|\lambda_{1}\right|}{(\tau-1 / 3)^{1 / 2}} \frac{\sqrt{3\left(1-\omega^{2}\right)}}{4-\omega^{2}},
$$

It is easy to easy to check that for any fixed $z \in(-\infty, \infty)$,
$\xi_{\omega}(x, z) \longrightarrow 0$ exponentially as $|x| \longrightarrow \infty$ and for any fixed $x \in(-\infty, \infty), \xi_{\omega}(x, z)$ is periodic in $z$.

## Special cases:

If $\omega=0$, then

$$
\xi_{0}(x, z)=-\lambda_{1} \operatorname{sech}^{2}\left(\left(\lambda_{1} /(\tau-1 / 3)\right)^{1 / 2} x / 2\right)
$$

is the solitary wave solution of the KdV equation (here $\tau$ can be < $1 / 3$ ).
More interesting case is that if $\omega \longrightarrow 1$, then

$$
\frac{3-\left(\left(\frac{\lambda_{1}}{\tau-1 / 3}\right)^{1 / 2} x\right)^{2}+\left(\frac{\lambda_{1}}{(\tau-1 / 3)^{1 / 2}} z\right)^{2}}{\left(3+\left(\left(\frac{\lambda_{1}}{\tau-1 / 3}\right)^{1 / 2} x\right)^{2}+\left(\frac{\lambda_{1}}{(\tau-1 / 3)^{1 / 2}} z\right)^{2}\right)^{2}},
$$

which is called lump solution (or localized solution) of KP-I equation since $\xi_{1}(x, z) \longrightarrow 0$ as $|x|+|z| \longrightarrow \infty$.

## Lump (or localized) wave



## Question:

Are these solutions approximations of some solutions of the exact fully nonlinear governing equations?
Here, we note that the KP-I equation is an approximation of the fully nonlinear equation and the solutions that were discussed are solutions of the KP-I equation.
Recent work on 3D waves:
Groves, et al. (2002, 2003),
Parau, Vanden-Broeck and Cooker (2005)
Kim and Akylas (2005), Milewski (2005)

Three dimensional waves
Consider the case that $\omega$ is near zero for $\xi_{\omega}(x, z)$. For KP-I equation, when $\omega$ goes from zero to a small positive number, a transversely inhomogeneous solution $\xi_{\omega}(x$, $z)$ spontaneously emerges from $\xi_{0}(x)$ that is homogeneous in the z-direction. This is termed as dimension-breaking phenomenon. So, for the KP-I equation, a family of periodically modulated solitary waves emerges from the KdV solitary wave in a dimension-breaking bifurcation.

Dimension-breaking for fully nonlinear equations
It was shown by Amick and Kirchgassner that for $\tau>1 / 3$, there is a two-dimensional solitary wave solution for the fully nonlinear equations.

Question: Can we obtain a dimensional breaking bifurcation from the twodimensional solitary wave using the exact equations after one dimension is added in the equations?

## Dimension-breaking for water waves (Groves,Haragus, s. 2002)

Let $\tau>1 / 3$ and $F^{-2}=1+\epsilon$ (i.e., $\lambda_{1}=1$ ). For $\epsilon>0$ small, the exact equations have a 2D solitary wave solution (Amick and Kirchgassner)

$$
\eta^{\star}(x ; \epsilon)=1-\epsilon \operatorname{sech}^{2}\left((\epsilon /(2(\tau-1 / 3)))^{1 / 2} x\right)+\mathrm{O}\left(\epsilon^{2}\right)
$$

Then there exist a constant $\omega_{0}$ in the interval ( 0 , $\left.(1 / 2)(\tau-1 / 3)^{-1 / 2}\right)$, a constant $\epsilon_{0}>0$ and a small neighborhood N of 0 in R such that for all $\epsilon \in\left(0, \epsilon_{0}\right)$, a family of solutions $\left\{\eta_{\mathrm{a}}(x, z ; \epsilon)\right\}_{\mathrm{a} \in \mathrm{N}}$ of the exact equations emerges from $\eta^{*}(x ; \epsilon)$ with

$$
\eta_{\mathrm{a}}(x, Z ; \epsilon)=\eta^{\star}(x ; \epsilon)+\epsilon \eta^{*}{ }_{\mathrm{a}}\left(\epsilon^{1 / 2} X, \epsilon Z ; \epsilon\right)
$$

where $\eta^{\star}{ }_{\mathrm{a}} \sim \mathrm{O}(|a|), \eta^{*}{ }_{\mathrm{a}}$ is periodic in $z$ with period $2 \pi /\left(\epsilon k_{\epsilon}+\mathrm{O}\left(\mathrm{a}^{2}\right)\right),\left|k_{\epsilon}-\omega_{0}^{2}\right|=\mathrm{O}\left(\epsilon^{1 / 4}\right)$.

## Existence of localized 3D waves (Grove, S. 2008)

 Let $\tau>1 / 3$ and $\mathrm{F}^{-2}=1+\epsilon$ there exist a constant $\epsilon_{0}>$0 such that for all $\epsilon \in\left(0, \epsilon_{0}\right)$, the exact fully nonlinear equations have a time-independent solution

$$
(\Phi, \eta)=\left(\epsilon^{1 / 2} \Psi\left(\epsilon^{1 / 2} x, Y, \epsilon Z ; \epsilon\right), 1+\epsilon \xi\left(\epsilon^{1 / 2} x, \epsilon Z ; \epsilon\right)\right)
$$

with $(\Psi, \xi)$ satisfying

$$
\begin{aligned}
0 & <\mathrm{C}_{0} \leq\left\|\Psi_{x}(x, Y, z ; \epsilon)\right\|_{w}^{1, \mathrm{p}_{\left(\mathrm{R}^{3}\right)}} \\
+ & \left\|\Psi_{Y}(x, Y, z ; \epsilon)\right\|_{w^{1}, \mathrm{p}}^{\left(\mathrm{R}^{3}\right)}+\left\|\Psi_{z}(x, Y, z ; \epsilon)\right\|_{w^{1, \mathrm{p}}\left(\mathrm{R}^{3}\right)} \\
& +\|\xi(x, z ; \epsilon)\|_{w^{2, \mathrm{p}}\left(\mathrm{R}^{2}\right)}+\|\xi(x, z ; \epsilon)\|_{w^{2, p}\left(\mathrm{R}^{2}\right)} \leq \mathrm{C}
\end{aligned}
$$

where $C_{0}, C$ are independent of $\epsilon$ and $p>1$ is large but fixed. Here $W^{\beta, p}\left(R^{2}\right)$ or $W^{\beta, p}\left(R^{3}\right)$ is the classical Sobolev space. Moreover, the smoothness of the solution can be obtained up to any fixed order.

## Other type of 3D waves (not based on KP equation, $\tau<1 / 3$ )

Let $\phi(t, x, y, z)$ be the velocity potential. Then $\phi$ and $\eta$ satisfy that in $D_{\eta}=\{0<z<H+\eta\}$

$$
\phi_{x x}+\phi_{y y}+\phi_{z z}=0, \quad \text { in } D_{\eta}
$$

with boundary conditions,

$$
\begin{aligned}
0= & \phi_{z}, \quad \text { on } z=0, \\
\eta_{t}= & \phi_{z}-\eta_{x} \phi_{x}-\eta_{y} \phi_{y}, \quad \text { on } z=H+\eta, \\
\phi_{t}= & -\frac{1}{2}\left(\phi_{x}^{2}+\phi_{y}^{2}+\phi_{z}^{2}\right)-g \eta+T\left[\frac{\eta_{x}}{\left(1+\eta_{x}^{2}+\eta_{y}^{2}\right)^{1 / 2}}\right]_{x} \\
& +T\left[\frac{\eta_{y}}{\left(1+\eta_{x}^{2}+\eta_{y}^{2}\right)^{1 / 2}}\right]_{y}+\varrho, \text { on } z=H+\eta
\end{aligned}
$$

If we are interested in traveling wave solutions, then

$$
\begin{aligned}
\eta(t, x, y) & =\eta(x+c t, y) \\
\phi(t, x, y, z) & =\phi(x+c t, y, z)
\end{aligned}
$$

Introduce non-dimensional variables

$$
\begin{gathered}
\left(x^{\prime}, y^{\prime}, z^{\prime}\right)=\frac{1}{h}(x+c t, y, z), \\
\eta^{\prime}\left(x^{\prime}, y^{\prime}\right)=\frac{1}{h} \eta(x+c t, y), \quad \phi^{\prime}\left(x^{\prime}, y^{\prime}, z^{\prime}\right)=\frac{1}{c h} \phi(x+c t, y, z),
\end{gathered}
$$

The governing equations are changed to

$$
\begin{aligned}
\mathrm{O}= & \phi_{x x}+\phi_{y y}+\phi_{z z}, \quad 0<z<1+\eta(x, y), \\
\mathrm{O}= & \phi_{z}, \quad \text { on } z=0, \\
\mathrm{O}= & \eta_{x}+\eta_{x} \phi_{x}+\eta_{y} \phi_{y}-\phi_{z}, \quad \text { on } z=1+\eta(x, y), \\
\mathrm{O}= & \phi_{x}+\frac{1}{2}\left(\phi_{x}^{2}+\phi_{y}^{2}+\phi_{z}^{2}\right)+\lambda \eta \\
& -b\left[\frac{\eta_{x}}{\left(1+\eta_{x}^{2}+\eta_{y}^{2}\right)^{1 / 2}}\right]_{x} \\
& -b\left[\frac{\eta_{y}}{\left(1+\eta_{x}^{2}+\eta_{y}^{2}\right)^{1 / 2}}\right]_{y}+\varrho, \quad \text { on } z=1+\eta(x, y),
\end{aligned}
$$

where $\lambda=F^{-2}$ and $\mathrm{b}=\tau$. To transform the system into a dynamical system, we introduce new dependent variables

$$
u=\phi_{x}, \quad \xi=\frac{\eta_{x}}{\left(1+\eta_{x}^{2}+\eta_{y}^{2}\right)^{1 / 2}}
$$

## The system is then changed to

$$
\begin{aligned}
\phi_{x} & =u, \\
u_{x} & =-\phi_{y y}-\phi_{z z}, \\
\eta_{x} & =\xi\left(\frac{1+\eta_{y}^{2}}{1-\xi^{2}}\right)^{1 / 2}, \\
\xi_{x} & =\frac{1}{b} u+\frac{\lambda}{b} \eta+\frac{1}{2 b}\left(u^{2}+\phi_{y}^{2}+\phi_{z}^{2}\right)-\left[\eta_{y}\left(\frac{1-\xi^{2}}{1+\eta_{y}^{2}}\right)^{1 / 2}\right]_{y}+\varrho
\end{aligned}
$$

with boundary conditions

$$
\begin{array}{ll}
\phi_{z}=0, & \text { on } z=0 \\
\phi_{z}=\xi(1+u)\left(\frac{1+\eta_{y}^{2}}{1-\xi^{2}}\right)^{1 / 2}+\eta_{y} \phi_{y}, & \text { on } z=1+\eta(x, y)
\end{array}
$$

## Flatten the upper unknown boundary by

$$
z=\frac{z}{1+\eta(x, y)}
$$

which maps the domain $D_{\eta}$ to a domain with $\tilde{z} \in[0,1]$. Then, the system is (still use $z$ )

$$
\phi_{x}=u+\frac{z \phi_{z} \xi}{1+\eta}\left(\frac{1+\eta_{y}^{2}}{z \eta_{y}} \xi^{2}\right)^{1 / 2},
$$

$$
u_{x}=-\left(\phi_{y}-\frac{z \eta_{y}}{1+\eta} \phi_{z}\right)_{y}+\frac{z \eta_{y}}{1+\eta}\left(\phi_{y}-\frac{z \eta_{y}}{1+\eta} \phi_{z}\right)_{z}
$$

$$
-\frac{\phi_{z z}}{(1+\eta)^{2}}+\frac{z u_{z} \xi}{1+\eta}\left(\frac{1+\eta_{y}^{2}}{1-\xi^{2}}\right)^{1 / 2},
$$

$$
\eta_{x}=\xi\left(\frac{1+\eta_{y}^{2}}{1-\xi^{2}}\right)^{1 / 2},
$$

$$
\xi_{x}=\frac{1}{b}\left(\left.u\right|_{z=1}+\lambda \eta\right)+\frac{1}{2 b}\left[u^{2}+\left(\phi_{y}-\frac{z \eta_{y}}{1+\eta} \phi_{z}\right)^{2}\right.
$$

$$
\left.+\frac{\phi_{z}^{2}}{(1+\eta)^{2}}\right]_{z=1}-\left[\eta_{y}\left(\frac{1-\xi^{2}}{1+\eta_{y}^{2}}\right)^{1 / 2}\right]_{y}+\varrho
$$

with the boundary conditions:
$\left.\phi_{z}\right|_{z=0}=0$,
$\left.\phi_{z}\right|_{z=1}=\frac{1+\eta}{1+\eta_{y}^{2}}\left[\xi\left(1+\left.u\right|_{z=1}\right)\left(\frac{1+\eta_{y}^{2}}{1-\xi^{2}}\right)^{1 / 2}+\left.\eta_{y} \phi_{y}\right|_{z=1}\right]$.
The Banach spaces to be used are defined as follows:
$H_{P}^{s}(I)=\left\{f \in H_{l o c}^{s}(\mathbf{R}) \mid f(y+P)=f(y)\right.$, for almost all $y \in \mathbf{R}\}$,
$H_{P}^{s}(\Sigma)=\left\{f \in H_{l o c}^{s}(\mathbf{R} \times(0,1)) \mid f(y+P, z)=\right.$ $f(y, z)$, for almost all $(y, z) \in \mathbf{R} \times(0,1)\}$,
and

$$
\mathcal{H}_{s}=H_{P}^{s+1}(\Sigma) \times H_{P}^{s}(\Sigma) \times H_{P}^{s+1}(I) \times H_{P}^{s}(I)
$$

## The system can be written

$$
\frac{d v}{d x}=F^{(b, \lambda)}(\varrho, v)
$$

$$
\begin{aligned}
\mathcal{D}= & \left\{v=(\phi, u, \eta, \xi) \in \mathcal{H}_{s+1} ;|\xi|<1, \eta\right\rangle-1,\left.\phi_{z}\right|_{z=0}=0, \\
& \left.\left.\phi_{z}\right|_{z=1}=\frac{1+\eta}{1+\eta_{y}^{2}}\left[\xi\left(1+\left.u\right|_{z=1}\right)\left(\frac{1+\eta_{y}^{2}}{1-\xi^{2}}\right)^{1 / 2}+\left.\eta_{y} \phi_{y}\right|_{z=1}\right]\right\} .
\end{aligned}
$$

Then we make a change of variables $G$ : $H_{s}$ $\mapsto H_{s}$ with $G(v)=G(\phi, u, \eta, \xi)=(\psi, u$, $\eta, \xi)=w$, which changes the boundary conditions to

$$
\psi_{z}=0 \quad \text { on } z=0,1
$$

Finally, the system is changed to

$$
\dot{w}=g^{(b, \lambda)}(\varrho, w)
$$

with boundary conditions $\left.\psi_{z}\right|_{z=0,1}=0$, where

$$
g^{(b, \lambda)}(\varrho, \cdot): W \rightarrow \mathcal{H}_{s}
$$

is a smooth vector field. Now, taking the linear part of the system, we have

$$
\dot{w}=K_{s} w+N(b, \lambda, \varrho, w)
$$

with the domain
$\mathcal{D}\left(K_{s}\right)=\left\{w=(\psi, u, \eta, \xi) \in \mathcal{H}_{s+1}:\left.\psi_{z}\right|_{z=0}=\left.\psi_{z}\right|_{z=1}=0\right\}$

## Spectrum of linear operator $K_{s}$

The spectrum of $K_{s}$ consists of isolated eigenvalues of finite algebraic multiplicity and $\sigma\left(K_{s}\right) \cap i R$ is a finite set. The eigenvalue $\kappa$ is precisely the solution of

$$
\left(\lambda-b \tau_{k}^{2}\right) \tau_{k} \sin \tau_{k}-\kappa^{2} \cos \tau_{k}=0
$$

with $\tau_{k}^{2}=\kappa^{2}-\frac{4 \pi^{2} k^{2}}{P^{2}}$. Moreover, there is a $C$ such that $\left\|\left(K_{s}-i \kappa I\right)^{-1}\right\|_{\mathcal{H}_{s} \rightarrow \mathcal{H}_{s}} \leq \frac{C}{|\kappa|}$
for each real number $\kappa$ with $|\kappa|>\sigma_{0}>0$.
Use Center manifolds by Mielke (1988) to obtain for any integer $r>0$, the system has a finite dimensional center manifold of class $C^{r}$.

## Eigenvalues of $K_{s}$

$K_{s}$ has a zero eigenvalue which has a geometric multiplicity 1 and algebraic multiplicity 2 if $\lambda \neq 1$. The other eigenvalues are


## Previous work:

Exact equations:
Groves (2001): periodic in the propagation direction.

Groves and Mielke (2001): Generalized solitary wave solutions .
Dias and looss (2003) gave a general spatial dynamical formulation.

## The existence theorem: (Deng and S)

For $\epsilon \in\left(0, \epsilon_{0}\right]$, assume that $\lambda=\lambda_{0}+\epsilon$ with (b, $\lambda_{0}$ ) on $C_{1}^{+}$. Then, there exists a continuous function $\rho_{1}$ of $\epsilon$ with $\rho=\epsilon^{9 / 2} \rho_{1}$ such that the original system has a solitary-wave solution that approaches to a periodic solution $\widehat{X}_{\epsilon, \varrho, J_{1}}$ at infinity (called a generalized solitary wave solution) in the propagation direction and is periodic in the transverse direction, provided that some conditions are satisfied. The part of solitary-wave solution satisfies the Schrodinger equation.

## The picture of the 3D wave:



## Solutions near the intersection of $\mathrm{C}_{1}, \lambda=1$



## Existence for $\lambda$ near 1 and $b<1 / 3$

For $\epsilon \in\left(0, \epsilon_{0}\right]$, assume that $\lambda=1-\epsilon$ and $b=$ $b_{0}+m(\epsilon)$ with $b_{0} \in(0,1 / 3)$. Then, there exists a continuous function $\rho_{1}$ of with $\rho=$ $\epsilon^{3 / 2} \rho_{1}$ such that the original system has a generalized solitary-wave solution that approaches to a periodic solution $\widehat{X}_{\epsilon, \varrho, J_{1}}$ at infinity in the propagation direction and is periodic in the transverse direction, provided that some conditions are satisfied. The part of solitary-wave solution satisfies a system of KdV-Schrodinger equations.

## The equations for the case with $\lambda$ near 1 and $b<1 / 3$

It is found that the first-order approximation of the solution for $\eta(x, y)=1+\epsilon A\left(\epsilon^{1 / 2} x\right)+\epsilon B\left(\epsilon^{1 / 2} x\right)$ $\cos (2 \pi y / P)$ where $A(x)$ and $B(x)$ satisfy

$$
\begin{aligned}
& A_{x x}-c_{1} A+c_{2} A^{2}+c_{3} B^{2}=0, \\
& B_{x x}-d_{1} B+d_{2} A B=0 .
\end{aligned}
$$

If $B=0$, the equations become the KdV equation.
If $A=0$, another terms has to be added $|B|^{2} B$ in the equation for $B$, which is the Schrödinger equation (time independent).


## Experiments:


b"ت


## Multi-Solitary Waves

From experiments, we can see that two solitary waves with same amplitude can propagate together, which can be explained intuitively as well. For surface waves on water without surface tension, it has been shown (Craig and Sternberg) that the Euler equations have no multi-solitarywave solutions. However, if the surface tension is small, there are solitary waves with oscillations at infinity. With these oscillations, it is possible to construct solutions with multi-humps. Similar problems have been discussed by Buffoni, Groves, et al. for solitary waves with decaying oscillations at infinity.

Existence of multi-hump solutions Here, we consider the following model equation:

$$
-\lambda_{1} \xi-(3 / 2) \xi^{2}+(\tau-1 / 3) \xi_{x x}-\epsilon \xi_{x x x x}=0
$$

where $\epsilon>0$ is small constant. This fifth-order KdV equation has been used to model the water-wave problem for $0<\tau<1 / 3$. It was shown that this equation has no solutions decaying to zero at infinity (Amick, McLeod, et al.), but has solitary-wave solutions with nondecaying oscillations at infinity (Hunter, Scheurle, et al.).

## Existence Theorem (Choi, Lee, oh, s., Whang)

It has been known that if $A=a \epsilon^{n}$ for a fixed $a \neq 0$ and an integer $n \geq 3$ and some appropriate chosen constants $v$ and $\delta$, the fifth-order KdV equation has a solution

$$
\begin{aligned}
\xi(x)=S(x) & +\epsilon R_{0}(x)+A\left[\cos \left(x \epsilon^{-1}-\delta \tanh (\nu x)\right)\right. \\
& \left.+A R^{+}\left(x \epsilon^{-1}-\delta \tanh (\nu x)\right)\right]
\end{aligned}
$$

where $S(x)$ is the solution of the KdV equation, $\mathrm{R}_{0}(\mathrm{x})$ decays exponentially at infinity and $\mathrm{R}^{+}(\mathrm{x})$ is periodic with periodic $2 \pi$. Moreover, for any given integer $m>0$, the fifth-order KdV equation has an $m$-hump solution that consists of $m$ identical functions obtained above patched together by a suitable choice of a that may depend on $\epsilon$ and the distance between adjacent humps, which may go to infinity as $\epsilon$ goes to zero.

## Future project:

Show that the exact Euler equations have multi-hump solutions using the solitary-wave solutions with oscillations at infinity for the case of small surface tension.

## Thanks for your attention!

