Lower semicontinuity for signed functionals with linear growth in the context of $\mathcal{A}$-quasiconvexity

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The functional

$$I(\mu) = \int_\Omega f(\mu_a(x))\,dx + \int_\Omega f^\infty\left(\frac{d\mu_s}{d|\mu_s|}\right)\,d|\mu_s|$$

where

- Ω open bounded subset of $\mathbb{R}^N$
- μ is a bounded Radon measure

$$\mu = \mu_a(x)dx + \mu_s$$

- $f : \mathbb{R}^d \to \mathbb{R}$ is $\mathcal{A}$-quasiconvex (Lipschitz continuous) and it has linear growth

$$|f(\xi)| \leq C(1 + |\xi|)$$

- $f^\infty(\xi) := \limsup_{t\to\infty} \frac{f(t\xi)}{t}$ (is also $\mathcal{A}$-quasiconvex)

Lower semicontinuity

with respect to

$$\mu_n \overset{*}{\rightharpoonup} \mu \text{ in } \mathcal{M}$$

$$|\mu_n| \overset{*}{\rightharpoonup} \Lambda \text{ such that } \Lambda(\partial\Omega) = 0$$

$$\mathcal{A}\mu_n = 0(*)$$

(*) Or more generally $\mathcal{A}\mu_n \to 0$ in $W^{-1,q}$ for some $1 < q < \frac{N}{N-1}$

$\mathcal{A}$-quasiconvexity

$$\mathcal{A} := \sum_{i=1}^{N} A^{(i)} \frac{\partial}{\partial x_i} \qquad (A^{(i)}\text{are } m \times d \text{ matrices})$$

with constant rank, i.e., there exists $r \in \mathbb{N}$ such that

$$\operatorname{rank}\left(\sum_{i=1}^{N} A^{(i)} w_i\right) = r$$

for all $w \in \mathbb{R}^N \setminus \{0\}$

Example $\mathcal{A} = \operatorname{curl}$

Previous results ($p = 1$)

- $\mathcal{A}$-quasiconvexity: weak-star convergence and the gap; Fonseca, Leoni and Müller ($f \geq 0$, just the absolutely continuous part)
- Relaxation of signed integrals in BV; Kristensen and Rindler ($\mathcal{A} = \mathrm{curl}$)
  etc...

$\mathcal{A}$-quasiconvex functions

$$f(a) \leq \int_Q f(a + w(x))\,dx \qquad Q = \text{unitary cube}$$

where $w \in L^1_{Q-per}(\mathbb{R}^N; \mathbb{R}^d)$, $\mathcal{A}w = 0$, $\int_Q w = 0$

$\mathcal{A}$-quasiconvex functions are convex in the directions of the characteristic cone (Murat, Tartar)

$$\mathcal{C} := \left\{ \lambda \in \mathbb{R}^d : \exists w \in \mathbb{R}^N \setminus \{0\} \left(\sum_{i=1}^N A^{(i)} w_i \right) \lambda = 0 \right\}$$

Example: $\mathcal{A} = \mathrm{curl}, \mathcal{C} =$rank-one directions

Projections

Proposition 0.1. *(Fonseca and Muller) Let $1 < q < \infty$. There exists a projection operator*

$$P : L^q_{Q_{\mathrm{per}}} \to L^q_{Q_{\mathrm{per}}}$$

such that

- *i)* $P^2 = P$
- *ii)* $\mathcal{A}P = 0$
- *iii)* $||u - Pu||_{L^q_{Q_{\mathrm{per}}}} \leq C||\mathcal{A}u||_{W^{-1,q}}$

$q = 1$ there is no projection proposition...

Consequence of $\mathcal{A}$-quasiconvexity

$$\liminf \int_Q f(w_n(x))\, dx \geq f(a)$$

for every $\{w_n\} \subset L^q(Q)$ such that

$$w_n \overset{*}{\rightharpoonup} a \text{ in } \mathcal{M}$$

$$|w_n| \overset{*}{\rightharpoonup} \Lambda \text{ such that } \Lambda(\partial Q) = 0$$

$$\mathcal{A} w_n \to 0 \text{ in } W^{-1,q} \text{ for some } 1 < q < \frac{N}{N-1}$$

Proof

It is enough to consider regular sequences, i.e.,

$$\liminf I(w_n) \geq I(\mu)$$

for $\{w_n\} \subset C^\infty$ with

$$w_n \overset{*}{\rightharpoonup} \mu \ \text{ in } \mathcal{M}$$

$$|w_n| \overset{*}{\rightharpoonup} \Lambda \ \text{ such that } \Lambda(\partial\Omega) = 0$$

$$\mathcal{A}w_n \to 0 \ \text{ in } W^{-1,q} \text{ for some } 1 < q < \frac{N}{N-1}$$

Proof

$$I(w_n) = \int_\Omega f(w_n(x))\,dx \leq C$$

thus

$$\lambda_n := f(w_n) \overset{*}{\rightharpoonup} \lambda$$

We have

$$-C(1+|\mu|) \leq \lambda \leq C(1+|\Lambda|)$$

(recall $w_n \overset{*}{\rightharpoonup} \mu, |w_n| \overset{*}{\rightharpoonup} \Lambda$)

Proof

$$\begin{aligned}\lim I(w_n) &= \lim \lambda_n(\Omega) = \lambda(\Omega)\\ &= \int_\Omega \frac{d\lambda}{d\mathcal{L}^N}\,dx + \int_\Omega \frac{d\lambda}{d|\mu_s|}\,d|\mu_s| + \lambda_s\end{aligned}$$

where λ_s is singular wr to $\mathcal{L}^N + |\mu_s|$

As $\lambda_s \geq 0$ we have

$$\lim I(w_n) = \lambda(\Omega) \geq \int_\Omega \frac{d\lambda}{d\mathcal{L}^N}\,dx + \int_\Omega \frac{d\lambda}{d|\mu_s|}\,d|\mu_s|$$

Proof

Thus to get the lsc result it is enough to prove the pointwise estimates

$$\frac{d\lambda}{d\mathcal{L}^N}(x) \geq f(\mu_a(x)) \text{ for a.e. } x \in \Omega$$

$$\frac{d\lambda}{d|\mu_s|}(x) \geq f^\infty\left(\frac{d\mu_s}{d|\mu^s|}(x)\right) \text{ for } |\mu^s|\text{-a.e. } x \in \Omega$$

Proof

Absolutely continuous part

$$
\begin{aligned}
\frac{d\lambda}{d\mathcal{L}^N}(x_0) &= \lim_{k} \frac{\lambda(Q(x_0;r_k))}{r_k^N} \\
&= \lim_{n,k} \frac{\int_{Q(x_0;r_k)} f(w_n(x))\,dx}{r_k^N} \\
&= \lim_{n,k} \int_Q f(w_n(x_0 + r_k y))\,dy
\end{aligned}
$$

Thus

$$v_{n,k}(y) := w_n(x_0 + r_k y) \overset{*}{\rightharpoonup} \mu_a(x_0)$$

and then it is enough to use the $\mathcal{A}$-quasiconvexity of f

Proof

Singular part

$$\begin{aligned}\frac{d\lambda}{d|\mu_s|}(x_0) &= \lim_k \frac{\lambda(Q(x_0;r_k))}{|\mu_s|(Q(x_0;r_k))} \\ &= \lim_{n,k} \frac{\int_{Q(x_0;r_k)} f(w_n(x))\,dx}{|\mu_s|(Q(x_0;r_k))} \\ &= \lim_{n,k} \int_Q f_k(v_{n,k}(y))\,dy\end{aligned}$$

where $t_k := \frac{|\mu_s|(Q(x_0;r_k))}{r_k^N} \to \infty$ and $f_k(\xi) := \frac{f(t_k\xi)}{t_k}$. We also have

$$v_{n,k}(y) := \frac{w_n(x_0 + r_k y)}{t_k} \overset{*}{\rightharpoonup} \tau \text{(not a constant)}$$

Proof

Singular part

$$\tau = v_{x_0}|\tau|$$

Using the equation $\mathcal{A}\tau = 0$ we can prove that

- If $v_{x_0} \notin \mathcal{C}$ then $\tau = v_{x_0} = \frac{d\mu_s}{d|\mu_s|}(x_0)$ (constant)

- If $v_{x_0} \in \mathcal{C}$ then $\tau = \tau(x.w^1, x.w^2, .., x.w^l)$ where w^i, $i = 1, .., l$ is an orthonormal basis for the space of vectors z such that

$$\left(\sum_{i=1}^{N} A^{(i)} z_i\right) v_{x_0} = 0$$

Proof

Singular part

Extend by periodicity τ (keeping the equation!!!!) to all $\mathbb{R}^N$ and then average to get a constant.

Curl

$$\mathcal{A} = \mathrm{curl}$$

In the singular part

$$\tau = Du$$

where $u = a\varphi(x.w)$ where $\varphi \in BV$.