Existence of travelling waves for a degenerate advection-diffusion equation

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CNA, September 11, 2012

INSTITUT de MATHEMATIQUES de TOULOUSE

Université Paul Sabatier TOULOUSE III

Université de Toulouse

Overview

Model

For $(t, x, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-1}$

$$\partial_t u - \nabla \cdot (\lambda \nabla u) + \alpha(y) \mathbf{e_x} \cdot \nabla u_x = 0$$

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Outline of the talk

1. **Motivations**
2. **Existence and qualitative properties**
   joint work with A. Novikov and J.-M. Roquejoffre
3. **Investigation of the free-boundary**
4. **Perspectives**

1 Motivations

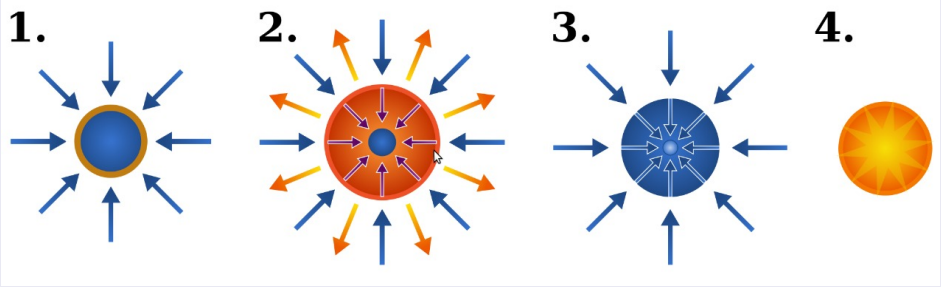
2 Existence and qualitative properties
- Formulation of the problem
- Results
- Sketch of the proof

3 Investigation of the free-boundary
- A generic picture...
- Comparaison with PME
- Heuristic scenario for corners

4 Perspectives

Physical motivations

Inertial Confinement Fusion (ICF)



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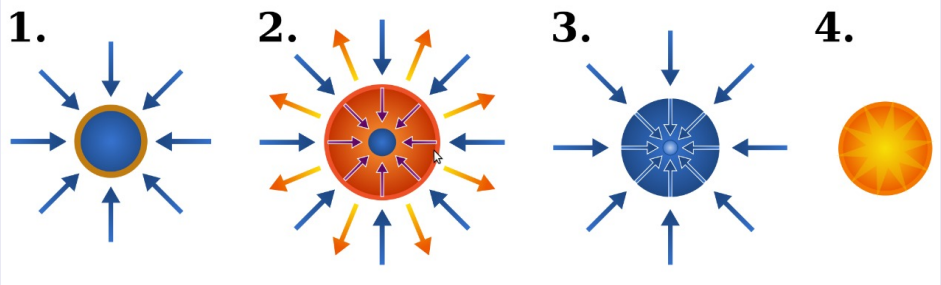

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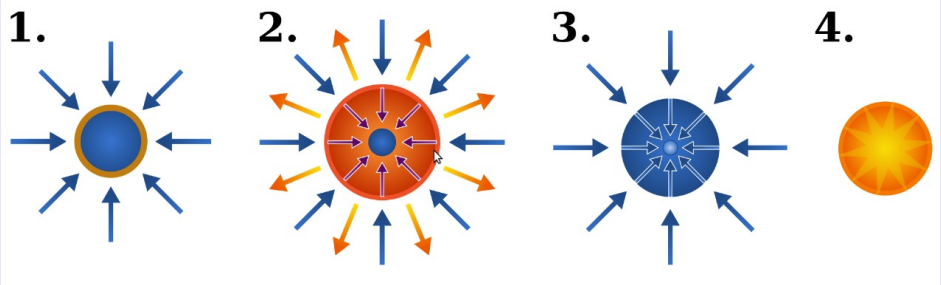

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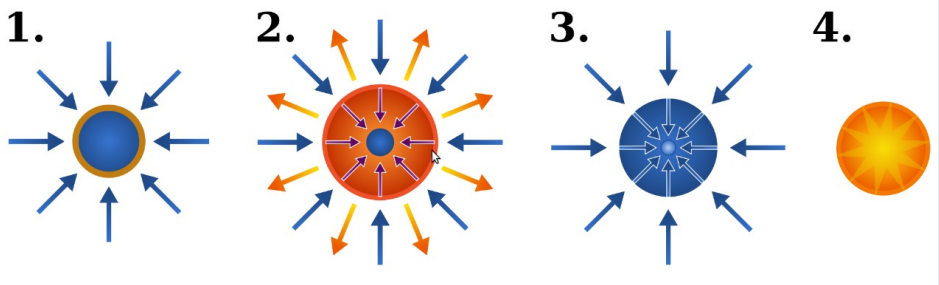

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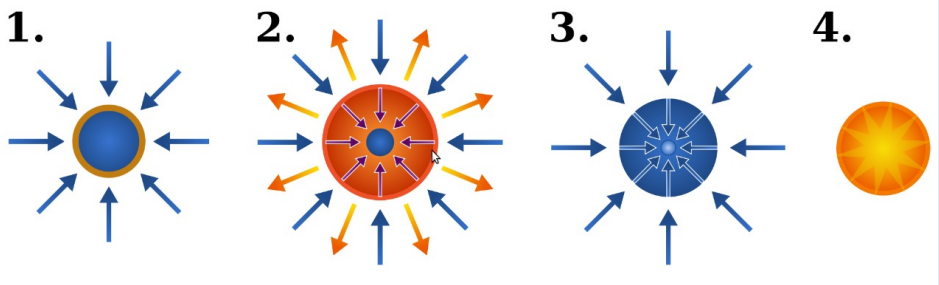

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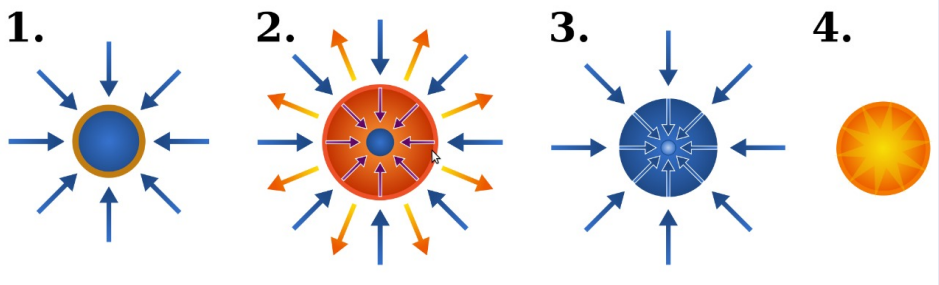

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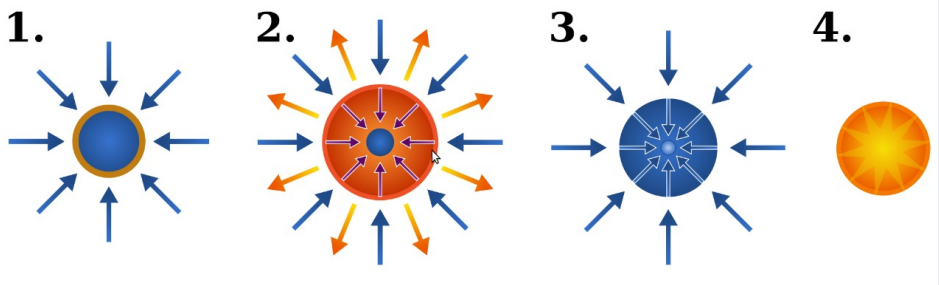

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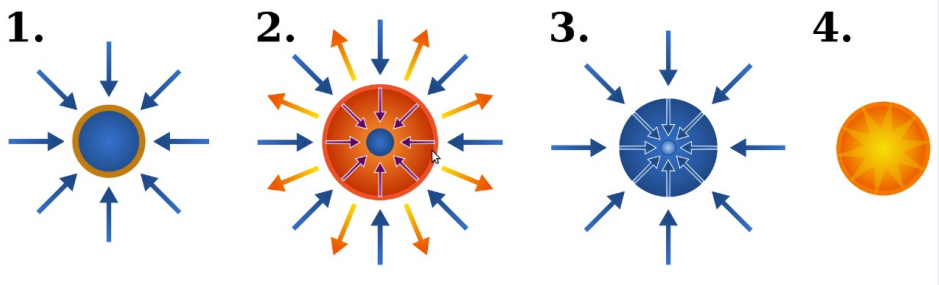

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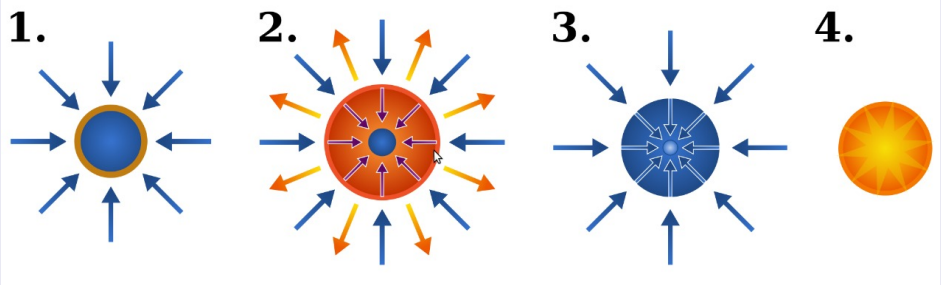

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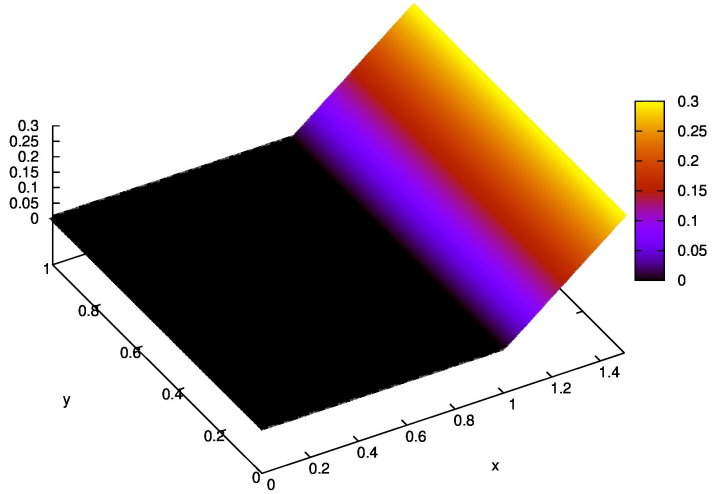

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- $x \to +\infty$: slope = speed, $p \sim cx$ (PME and ICF)

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Definition 1 : weak solutions

$p \geq 0$ continuous is a weak solution iff

$$\forall \varphi \in \mathcal{C}_c^\infty, \qquad \int u^{m+1}\Delta\varphi + (c+\alpha)u\varphi_x = 0$$

Notions of solution (continued)

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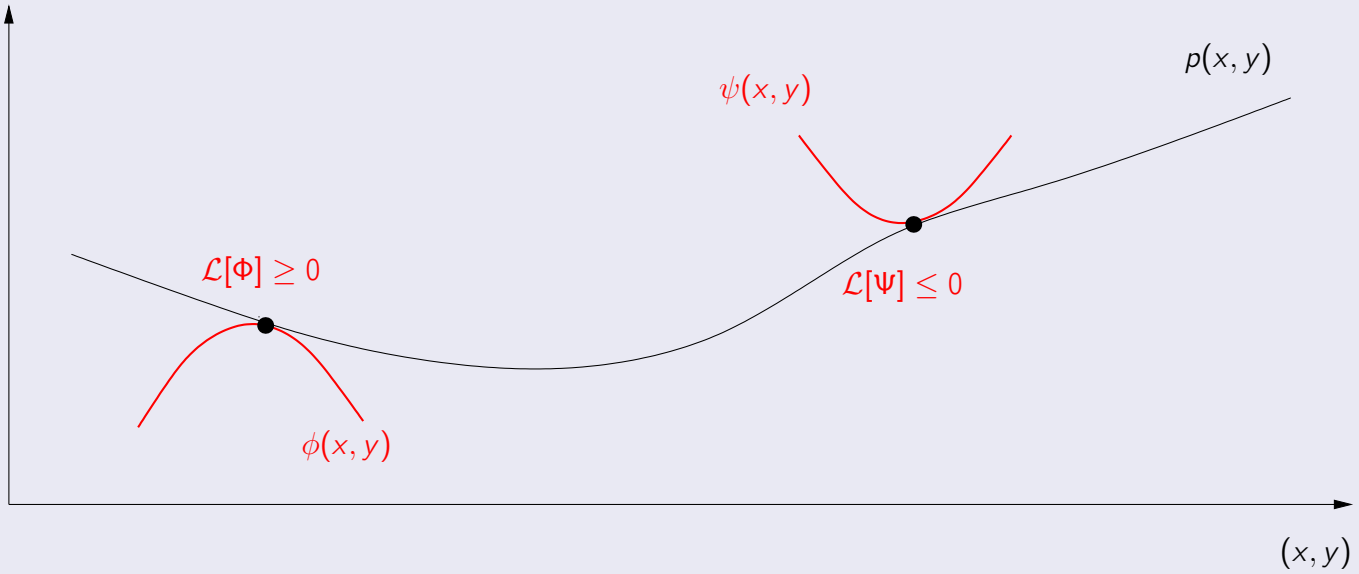

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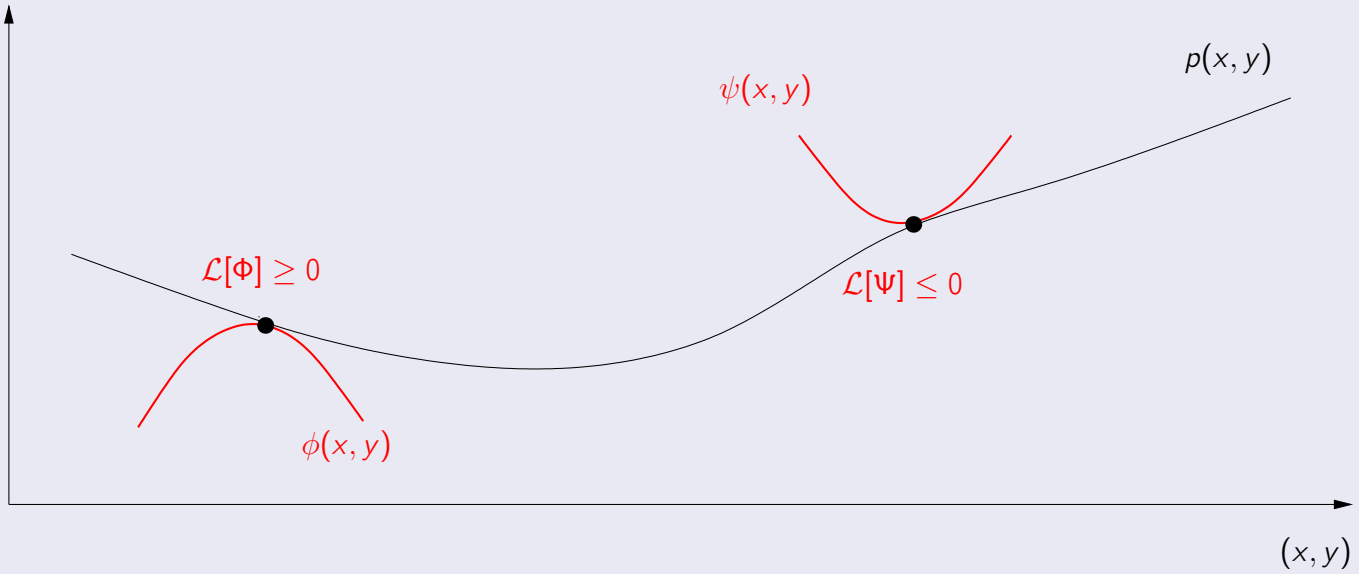

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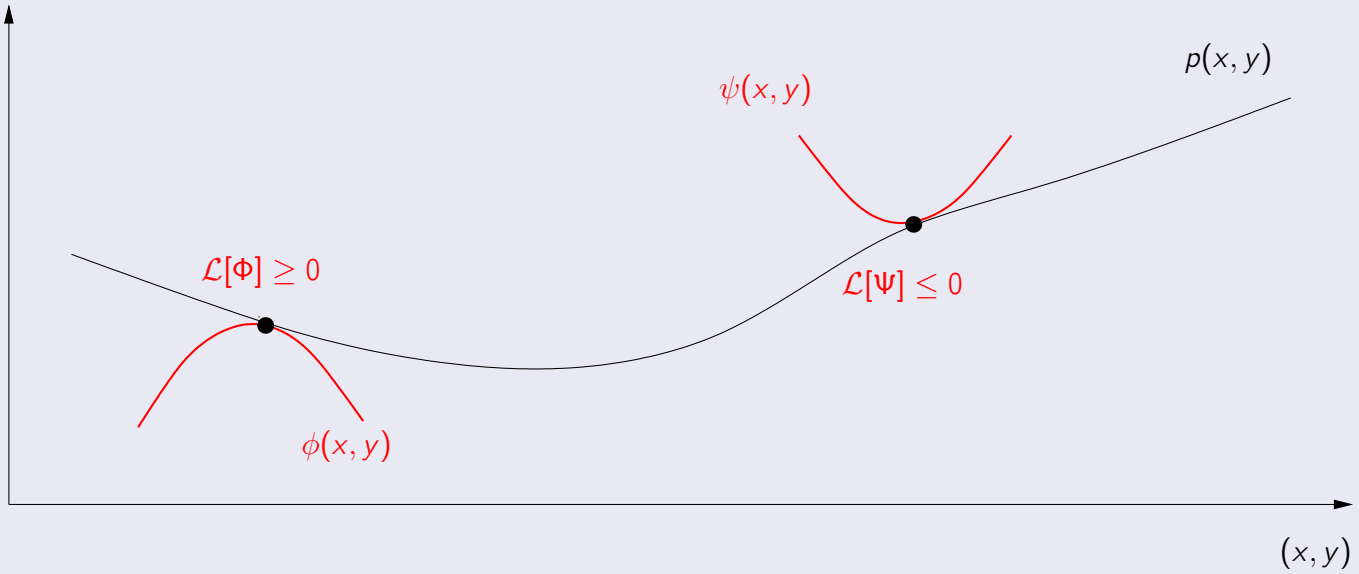

In general "viscosity $\neq$ weak". Here OK because $p \in$ Lipschitz.

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- Formulation of the problem
- Results
- Sketch of the proof

3 Investigation of the free-boundary
- A generic picture...
- Comparaison with PME
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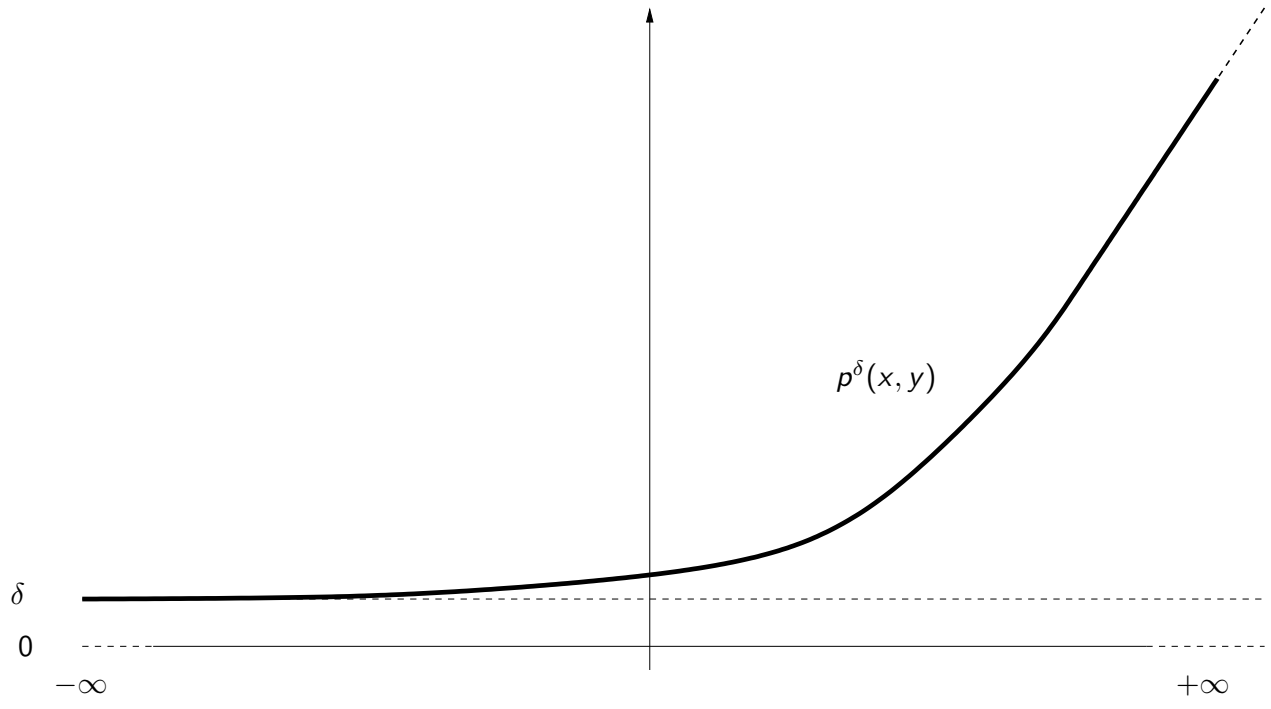

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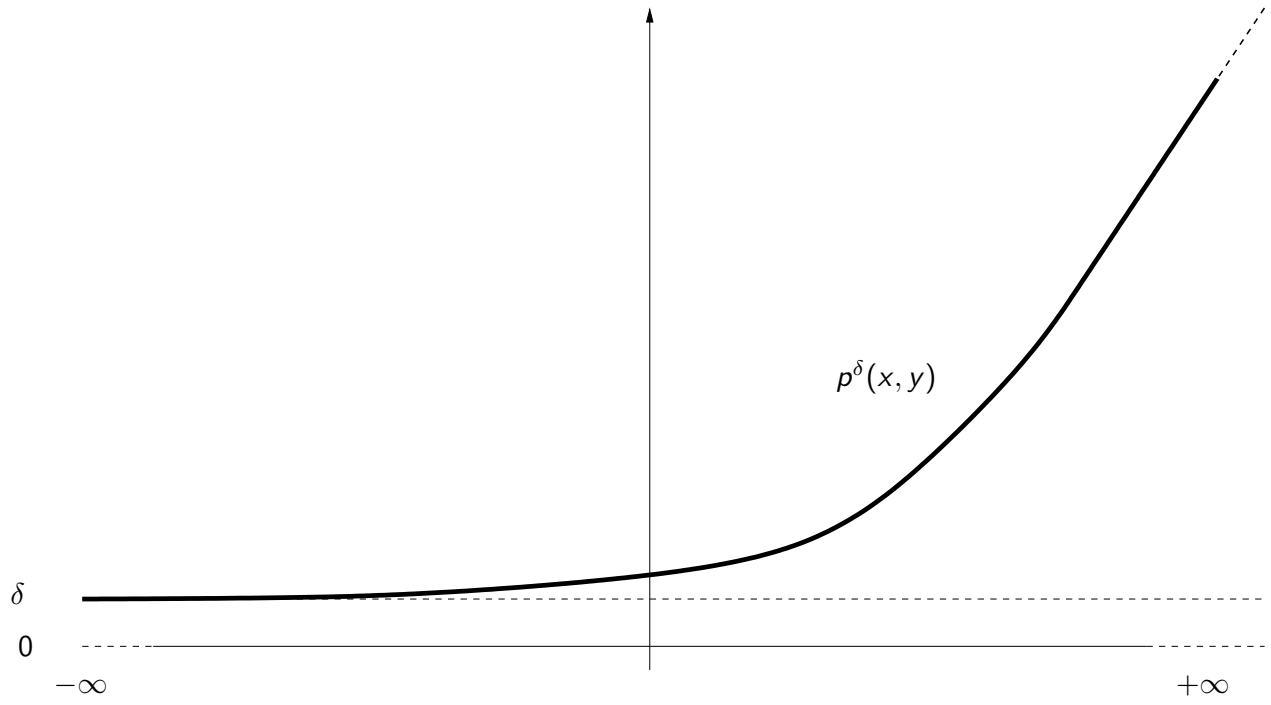

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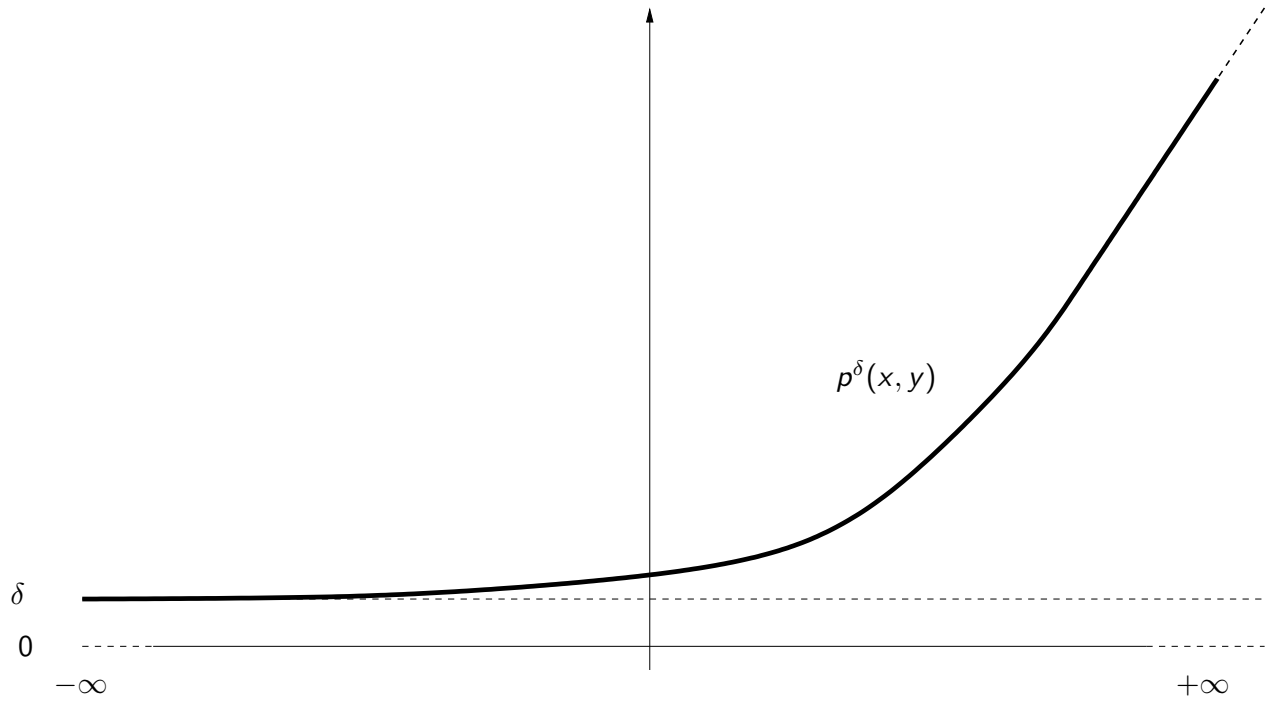

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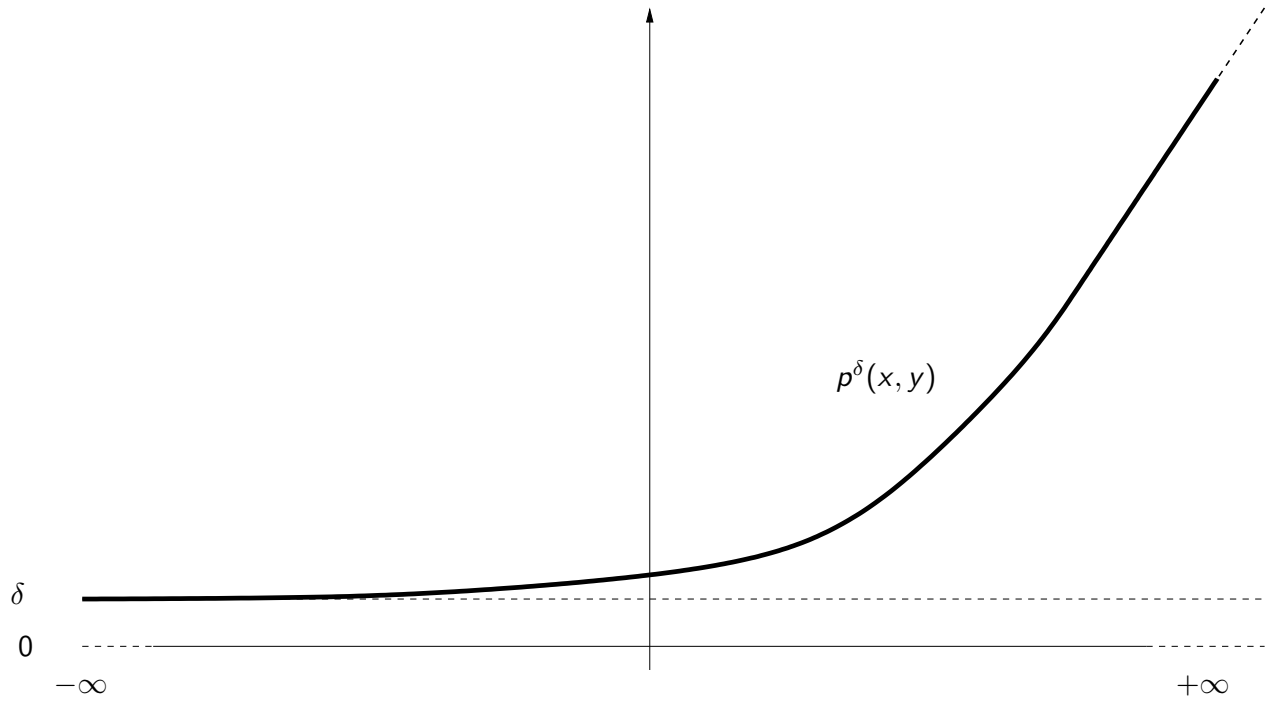

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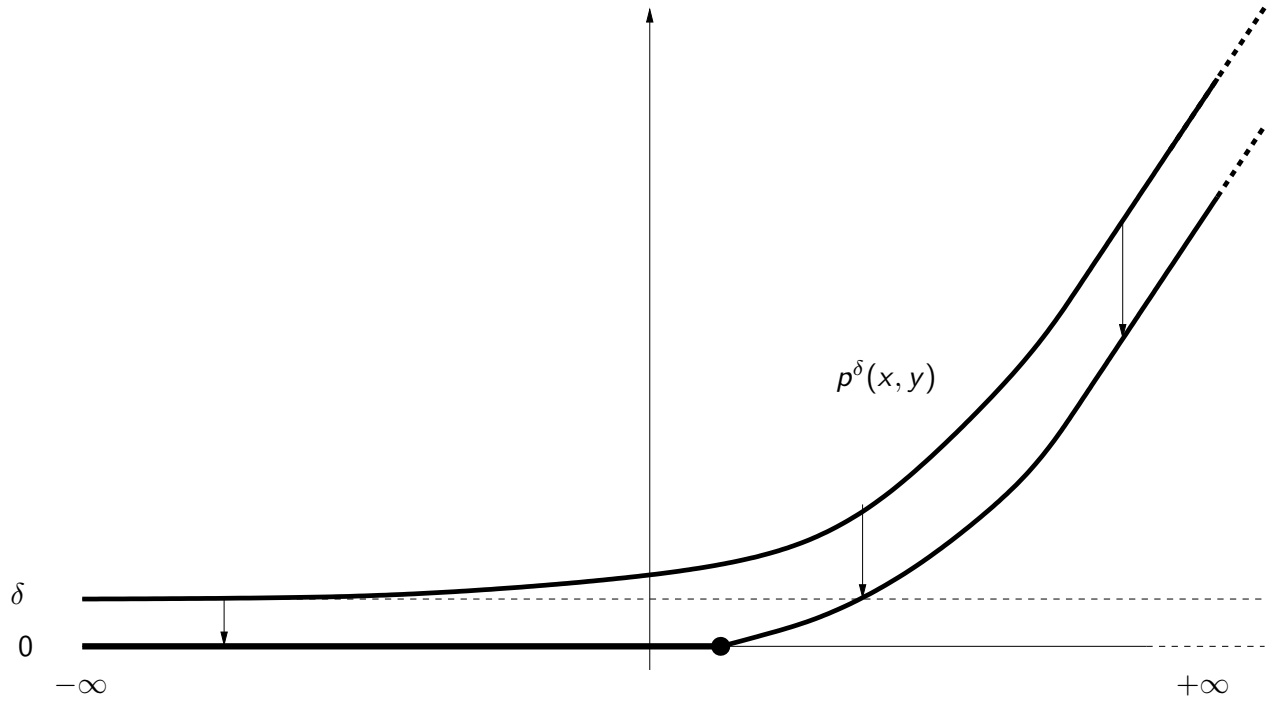

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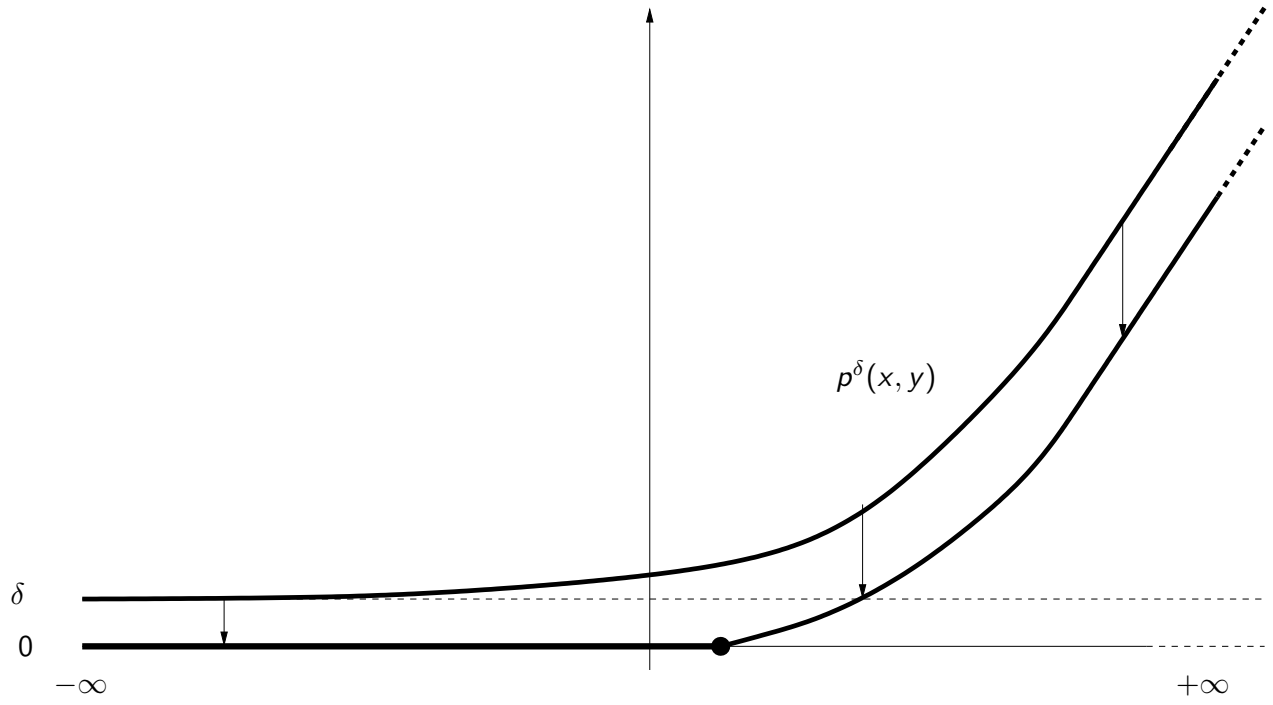

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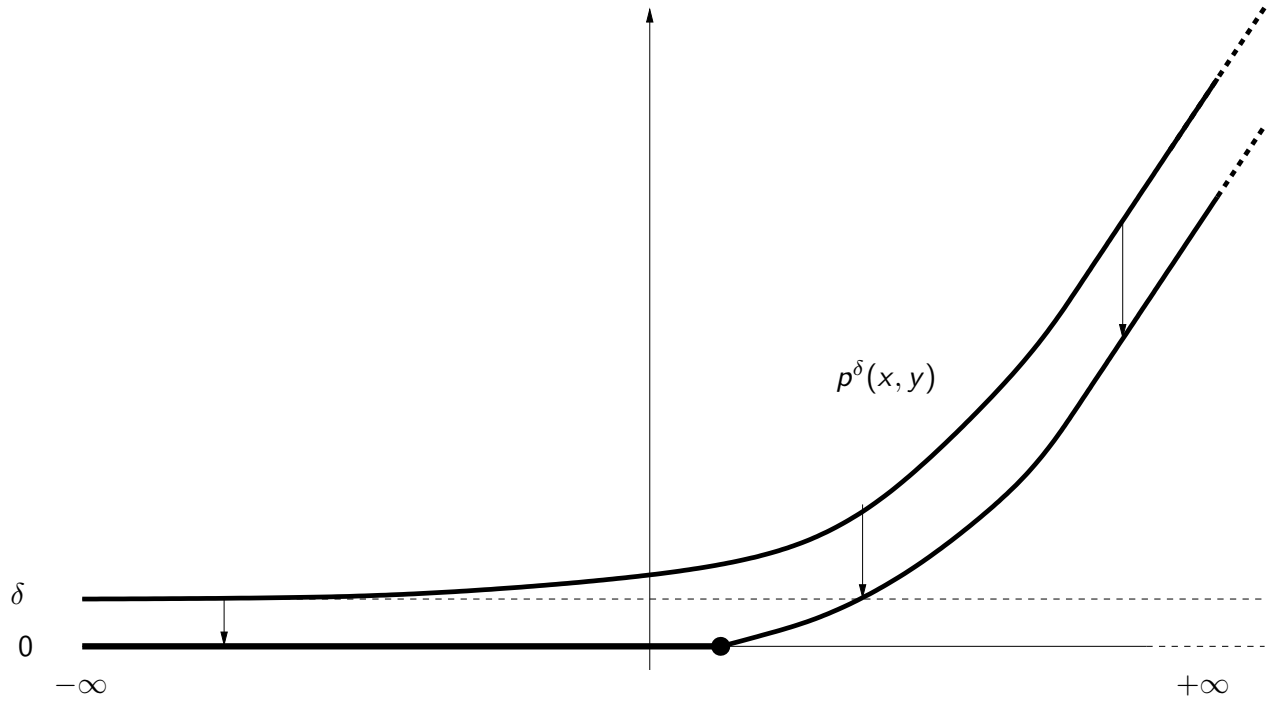

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$$I(y) := \inf\big(x, \quad p(x,y) > 0\big), \qquad p(x,y) > 0 \Leftrightarrow x > I(y)$$

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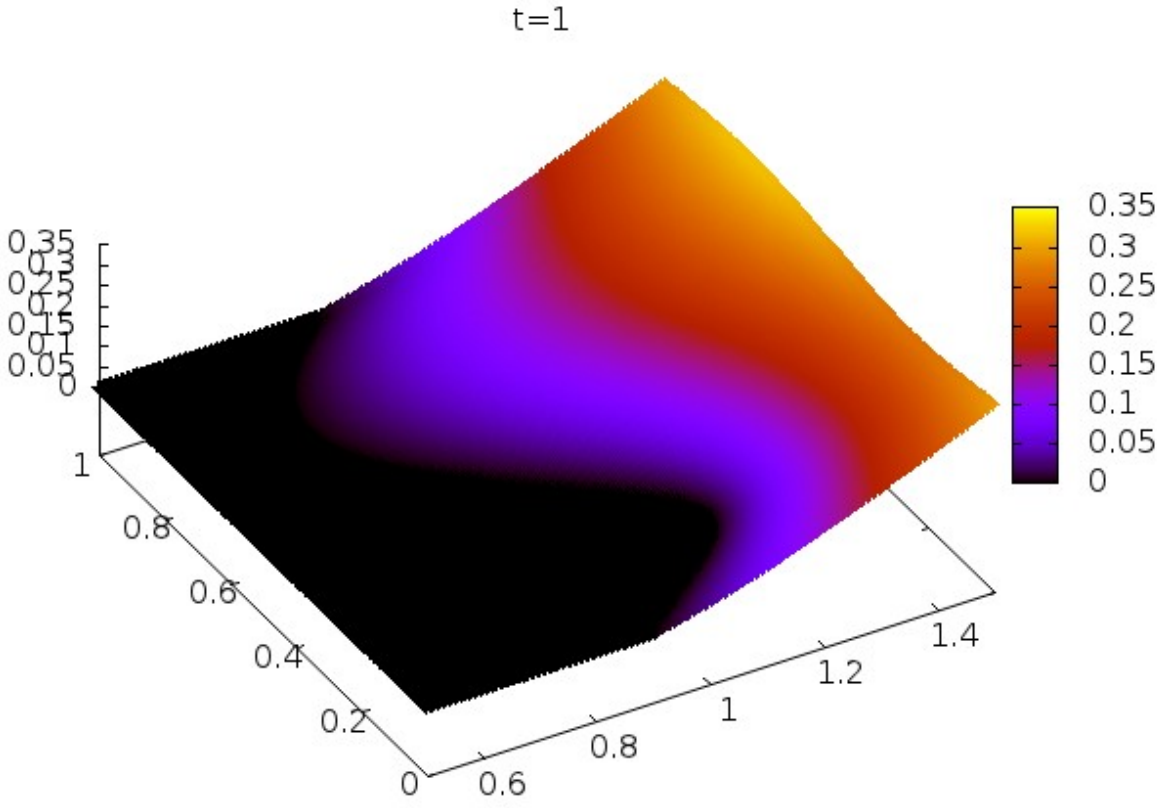
2 Existence and qualitative properties
- Formulation of the problem
- Results
- Sketch of the proof

3 Investigation of the free-boundary
- A generic picture...
- Comparaison with PME
- Heuristic scenario for corners

4 Perspectives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0.4
0.2
0
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0.1
0.05
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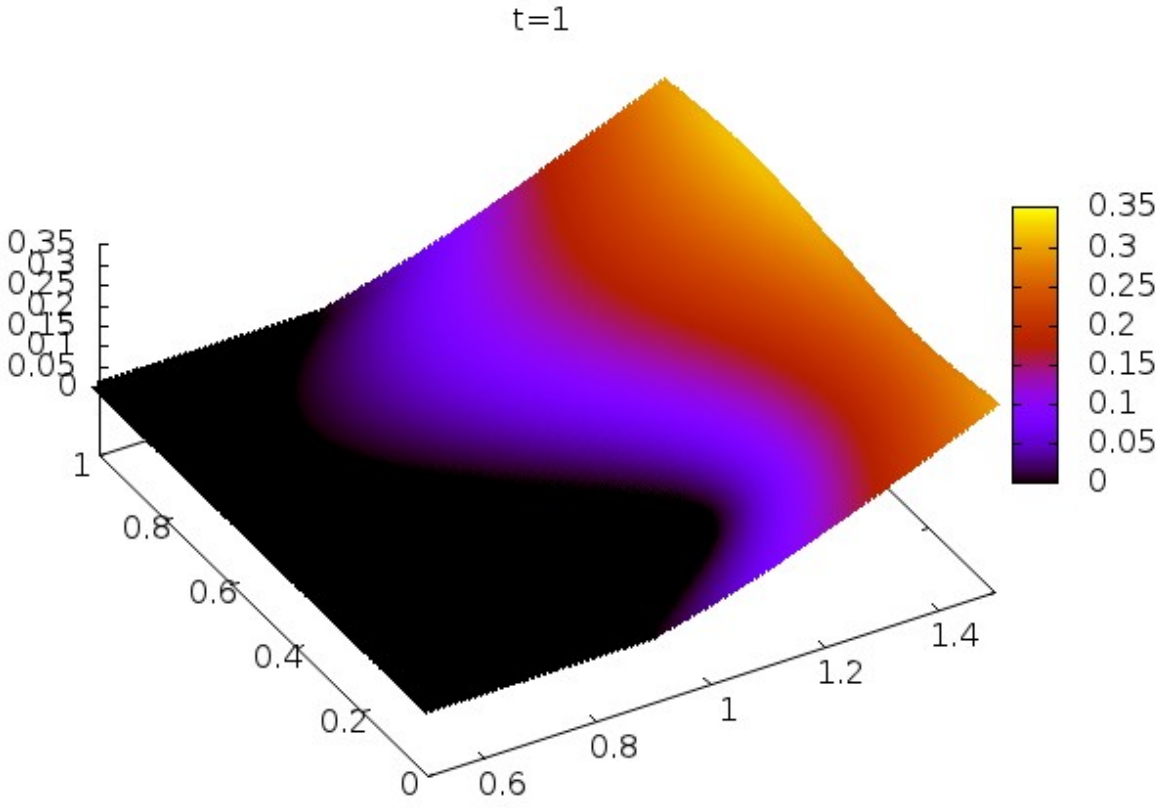
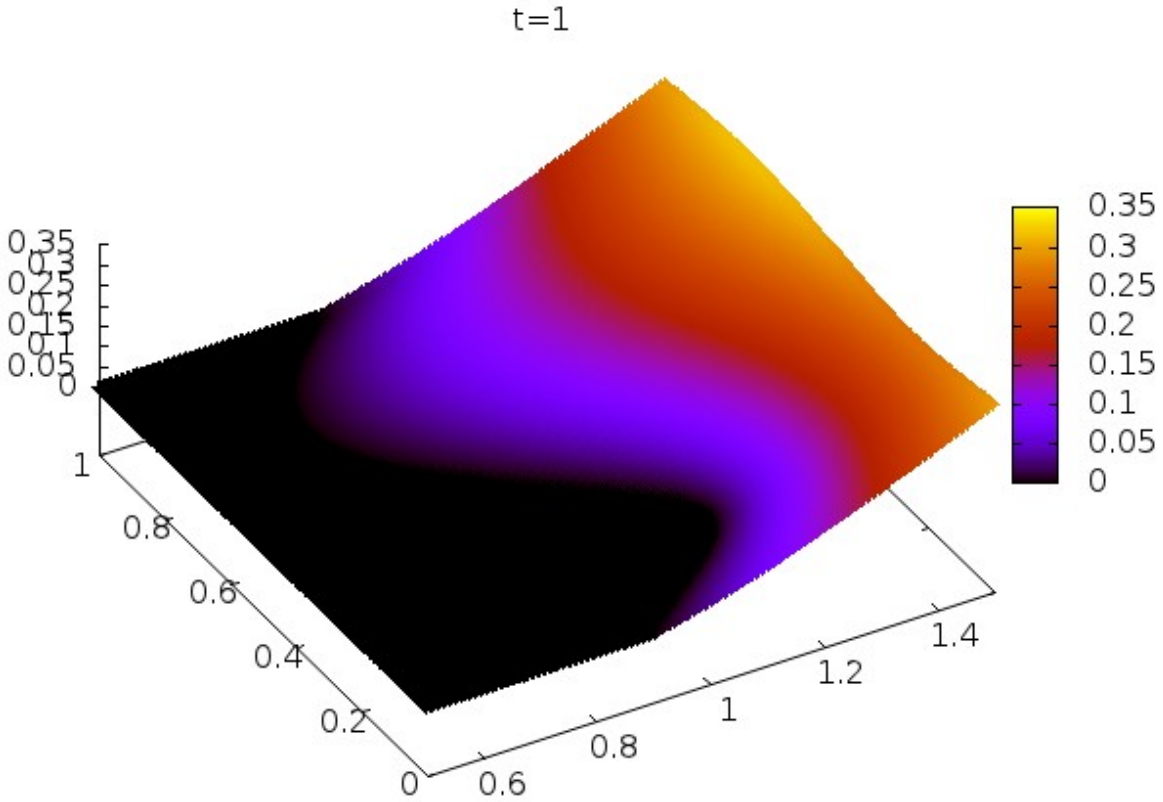


Reminder

Free-boundary $p(x, y) > 0 \Leftrightarrow x > l(y)$, periodical and



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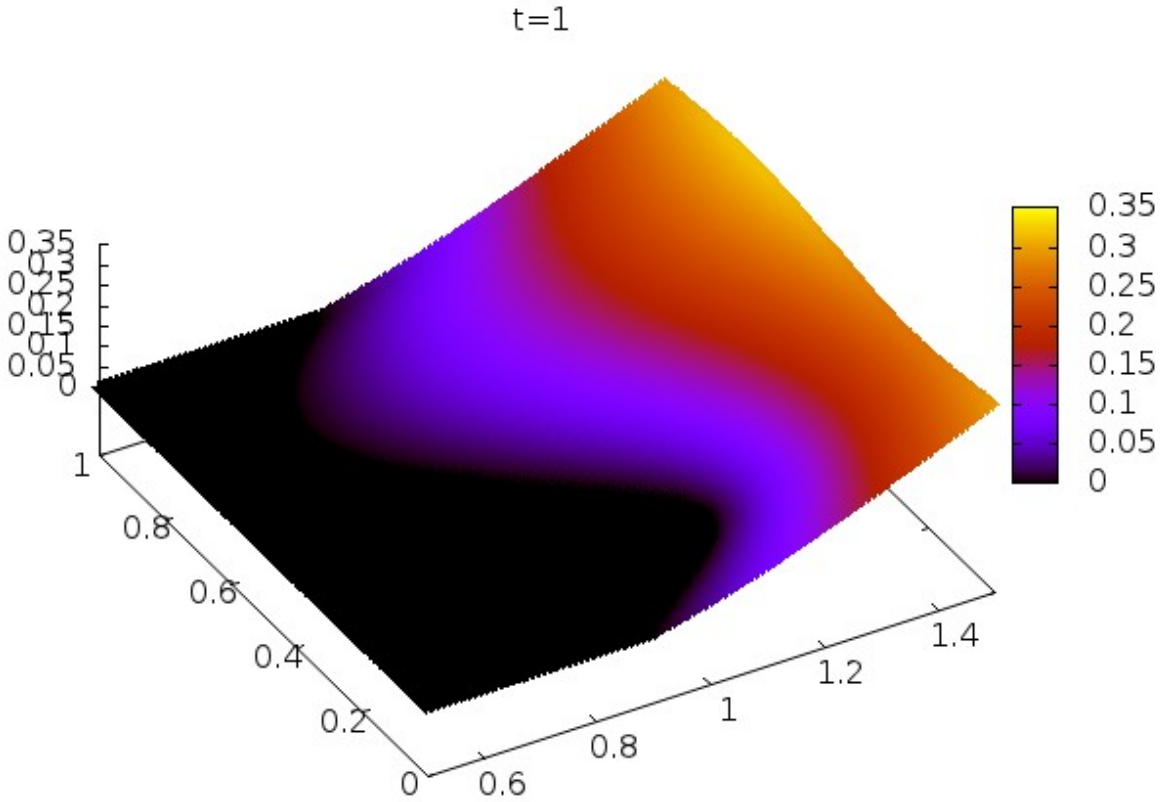

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The Porous Media Equation

$$\partial_t p - mp\Delta p = |\nabla p|^2 \tag{PME}$$

Differential equation

Free-boundary $\Gamma_t = \partial\{p(t,.) > 0\}$ and $p|_{\Gamma_t} = 0$:

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Wave solution $\Rightarrow$ trivial time-evolution. Geometrical description ?

Case $\alpha(y) \neq 0$

$$-mp\Delta p + (c + \alpha)p_x = |\nabla p|^2$$

Definition

Non-degeneracy (strong) : $p_x \geq a > 0$ in the neighborhood of Γ

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Lipschitz regularity = optimal ? $\exists$ corners ?

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Geometrical equation (HJ) : stationary equivalent of the free-boundary differential equation in the PME.

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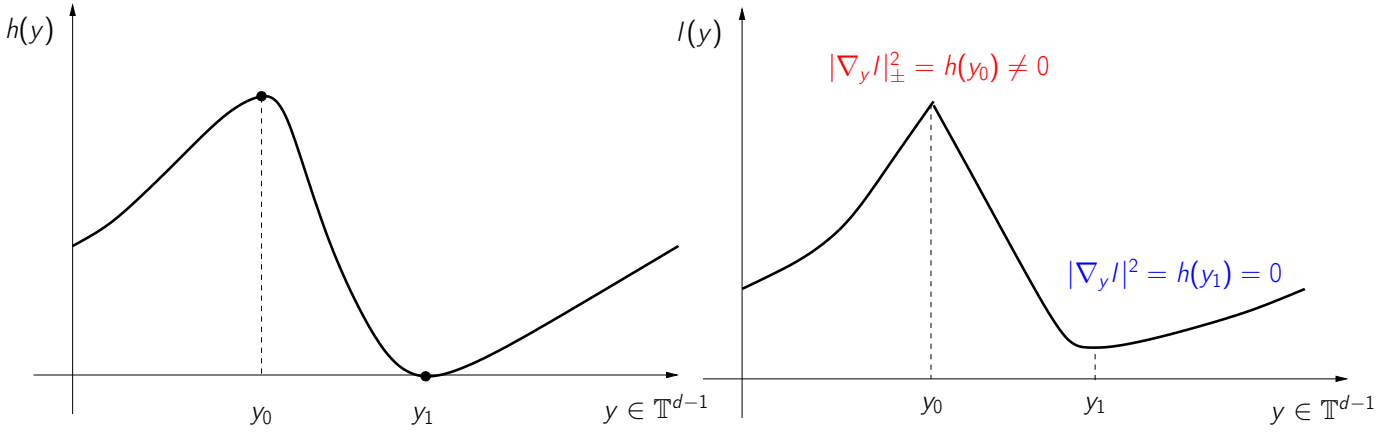

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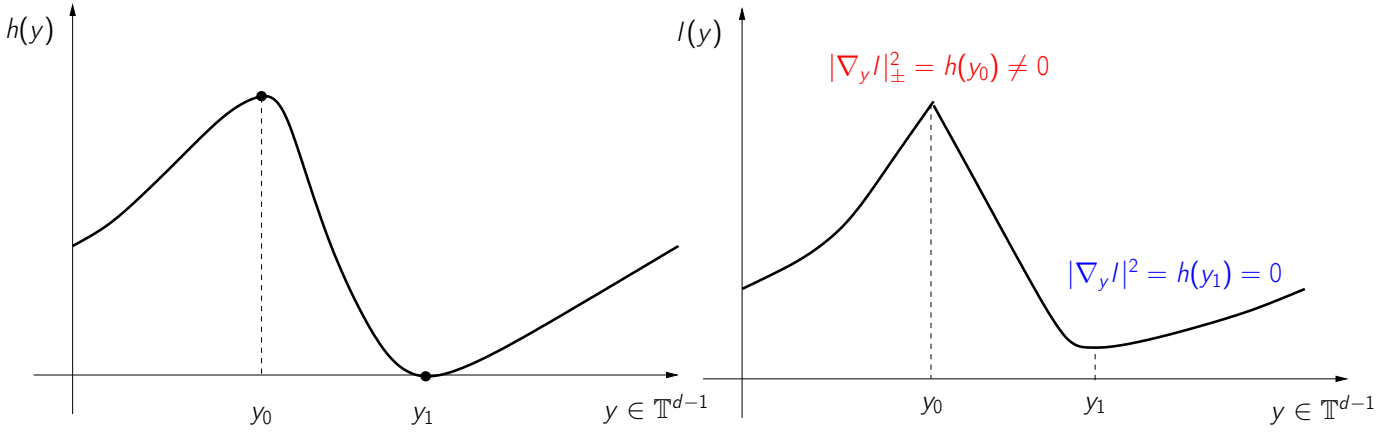

Existence of corners

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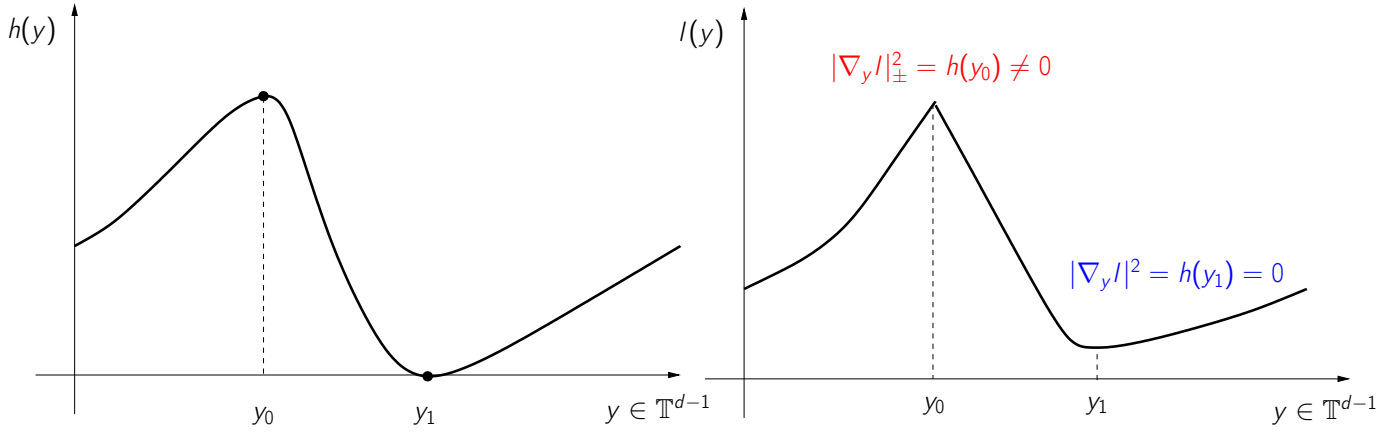

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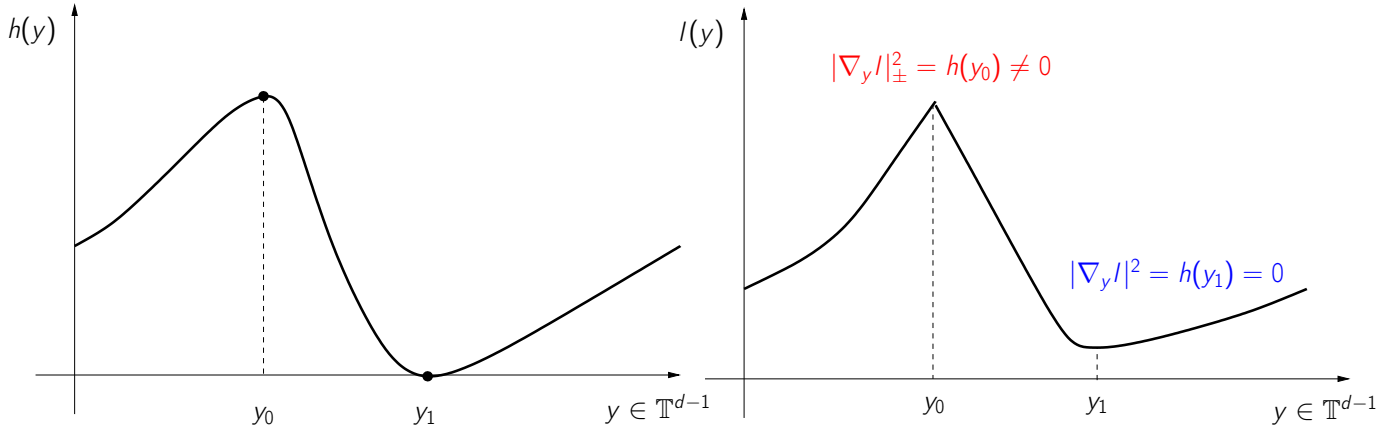

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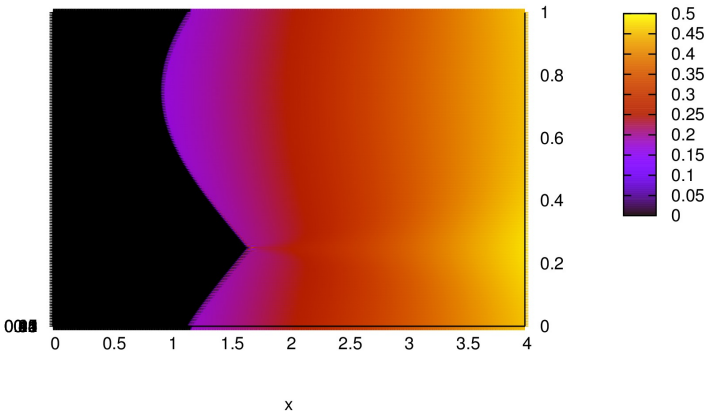
- Here $h(y) = \frac{c+\alpha(y)}{p_x|_\Gamma(y)} - 1$. Zeros of h ?

$m < 1$

$m > 1$

$m < 1$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4
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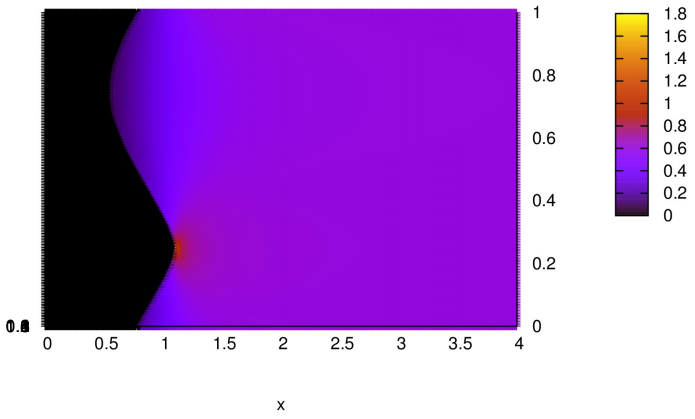
0.3
0.25
0.2
0.15
0.1
0.05
0

$m > 1$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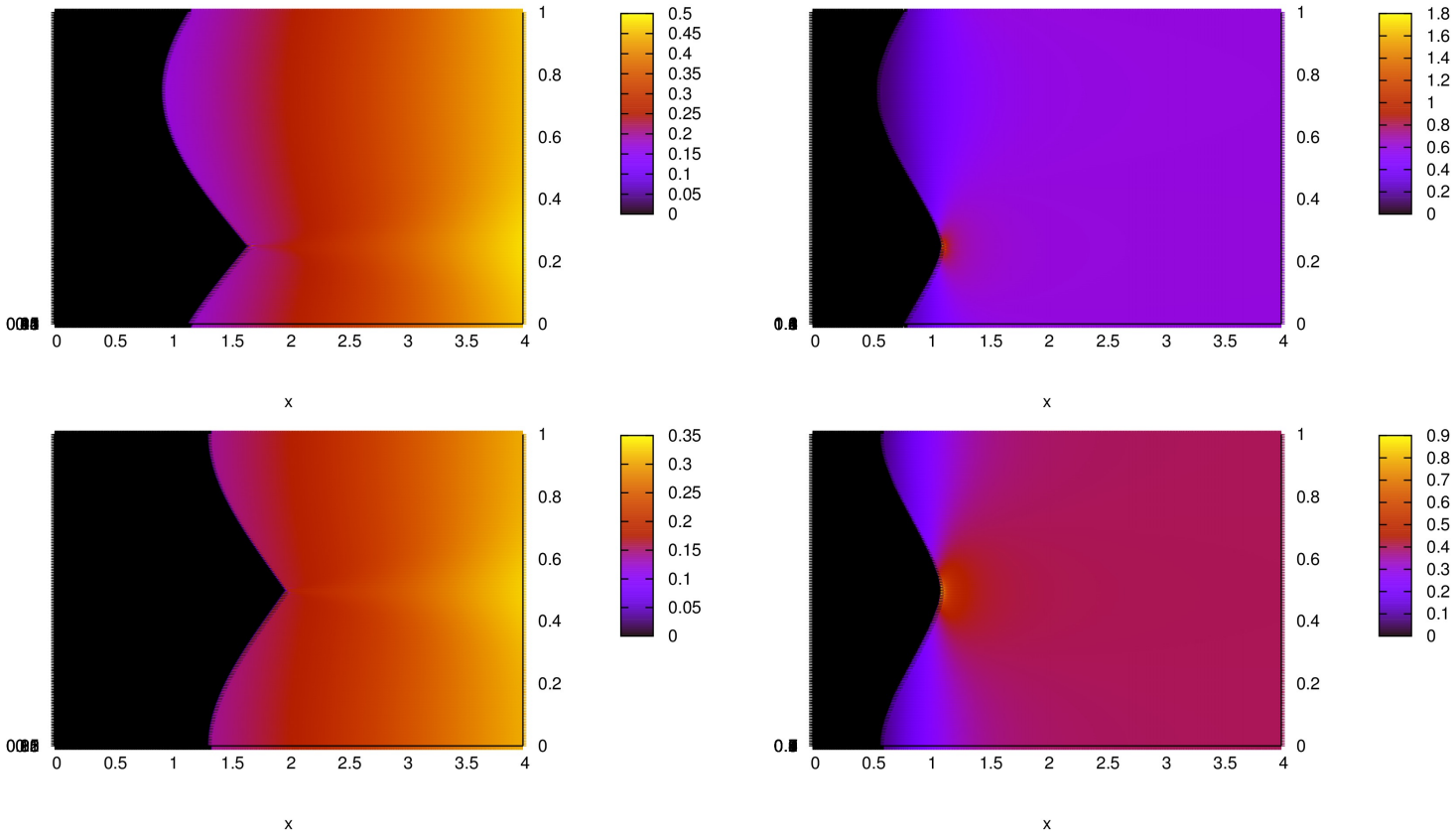

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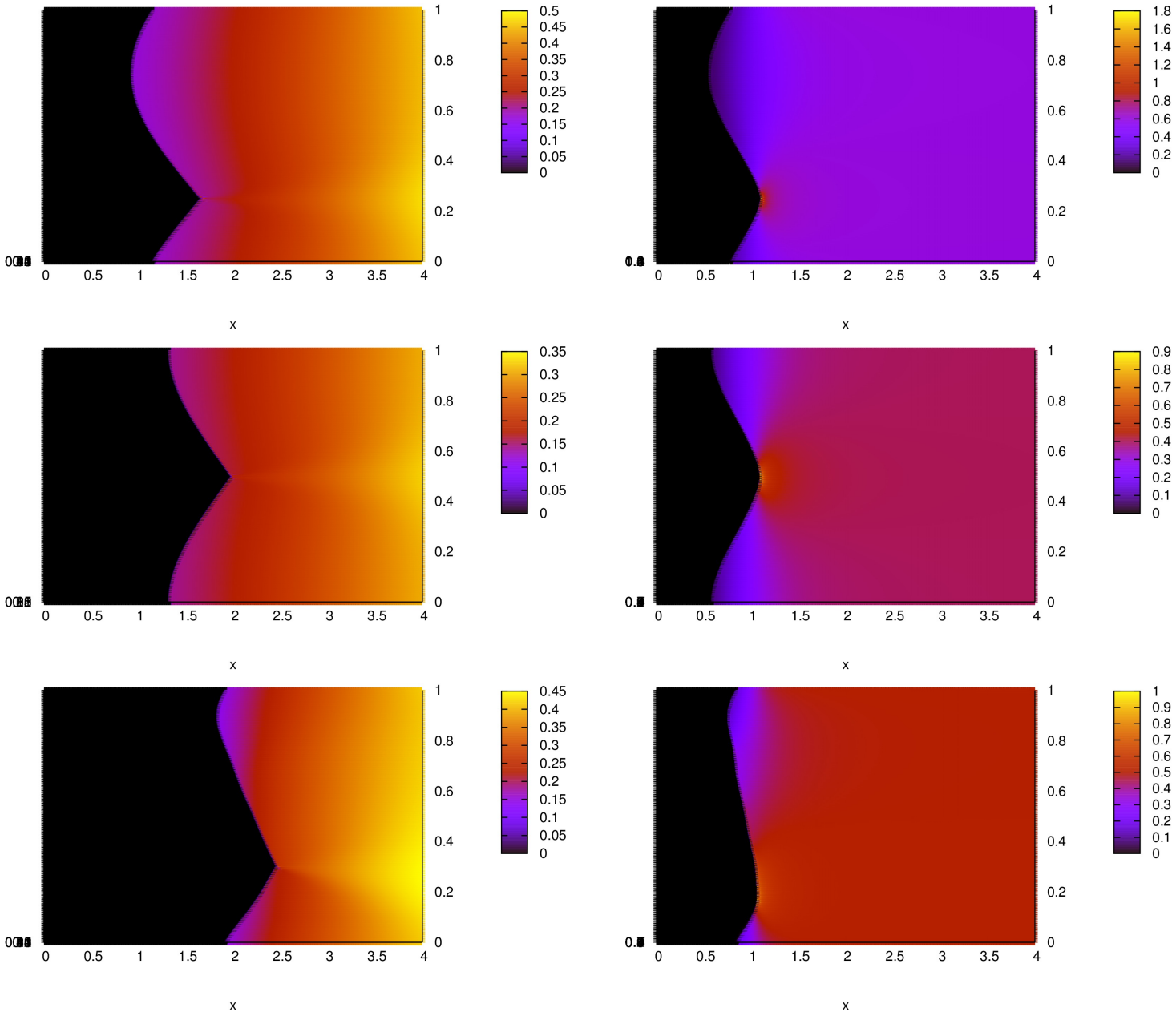

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Possible extensions

- Stability of the wave for the Cauchy problem
- Construction of particular explicit solutions
- More general flows, non-periodical
- Cell flows

Thank you for listening !