Nonlinear elliptic systems with general growth

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– Tuesday, September 18^{th}, 2012, 1:30 p.m. –

How do the systems we find in nature evolve and why?

Some of the most challenging of these involve interactions among many communicating length and time scales and, even when equations for their motion have been written down on paper, little is understood about their nature.

David Kinderlehrer

at his webpage: http://www.math.cmu.edu/math/faculty/kinderlehrer.php

We also impose mathematical restrictions on the written down equations!

In PDEs it is known the notion of "natural growth conditions", which are mathematical conditions useful to state and to prove mathematical results.

On the contrary, for applications, one should consider more "general growth conditions", for which the analysis is sometime much more complicated.

Mathematically, we are led to classify problems in PDEs by mean of power (polynomial) growth.

This book addresses fundamental questions related to lower semi-continuity and relaxation of functionals within the unconstrained setting, mainly in L^p spaces. It prepares the ground for the second volume in which the variational treatment of functionals involving fields and their derivatives will be undertaken within the framework of Sobolev spaces.

Irene Fonseca and Giovanni Leoni

Modern Methods in the Calculus of Variations: L^p Spaces

Springer Monographs in Mathematics, Springer, 2007.

PDEs AND SYSTEMS

We consider partial differential equations in divergence form, for $x \in \Omega$, open set of $\mathbb{R}^n$, $n \geq 2$, of the type

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a^i \left(x, u\left(x\right), Du\left(x\right)\right) = b\left(x, u\left(x\right), Du\left(x\right)\right)$$

with the vector field $a^i\left(x, s, \xi\right)$ locally Lipschitz continuous in $\Omega \times \mathbb{R} \times \mathbb{R}^n$.

More generally we consider systems of m partial differential equations in divergence form, of the type

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a^i_\alpha \left(x, u\left(x\right), Du\left(x\right)\right) = b_\alpha\left(x, u\left(x\right), Du\left(x\right)\right), \quad \alpha = 1, 2, \ldots, m$$

for maps $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$. Here Du and $\left(a^i_\alpha\right)$ are maps with values in the set of $m \times n$ matrices.

INTEGRALS OF THE CALCULUS OF VARIATIONS

Special (but relevant) case when the previous pde is the *Euler's first variation* of an *integral of the calculus of variations* of the type

$$\int_{\Omega} f\left(x, u\left(x\right), Du\left(x\right)\right)\, dx$$

with $\frac{\partial f(x,s,\xi)}{\partial \xi^i} = a^i\left(x, s, \xi\right)$, $\frac{\partial f(x,s,\xi)}{\partial s} = b\left(x, s, \xi\right)$ for the case of equations.

While, if $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, $m > 1$, then the Euler's first variation gives origin to a system with

$$\frac{\partial f\left(x, s, \xi\right)}{\partial \xi^i_\alpha} = a^i_\alpha\left(x, s, \xi\right) , \quad \frac{\partial f\left(x, s, \xi\right)}{\partial s_\alpha} = b_\alpha\left(x, s, \xi\right)$$

NATURAL GROWTH CONDITIONS

The classical models of equations and systems, related to the $p-$Laplacian for some $p > 1$, are either

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(|Du(x)|^{p-2} \frac{\partial u_\alpha}{\partial x_i} \right) = 0\,,, \qquad \alpha = 1, 2, \ldots, m,$$

or

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left(1 + |Du(x)|^2\right)^{\frac{p-2}{2}} \frac{\partial u_\alpha}{\partial x_i} \right) = 0\,,, \qquad \alpha = 1, 2, \ldots, m,$$

for which classically we say that we have *natural growth conditions*. The classical literature on *regularity* is very large. For the case of equations ($m = 1$) a classical book of references is by Ladyzhenskaya-Ural'tseva (largely based on the work by De Giorgi).

Under the terminology of *"natural growth conditions"* we refer to integrals

$$\int_{\Omega} f\left(x, u\left(x\right), Du\left(x\right)\right)\, dx$$

with growth conditions on the integrand f of the type

$$m\left|\xi\right|^{p} \leq f\left(x, s, \xi\right) \leq M\left(1 + \left|\xi\right|^{p}\right)$$

for all $x \in \Omega$, $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$, for some $p > 1$ and $m > 0$. Growth conditions also for the first and the second derivatives of f. In particular, for the matrix $D^2 f = \left(\frac{\partial^2 f}{\partial \xi_i \partial \xi_j}\right)$, the *natural growth conditions* require that

$$m\left(1 + \left|\xi\right|^2\right)^{\frac{p-2}{2}} \left|\lambda\right|^2 \leq \sum_{i,j=1}^{n} \frac{\partial^2 f\left(x, s, \xi\right)}{\partial \xi_i \partial \xi_j} \lambda_i \lambda_j \leq M\left(1 + \left|\xi\right|^2\right)^{\frac{p-2}{2}}$$

and similarly for the other derivatives $\frac{\partial f_{\xi_i}(x,s,\xi)}{\partial x_j}$, $\frac{\partial f_{\xi_i}(x,s,\xi)}{\partial s}$.

SOME SCALAR MODELS

We fist introduce the problem for equations, later we will study systems too.

We consider integrals of the calculus of variations which satisfy either the so called p, q *−growth conditions* for some p, q exponents with $p \leq q$, or *non-standard* or *general* growth conditions.

We begin with an integral of the calculus of variations with *anisotropic growth conditions*: the integrand $f(x, s, \xi)$, with respect to $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$, is a polynomial of different growth in the ξ_i, for instance ...

$$\int_\Omega \left\{ |Du|^2 + (u_{x_n})^4 \right\} dx \, .$$

This is a *"perturbation"* of the Dirichlet integral – a *perturbation* of harmonic functions – but a relevant perturbation, involving the principal part and not lower order terms.

Note that this integral, with *convex* integrand

$$f(\xi) = \xi_1^2 + \xi_2^2 + \ldots + \xi_n^2 + \xi_n^4$$

which is an *analytic* function (a polynomial!), may have not smooth minimizers, not even bounded!

We will see how to construct an unbounded minimizer to the corresponding integral.

However, let us first mention, for instance, some other examples for $p > 1$:

$$\int_\Omega |Du(x)|^{p(x)}\, dx\,,$$

$$\int_\Omega |Du(x)|^p \log(1 + |Du(x)|)\, dx\,,$$

$$\int_\Omega \{a(x)\,|Du(x)|^p + b(x)\,|Du(x)|^q\}\, dx\,,$$

with $q > p$ and

$$\begin{cases} a(x)\,, b(x) \geq 0 \\ a(x) + b(x) > 0\,. \end{cases}$$

EXAMPLE

We consider a set

$$\Omega \subset \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \; x_n \geq \text{const} > 0\}$$

and there the PDE

$$\sum_{i=1}^{n-1} \frac{\partial}{\partial x_i} \left(\left(\sum_{s=1}^{n-1} \left(\frac{\partial u}{\partial x_s} \right)^2 \right)^{\frac{p-2}{2}} \frac{\partial u}{\partial x_i} \right) + \frac{\partial}{\partial x_n} \left(\left| \frac{\partial u}{\partial x_n} \right|^{q-2} \frac{\partial u}{\partial x_n} \right) = 0 \, .$$

We look for a (pointwise) solution of the form

$$u\,(x_1, x_2, \ldots, x_n) = c \left(\sum_{s=1}^{n-1} x_s^2 \right)^{\alpha} \cdot x_n^{\beta}, \qquad (x_n > 0) \;,$$

for some exponents $\alpha, \beta \in \mathbb{R}$ and for some constant c.

By a computation we find

$$\alpha = \boxed{-}\frac{p}{2\,(q-p)}, \qquad \beta = \frac{q}{q-p}\,.$$

Thus, for a particular constant c, the function

$$u\,(x_1, x_2, \ldots, x_n) = c\,\frac{(x_n)^{\frac{q}{q-p}}}{\left(\sum_{s=1}^{n-1} x_s^2\right)^{\frac{p}{2(q-p)}}}$$

is a pointwise solution to the pde.

Note that $u\,(x_1, x_2, \ldots, x_n)$ is singular on the line

$$x_1 = x_2 = \ldots = x_{n-1} = 0\,.$$

Which are the *summability properties* of the gradient Du?

Then we have to test a *summability property* of the gradient Du in order to prove that u is a *weak* solution to the pde too.

We find the further conditions on p, q:

$$\begin{cases} n > 2, \quad p \in (1, n-1) \\ q > \frac{(n-1)p}{n-1-p} \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \frac{\partial u}{\partial x_s} \in L^p(\Omega), \quad \forall s = 1, 2, \ldots, n-1 \\ \frac{\partial u}{\partial x_n} \in L^q(\Omega) \end{cases}$$

By these summability properties (q large in dependence of p) we can show that u is a *weak* solution to the pde too. Note that, if n is *sufficiently large*, then $u \in W^{1,q}(\Omega)$ too.

SOME EXAMPLES WITH GENERAL GROWTH

Here we give examples of integrals of the calculus of variations with *growth depending on* $x \in \Omega$, trough an exponent $p(x)$

$$\int_{\Omega} |Du(x)|^{p(x)}\, dx \quad \text{or} \quad \int_{\Omega} \left(1 + |Du(x)|^2\right)^{\frac{p(x)}{2}} dx\,.$$

Starting from Zhikov, and then Acerbi-Mingione and others. Rosario Mingione pointed out to me a group of mathematicians in Finland actually working on regularity for this problem; we quote: Petteri Harjulehto, Peter Hästö, Mika Koskenoja, Mikko Pere, Susanna Varonen, working at the Department of Mathematics, University of Helsinki.

Note also that, if the exponent $p(x)$ is continuous, then the *local* regularity problem (regularity in a *small* ball $B_r \subset \Omega$) can be studied under the so called $p, q-$*growth conditions,* with $q > p$ and ratio $\frac{q}{p}$ arbitrarily close to 1 depending on the radius r of the ball $B_r \subset \Omega$ and on the modulus of continuity of $p(x)$.

The *$p,q-$growth conditions* are related to integrals

$$\int_\Omega f\left(x,u\left(x\right),Du\left(x\right)\right)\,dx$$

with growth conditions on f of the type

$$m\left|\xi\right|^{\boxed{p}} \leq f\left(x,s,\xi\right) \leq M\left(1+\left|\xi\right|^{\boxed{q}}\right),$$

for all $x \in \Omega$, $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$; similarly for the first and the second derivatives of f.

A functional which satisfies *$p,q-$growth conditions* (again, with $q > p$ and ratio $\frac{q}{p}$ arbitrarily close to 1) is

$$\int_\Omega \left|Du\left(x\right)\right|^p \log\left(1+\left|Du\left(x\right)\right|\right)\,dx\,.$$

This functional has been studied classically in the literature by mean of Orlichz spaces. More recently we can quote, in particular for the case $p = 1$, Esposito-Mingione, Mingione-Siepe, Fuchs-Mingione, Bildhauer.

An example (Fusco-Sbordone, Talenti) of functional which satisfies $p, q-$*growth conditions* (but with *ratio* $\frac{q}{p}$ **not** *arbitrarily close to* 1) is $\int_\Omega g(|Du(x)|)\,dx$

$$\text{with} \qquad g(t) = \begin{cases} |\xi|^{p+1-\sin\log\log|\xi|}, & \text{if } |\xi| > \mathrm{e} \\ \mathrm{e}\,|\xi|^p, & \text{if } |\xi| \le \mathrm{e} \end{cases}.$$

Note that this is a case with integrand $f(Du(x)) = g(|Du(x)|)$ depending only on the *modulus of the gradient*.

Integrals which have not polynomial growth, but *general* (exponential) *growth* are, for example,

$$\int_\Omega \mathrm{e}^{|Du(x)|^p}\,dx \qquad \text{or} \qquad \int_\Omega \mathrm{e}^{\left(1+|Du(x)|^2\right)^{\frac{p}{2}}}\,dx$$

(references: Lieberman, Duc-Eells, Marcellini, Cellina).

REGULARITY

In order to fix a case to consider in more details, we study the partial differential equation in divergence form

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a^i (x, Du(x)) = b(x), \qquad x \in \Omega .$$

We assume that the vector field $\left(a^i (x, \xi)\right)_{i=1,\dots,n}$ is locally Lipschitz continuous in $\Omega \times \mathbb{R}^n$ for some $n \geq 2$, and that it satisfies the so called $p, q-$*growth conditions* (for every $\lambda, \xi \in \mathbb{R}^n, \; x \in \Omega$) for some p, q exponents with $p \leq q$

$$\sum_{i,j=1}^{n} \frac{\partial a_i}{\partial \xi_j} \lambda_i \lambda_j \geq m \left(1 + |\xi|^2\right)^{\frac{p-2}{2}} |\lambda|^2, \quad \sum_{i,j=1}^{n} \left|\frac{\partial a_i}{\partial \xi_j}\right| \leq M \left(1 + |\xi|^2\right)^{\frac{q-2}{2}} .$$

Moreover

$$\sum_{i,j=1}^{n}\left|\frac{\partial a_i(x,\xi)}{\partial \xi_j}-\frac{\partial a_j(x,\xi)}{\partial \xi_i}\right|\leq M\left(1+|\xi|^2\right)^{\frac{\frac{p+q}{2}-2}{2}},$$

$$\sum_{i,s=1}^{n}\left|\frac{\partial a_i(x,\xi)}{\partial x_s}\right|\leq M\left(1+|\xi|^2\right)^{\frac{\frac{p+q}{2}-1}{2}}.$$

Finally, we assume that

$$b\in L^{\infty}_{\mathrm{loc}}(\Omega)\ .$$

REGULARITY THEOREM 1. *Under the previous assumptions ($p, q-$growth conditions), if $p \geq 2$ and*

$$\frac{q}{p} < \frac{n}{n-2} \quad \text{in the case } \; n > 2,$$

(without restrictions on p, q if $n = 2$), then every weak solution $u \in W^{1,q}_{\text{loc}}(\Omega)$ to the pde

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a^i (x, Du(x)) = b(x), \qquad x \in \Omega$$

is of class $W^{1,\infty}_{\text{loc}}(\Omega)$, i.e. the gradient Du is locally bounded in Ω, and there exist $c, \beta > 0$ and $\theta \geq 1$ such that the following estimate for the gradient holds: for every ϱ, R such that $0 < \rho < R \leq \varrho + 1$

$$\sup_{x \in B_\varrho} \left(1 + |Du(x)|^2\right)^{\frac{1}{2}} \leq \frac{c}{(R - \varrho)^{\theta\beta}} \left(\int_{B_R} \left[\left(1 + |Du(x)|^2\right)^{\frac{1}{2}} \right]^q dx \right)^{\frac{\theta}{q}}.$$

Then the behavior at infinity of the vector field $\left(a^i(x,\xi)\right)_{i=1,\dots,n}$ becomes irrelevant.

In fact, when $0 < \rho < R$, we have

$$\sup_{x\in B_\varrho}\left(1+|Du(x)|^2\right)^{\frac{1}{2}} = c(\varrho) < +\infty$$

and thus $\left|a^i(x,\xi)\right| \leq C(\varrho)$ for every $|\xi| \leq M(\varrho)$).

Therefore more regularity applies: in particular, if for some $k \in \{1,2,\dots\}$,

$$\left(a^i\right)_{i=1,\dots,n} \in C^{k,\alpha}_{\text{loc}}\left(\Omega\times\mathbb{R}^n\right), \qquad b \in C^{k-1,\alpha}_{\text{loc}}(\Omega),$$

then

$$u \in C^{k+1,\alpha}_{\text{loc}}(\Omega).$$

In the regularity result of Theorem 1, is it true that $u \in W^{1,q}(\Omega)$ too?

Let us recall the conclusion of Theorem 1:

REGULARITY THEOREM 1. *. . . if . . .*

$$\frac{q}{p} < \frac{n}{n-2}$$

. . . then every weak solution $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ *to the pde*

. . . is of class $W^{1,\infty}_{\mathrm{loc}}(\Omega)$ *. . .*

and there exist $c, \beta > 0$ *and* $\theta \geq 1$ *such that*

$$\sup_{x \in B_\varrho} \left(1 + |Du(x)|^2\right)^{\frac{1}{2}} \leq \frac{c}{(R-\varrho)^{\theta\beta}} \left(\int_{B_R} \left[\left(1 + |Du(x)|^2\right)^{\frac{1}{2}} \right]^q dx \right)^{\frac{\theta}{q}} .$$

To answer to this question we use the *interpolation inequality*:

$$\|v\|_{L^q} \leq \|v\|_{L^p}^{\frac{p}{q}} \cdot \|v\|_{L^\infty}^{1-\frac{p}{q}} \ .$$

Then, with the notation $v(x) = \left(1 + |Du(x)|^2\right)^{\frac{1}{2}}$, we get

$$\begin{cases} \|v\|_{L^q} \leq \|v\|_{L^p}^{\frac{p}{q}} \cdot \|v\|_{L^\infty}^{1-\frac{p}{q}} \\ \|v\|_{L^\infty(B_\varrho)} \leq \frac{c}{(R-\varrho)^{\theta\beta}} \|v\|_{L^q(B_R)}^{\theta} \end{cases}$$

$$\Downarrow$$

$$\|v\|_{L^\infty(B_\varrho)} \leq \frac{c}{(R-\varrho)^{\theta\beta}} \|v\|_{L^q(B_R)}^{\theta}$$

$$\leq \frac{c}{(R-\varrho)^{\theta\beta}} \|v\|_{L^p(B_R)}^{\theta\frac{p}{q}} \cdot \|v\|_{L^\infty(B_R)}^{\theta\left(1-\frac{p}{q}\right)} \ .$$

Therefore:

$$\|v\|_{\boxed{L^\infty(B_\varrho)}} \leq \frac{c}{(R-\varrho)^{\theta\beta}} \|v\|_{L^p(B_R)}^{\theta\frac{p}{q}} \cdot \|v\|_{\boxed{L^\infty(B_R)}}^{\theta\left(1-\frac{p}{q}\right)} .$$

With *abuse* (not only of notations!) in the conclusion of Theorem 1 we identify ϱ and R, and we do not consider the denominator $(R-\varrho)^{\theta\beta/q}$!!

Thus, formally, $\|v\|_{L^\infty}^{1-\theta\left(1-\frac{p}{q}\right)} \leq c\,\|v\|_{L^p}^{\theta\frac{p}{q}}$. Here we get the condition

$$1-\theta\left(1-\frac{p}{q}\right) > 0 .$$

Thus we need to know a precise expression of the exponent θ. By a more precise analysis we find (if $n > 2$) $\theta = \frac{2q}{np-(n-2)q}$. Note that $\theta = 1$ if $q = p$. Moreover $1-\theta\left(1-\frac{p}{q}\right) > 0$ if and only if $\frac{q}{p} < \frac{n+2}{n}$.

REGULARITY THEOREM 2 (Interpolation). *Under the previous assumptions ($p,q-$growth conditions), if $q \geq p \geq 2$ and $\frac{q}{p} < \frac{n+2}{n}$, then every weak solution $u \in W^{1,p}_{\text{loc}}(\Omega)$ to the pde*

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a^i(x, Du(x)) = b(x), \qquad x \in \Omega$$

is of class $W^{1,\infty}_{\text{loc}}(\Omega)$, and there exist $c, \alpha, \beta > 0$ and $\theta \geq 1$ (given above) such that, for every ϱ and R such that $0 < \rho < R \leq \varrho + 1$,

$$\left\| \left(1 + |Du(x)|^2\right)^{\frac{1}{2}} \right\|_{L^q(B_\varrho)} \leq \left(\frac{c}{(R-\varrho)^{\beta \frac{q-p}{p}}} \left\| \left(1 + |Du(x)|^2\right)^{\frac{1}{2}} \right\|_{L^p(B_R)}^{\frac{1}{\theta}} \right)^{\alpha},$$

$$\left\| \left(1 + |Du(x)|^2\right)^{\frac{1}{2}} \right\|_{L^\infty(B_\varrho)} \leq \left(\frac{c}{(R-\varrho)^{\beta \frac{q}{p}}} \left\| \left(1 + |Du(x)|^2\right)^{\frac{1}{2}} \right\|_{L^p(B_R)} \right)^{\alpha}.$$

EXISTENCE

Under the previous assumptions ($p, q-$growth conditions), if $q \geq p \geq 2$ *and*

$$\frac{q}{p} < \frac{n+2}{n} \,,$$

then there exists a weak solution $u \in W^{1,\boxed{q}}_{\text{loc}}(\Omega)$ *to the Dirichlet problem*

$$\begin{cases} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} a^i\left(x, Du\left(x\right)\right) = b\left(x\right), & x \in \Omega \\ u\left(x\right) = u_0\left(x\right), & x \in \partial\Omega \,, \end{cases}$$

provided the boundary datum has the summability property: $u_0 \in W^{1,r}(\Omega)$, with $r = \frac{p(q-1)}{p-1}$. *Moreover* u *is of class* $W^{1,\infty}_{\text{loc}}(\Omega)$ *and, if* $a^i \in C^{1,\alpha}_{\text{loc}}(\Omega \times \mathbb{R}^n)$ *for every* $i = 1, 2, \ldots, n$ *and* $b \in C^{0,\alpha}_{\text{loc}}(\Omega)$ *then*

$$u \in \left(u_0 + W^{1,p}_0(\Omega)\right) \cap W^{1,q}_{\text{loc}}(\Omega) \cap C^{2,\alpha}_{\text{loc}}(\Omega) \,.$$

SOME REFENCES

De Giorgi, Ladyzenskaya-Ural'ceva, Morrey, Moser, Nash, Serrin, Stampacchia. J.L.Lions, Giaquinta, Giusti, DiBenedetto-Trudinger, among others. Uhlenbeck gave a specific regularity result for the integral $\int_\Omega |Du(x)|^p \, dx$ $(p \geq 2)$ in the vector valued case. Some (relatively) more recent Lipschitz continuity results under *natural growth conditions* have been given by Chipot-Evans, Di Benedetto, Manfredi.

Let us give some references (some others will be given later) when the *natural growth conditions* are not satisfied.

Regularity for elliptic equations and for integrals of the calculus of variations with *anisotropic growth conditions* started in 1989 by Marcellini. The counterexample to regularity was given by Giaquinta, Hong Min Chung and Marcellini.

Further results for anisotropic integrals and equations have been given by Acerbi-Fusco, Boccardo-Marcellini-Sbordone, Fusco-Sbordone, Hong Min Chung, Korolev, Stroffolini.

Non-standard growth conditions have been considered by Bhattacharya-Leonetti, Choe, Leonetti, Moscariello-Nania, Tang Qi. Variational problems under non-standard growth conditions, in the vector-valued case too, have been studied by Acerbi-Fusco, Bildhauer, Dall'Aglio-Mascolo-Papi, Esposito-Leonetti-Mingione, Leonetti-Mascolo-Siepe, Marcellini.

Mingione, in collaboration with Acerbi, Coscia, Esposito, Fonseca, Fuchs, Fusco, Leonetti, Maly, Mucci, Seregin, Siepe, C.Trombetti, gave new contributions to this subject. Bernd Kawohl found some interest in regularity for elliptic equations with *anisotropic growth conditions*. Michael Bildhauer in a book (2003) studied regularity of several different variational problems under general growth conditions. *More recently many others ...*

VECTOR-VALUED MAPS

In the general *vectorial setting* it is well known that, in general, we can look for the so called partial regularity, since the pioneering work of Morrey and De Giorgi. The related mathematical literature is wide. In particular Yoshikazu Giga worked in this field too. However only a few contributions are available for *general growth*: we refer to the papers by Acerbi-Fusco (1994) and by Esposito-Leonetti-Mingione (1999). The book by Bildhauer (2003) gives an overview and a detailed list of references.

If some additional structure conditions are assumed then several results can be found in the mathematical literature on the subject (for everywhere regularity). For instance, as a generalization of the "$p-$*growth*" case considered by Uhlenbeck (1977), Marcellini (1996) proposed an approach to the regularity of minimizers of the integrals with the integrands of the form $f(\xi) = g(|\xi|)$.

Precisely, with the integrand of the form $f(\xi) = g(|\xi|)$, where $g : [0, +\infty) \to [0, +\infty)$ is an increasing convex function, without growth assumption on $g(t)$ as $t \to +\infty$:

$$F(u) = \int_\Omega g(|Du|)dx,$$

For example, the regularity result can be applied to the *exponential growth*, such as any finite composition of the type (with $p_i \geq 1, \forall i = 1, 2 \dots k$)

$$g(|\xi|) = (\exp(\dots(\exp(\exp|\xi|^2)^{p_1})^{p_2})\dots)^{p_k}.$$

However, some other restrictions ware imposed, such as, for instance, the fact that $t \in (0, +\infty) \to \frac{g'(t)}{t}$ is assumed to be an increasing function. To exemplify, the model case $g(t) = t^p$ gives the restriction $p \geq 2$.

Afterwards, Leonetti-Mascolo-Siepe (2003) consider the case of subquadratic $p, q-$growth conditions, i.e. they assume $1 < p < q < 2$. Their result includes energy densities f of the type $f(\xi) = |\xi|^p \log^\alpha(1 + |\xi|)$ with $p < 2$.

Fuchs-Mingione concentrate on the case of nearly-linear growth. Typical examples are the logarithmic case $f(\xi) = |\xi| \log(1 + |\xi|)$ and its iterated version, for $k \in$ arbitrary,

$$\begin{cases} f_k(\xi) & = & |\xi| L_k(|\xi|), \\ L_{s+1}(t) & = & \log(1 + L_s(t)), \ L_1(t) = \log(1 + t) \end{cases}$$

Bildhauer considers linear behavior; he gives conditions that can keep γ-elliptic linear growth with $\gamma < 1 + \frac{2}{n}$. Examples of γ-elliptic linear integrands are given by

$$g_\gamma(t) = \int_0^t \int_0^s (1 + z^2)^{-\frac{\gamma}{2}} dz ds, \ \forall t \geq 0.$$

For $\gamma = 1$, $g_\gamma(t)$ behaves like $t \log(1 + t)$ and in the limit case $\gamma = 3$, $g_\gamma(t)$ becomes $(1 + t^2)^{1/2}$.

Hence the functions $g_\gamma(t)$ provide a one parameter family connecting *logarithmic examples* with the *minimal surface integrand*.

Marcellini and Papi gave new conditions which include different kind of growths: *more general conditions on the function g embracing growths moving between linear and exponential functions.*

The conditions are the following (we consider explicitly the case $n \geq 3$, while if $n = 2$ then the exponent $\frac{n-2}{n}$ can be replaced by any real number):

Let $t_0, H > 0$ and let $\beta \in \left(\frac{1}{n}, \frac{2}{n}\right)$. For every $\alpha \in \left(1, \frac{n}{n-1}\right]$ there exist $K = K(\alpha)$ such that, for all $t \geq t_0$,

$$Ht^{-2\beta}\left[\left(\frac{g'(t)}{t}\right)^{\frac{n-2}{n}} + \frac{g'(t)}{t}\right] \leq \ g''(t) \ \leq K\left[\frac{g'(t)}{t} + \left(\frac{g'(t)}{t}\right)^{\alpha}\right].$$

The exponent α in the right hand side is a parameter to play; i.e., to use to test more functions g. The condition in the left-hand side of permits to achieve functions, for instance, with second derivative going to zero as a power $t^{-\gamma}$, (i.e. γ-elliptic), where γ is not too large and is related to the dimension n, i.e. $\gamma < 1 + \frac{2}{n}$.

As well as functions considered previously, others examples in the linear case include, for $r \in (0, 1)$,

$$g_r(t) = h(t) - t^r, \ \forall t \geq 1, \ n < \frac{2}{1-r},$$

and also

$$g_r(t) = h(t) + (1 - t^r)^{\frac{1}{r}}, \ \forall t \geq 1, \ n < \frac{2}{r},$$

where $h(t)$ is a convex function such that, for suitable constants C_1, C_2,

$$C_1(1 + t) \leq h(t) \leq C_2(1 + t).$$

We observe that the functions

$$g_k(t) = (1 + t^k)^{\frac{1}{k}},$$

related to *minimal surfaces*, are convex if $k \geq 1$, and

$$g_k''(t) = (k-1)\, t^{k-2}(1 + t^k)^{\frac{1}{k}-2} = \mathcal{O}\left(\frac{1}{t^{k+1}}\right)$$

when $t \to +\infty$, so that they *do not satisfy* left-hand side of the previous condition.

As far as $p, q-$growth is concerned, we like to observe that the previous growth condition is satisfied without assuming any restriction on p and q. For example, fixed $1 < p < q$, we can consider the function (cfr. Dall'Aglio-Mascolo-Papi), where τ_0 is such that $\sin \log \log \log \tau_0 = -1$,

$$g(t) = \begin{cases} t^p & \text{if } \; t \leq \tau_0, \\ t^{\frac{p+q}{2} + \frac{q-p}{2} \sin \log \log \log t} & \text{if } \; t > \tau_0, \end{cases} .$$

In the paper by Marcellini and Papi to appear, the following two results are proved, the first one valid under *general growth conditions*, the second one specific for the *linear case*.

Theorem A (General growth). *Let* $g : [0, +\infty) \to [0, +\infty)$ *be a convex function of class* $W^{2,\infty}_{\mathrm{loc}}$ with $g(0) = g'(0) = 0$, *satisfying the general growth condition. Let* $u \in W^{1,1}_{\mathrm{loc}}(\Omega; R^m)$ *be a minimizer. Then*

$$u \in W^{1,\infty}_{\mathrm{loc}}(\Omega; R^m)\,.$$

Moreover, for every $\epsilon > 0$ *and* $R > \rho > 0$ *there exists a constant* $C = C(\epsilon, n, \rho, R)$ *such that*

$$\|Du\|^{2-\beta n}_{L^\infty(B\rho;\mathbb{R}^{m\times n})} \leq C \left\{ \int_{B_R} (1 + g(|Du|))\, dx \right\}^{\frac{1}{1-\beta}+\epsilon}.$$

Theorem B (Linear growth). *Let $g : [0, +\infty) \to [0, +\infty)$ be a convex function of class $W^{2,\infty}_{\mathrm{loc}}$ with $g(0) = g'(0) = 0$. If g has the linear behavior at infinity*

$$\lim_{t \to +\infty} \frac{g(t)}{t} = l \in (0, +\infty)$$

and if its second derivative satisfies the inequalities

$$H \frac{1}{t^{\gamma}} \leq g''(t) \leq K \frac{1}{t} \,, \qquad \forall t \geq t_0 \,,$$

for some positive constants H, K, t_0 and for some $\gamma \in \left[1, 1 + \frac{2}{n}\right)$, then $u \in W^{1,\infty}_{loc}(\Omega; R^m)$ and, for every $R > \rho > 0$, there exists a constant $C = C(n, \rho, R, l, H, K)$ such that

$$\| Du \|^{\frac{2-n(\gamma-1)}{2}}_{L^{\infty}(B\rho; \mathbb{R}^{m \times n})} \leq C \int_{B_R} (1 + g(|Du|)) \, dx \,.$$

ANISOTROPIC INTEGRANDS

An integral with anisotropic integrand is a special relevant case which can be treated under sharp assumptions. For instance, we mean integrals of the calculus of variations of the form

$$F(u) = \int_\Omega \sum_{i=1}^{n} |u_{x_i}(x)|^{p_i(x)}\, dx$$

for some bounded measurable functions $p_i(x)$.

By the previous examples we know that they may have not smooth, even unbounded, minimizers. This happens also in the case of *constant* exponents p_i, $i = 1, \ldots, n$, if they are spread out; i.e., if the ratio $\max\{p_i\} / \min\{p_i\}$ is not close enough to 1 in dependence on n. For example when $n > 3$ and

$$p_1 = \ldots = p_{n-1} = 2, \qquad p_n = q > \frac{2(n-1)}{n-3}\,.$$

Following an original idea by Boccardo-Marcellini-Sbordone, Cupini-Marcellini-Mascolo (2009 and 2012) considered exponents p_i, $i = 1, \ldots, n$, and q greater than or equal to 1, such that

$$p_i \leq p_i(x) \leq q, \quad \text{a.e. } x \in B_r, \quad 1 \leq i \leq n,$$

where B_r is a ball of radius $r > 0$ contained in Ω.

Then let $\overline{p}$ be the *harmonic average* of the $\{p_i\}$; i.e.,

$$\frac{1}{\overline{p}} := \frac{1}{n} \sum_{i=1}^{n} \frac{1}{p_i}$$

and let $\overline{p}^*$ be the *Sobolev conjugate exponent* of $\overline{p}$; i.e.,

$$\overline{p}^* = \frac{n\overline{p}}{n - \overline{p}}$$

if $\overline{p} < n$, while $\overline{p}^*$ is any fixed real number greater than $\overline{p}$, if $\overline{p} \geq n$.

Theorem (Cupini-Marcellini-Mascolo – 2009-2012). Let u be a local minimizer of the **anisotropic integrals** of the calculus of variations, as described above, and let $\boxed{q < \overline{p}^*} = \frac{n\overline{p}}{n-\overline{p}}$.

Then u is locally bounded in Ω and the following estimate holds

$$\|u - u_r\|_{L^\infty(B_{r/(2\sqrt{n})}(x_0))} \leq$$

$$c\left\{1 + \int_{B_r(x_0)} \sum_{i=1}^{n} |u_{x_i}(x)|^{p_i(x)}\, dx\right\}^{\frac{1+\theta}{p}},$$

for some constant $c > 0$, where $u_r = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u\, dx$, $p = \min_{1\leq i\leq n}\{p_i\}$ and $\theta = \frac{\overline{p}^*(q-p)}{p(\overline{p}^*-q)}$.

Observe that if

$$p_1 = \ldots = p_{n-1} = 2 \quad \text{and} \quad p_n = q \geq 2$$

then the assumption $q < \overline{p}^*$ gives

$$q < 2\,(n-1)\,/\,(n-3)\,.$$

This inequality is exactly the opposite of condition in the counterexample to regularity (apart from the equality which is not achieved), since the borderline case $q = \overline{p}^*$ is not included in the Theorem.

Thus, our regularity result is essentially sharp.

As a consequence of the previous theorem and of the quoted result by Lieberman, we get the following gradient estimate under anisotropic growth. As before $\overline{p}(x)$ is the harmonic average of the $\{p_i(x)\}$ and $\overline{p}^*(x)$ is the Sobolev conjugate exponent of $\overline{p}(x)$

Corollary (Cupini-Marcellini-Mascolo)**.** Let u be a local minimizer of the integral

$$F(u) = \int_\Omega \sum_{i=1}^{n} |u_{x_i}(x)|^{p_i(x)}\, dx$$

with exponents $p_i(x)$ locally Lipschitz continuous in Ω. Let $x_0 \in \Omega$. If $p_i(x_0) < \overline{p}^*(x_0)$ for every $i = 1, \ldots, n$, then u is Lipschitz continuous in a neighborhood of x_0.

PARABOLIC EQUATIONS AND SYSTEMS

Bögelein-Duzaar-Marcellini (*J. Math. Pures Appl.*, to appear) consider parabolic equations of the type

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} a^i\left(x, t, Du\left(x, t\right)\right)$$

on a parabolic space-time cylinder Ω_T. The vector-field a is assumed to satisfy non-standard $p, q-$growth conditions. When

$$2 \leq p \leq q < p + \frac{4}{n}$$

it is established that any weak solution

$$u \in L^p(0, T; W^{1,p}(\Omega)) \cap L^q_{\mathrm{loc}}(0, T; W^{1,q}_{\mathrm{loc}}(\Omega))$$

admits a locally bounded spatial gradient Du.

Moreover, it is shown that the stronger assumption

$$2 \leq p \leq q < p + \frac{4}{n+2}$$

guarantees an existence result for the Cauchy-Dirichlet problem associated to the parabolic equation.

The results cover for example the parabolic p_i-Laplacian; i.e., anisotropic parabolic equations of the type

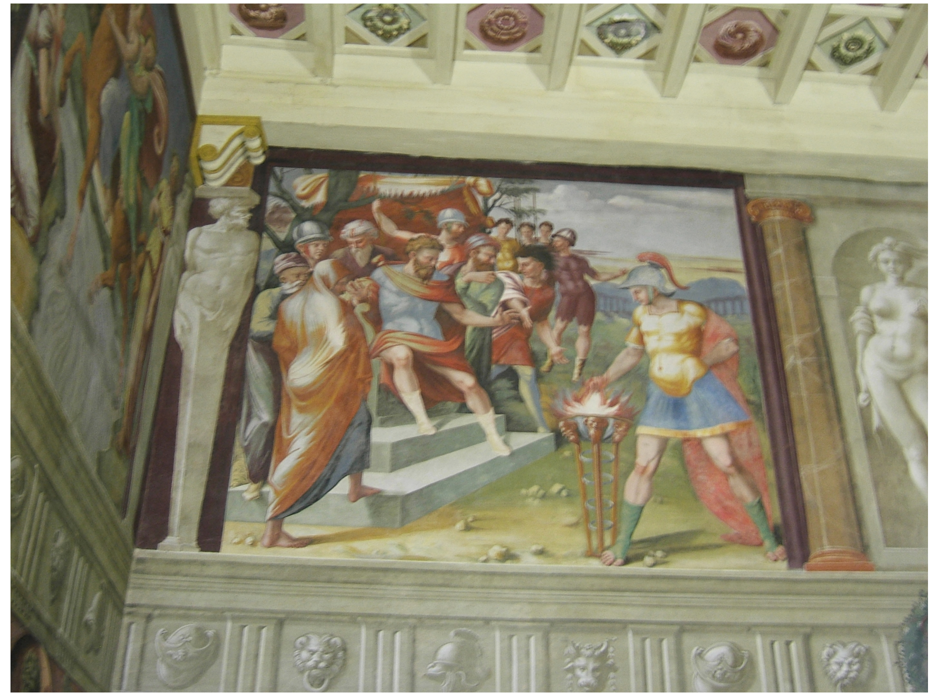
$$\partial_t u - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(|D_i u|^{p_i-2} D_i u \right) = 0$$

with suitable growth exponents p_i.

Results for systems under $p, q-$growth conditions are in preparation.

Thanks to all the audience

In particular thanks to my colleagues and friends David, Giovanni and Irene.



Cortona, Italy — May 2007

Have a nice day!! CMU – *September 18th*, 2012