Quasistatic crack growth in finite elasticity

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Joint work with Gianni Dal Maso

Aim of the work

- Crack growth: We want to determine the evolution of a brittle body

  Griffith's principle: Competition between
  - ⋆ the energy spent to produce new crack (surface energy)
  - ⋆ the energy released by crack's growth (volume energy)

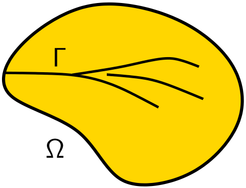
- Quasistatic: The configuration is a global minimizer of the total energy

  A law of energy balance holds and the process is rate-independent

- Finite elasticity: Orientation-reversing deformations are penalized

  All previous existence results are incompatible with non-interpenetration

Setting of the problem



$\Omega \subseteq \mathbb{R}^n$ elastic body

$\Gamma \subseteq \Omega$ crack of measure $\mathcal{H}^{n-1}(\Gamma) < +\infty$

$u \colon \Omega \setminus \Gamma \longrightarrow \mathbb{R}^n$ deformation of the elastic part

$t \in [0, T]$ time

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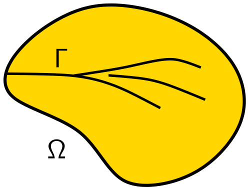

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$$\mathcal{E}(u, \Gamma) := \mathcal{W}(u) + \mathcal{H}^{n-1}(\Gamma)$$

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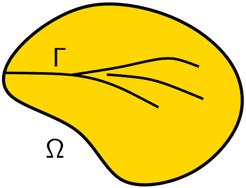

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(where ψ is extended to the whole of Ω)

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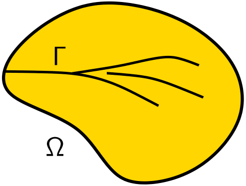

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Problem

Find $t \mapsto (u(t), \Gamma(t))$ such that $(u(t), \Gamma(t))$ minimizes $\mathcal{E}(u, \Gamma)$
taking into account the irreversibility of fracture

Minimum energy configurations

Fixed $t \in [0, T]$, we consider the solutions to the static problem

Equilibria (Griffith)

$(u(t), \Gamma(t))$ should be stationary for $\mathcal{E}(v, \Gamma)$
on the set $\{(v, \Gamma) \colon \; v\big|_{\partial\Omega} = \psi(t) \,, \quad \Gamma \supseteq \Gamma(t)\}$

The notion of critical point is not defined in this setting!

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Among the stationary points, we consider only the global minima

Unilateral minimum problem

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We consider quasistatic evolutions

Evolution problem

Find $t \mapsto (u(t), \Gamma(t))$ with the unilateral minimum property at every t

Strategy

$$\mathcal{E}(u,\Gamma) := \mathcal{W}(u) + \mathcal{H}^{n-1}(\Gamma)$$

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3. Approximate solutions:

- $(u_k^0, \Gamma_k^0) := (u_0, \Gamma_0)$
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5. Global stability: $(u(t), \Gamma(t))$ unilateral minimum for every t
6. Energy-dissipation balance

$\Rightarrow$ We must control the bulk energy $\mathcal{W}(u)$ in the passage to the limit

Bulk energy with polynomial growth

$$\mathcal{W}(u) := \int_\Omega W(\nabla u(x))\,\mathrm{d}x$$

In the limit process one makes use of the lower semicontinuity of $\mathcal{W}$

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  Linearized elasticity:
  $$W(\nabla u) = |\nabla u - \mathrm{I}|^2$$
- Dal Maso-Francfort-Toader (2005)
  Nonlinear elasticity: W quasiconvex with polynomial growth
  $$C_1|\nabla u|^p \leq W(\nabla u) \leq C_2|\nabla u|^p\,, \quad p > 1$$

Finite elasticity

$$\mathcal{W}(u) := \int_\Omega W(\nabla u(x))\, \mathrm{d}x$$

Local non-interpenetration

$$F \in \mathbb{M}^{n\times n} \qquad \begin{array}{ll} W(F) = +\infty & \text{if } \det F \le 0 \\ W(F) \to +\infty & \text{if } \det F \to 0^+ \end{array}$$

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Difficulty

Non-interpenetration is incompatible

with the assumption of polynomial growth made previously

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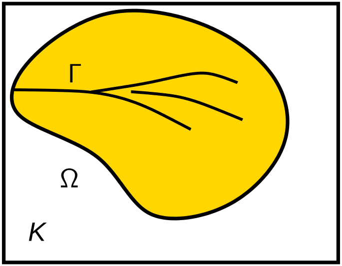
Goal

Existence result compatible with non-interpenetration,
without assuming polynomial growth

Functional setting



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$$u \in SBV\ (\Omega; K)$$

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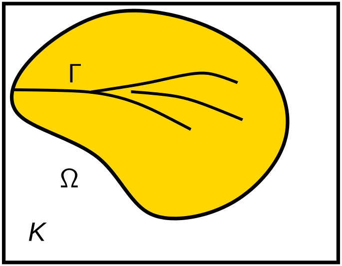

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$$J_u \subseteq \Gamma$$

Properties of *SBV* functions (De Giorgi-Ambrosio, 1988)

For every $u \in SBV(\Omega; K)$ we can define

- the approximate gradient $\nabla u \in L^1(\Omega; \mathbb{M}^{n\times n})$
- the jump set J_u, a rectifiable subset of Ω

In $\Omega \setminus J_u$ the function u is approximately continuous

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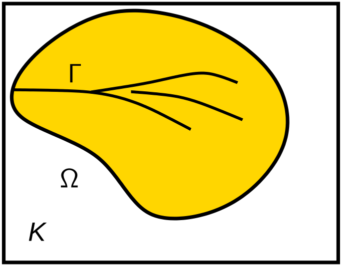

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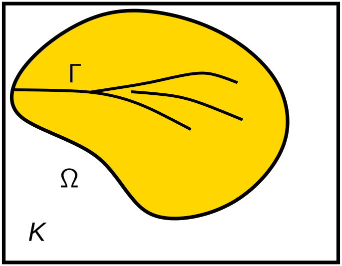

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Weak* convergence in $SBV^p(\Omega; K)$

- $u_k \longrightarrow u$ a.e.
- $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^p(\Omega; \mathbb{M}^{n\times n})$
- $\mathcal{H}^{n-1}(J_{u_k})$ bounded

This notion of convergence is compact (Ambrosio, 1989)

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- $W : \mathbb{M}^{n\times n} \longrightarrow [0,+\infty]$ continuous and polyconvex

$$W(F) = G\,(\mathrm{adj}_1 F, \dots, \mathrm{adj}_k F, \dots, \mathrm{adj}_n F)$$

with	G	continuous and convex
	$\mathrm{adj}_k F$	vector whose components are the minors of F of order k

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- $W : \mathbb{M}^{n\times n} \longrightarrow [0,+\infty]$ continuous and polyconvex
- Lower bound:

$$W(F) \geq \sum_{k=1}^{n} \beta_k |\mathrm{adj}_k F|^{p_k}$$

with $\beta_k > 0$ for every k

$p_1 = p \geq 2, \quad p_k \geq p'$ for $k = 2, \dots, n-1, \quad p_n > 1$

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Theorem (Fusco-Leone-March-Verde, 2006)

The functional $\mathcal{W}$ is lower semicontinuous with respect to weak convergence in $SBV^p(\Omega;K)$*

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$$\forall\, \varepsilon > 0\ \exists\, \delta > 0: \quad |F' - I| < \delta \quad \Rightarrow \quad |K(F'F) - K(F)| \leq \varepsilon (W(F)+1)$$

  (see Ball, Francfort-Mielke, ...)

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Example (Ogden materials, 1972)

For $n = 3$, we want:

$$W(F) \geq \beta_1 |F|^{p_1} + \beta_2 |\mathrm{cof}\, F|^{p_2} + \beta_3 |\det F|^{p_3}$$

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Example (Ogden materials, 1972)

For $n = 3$, $\beta_1, \beta_2, \beta_3, \gamma > 0$, $p_1 = p \ge 2$, $p_2 \ge p'$, $p_3 > 1$, $q > 0$,
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Theorem (Dal Maso-L., 2010)

Under the previous assumptions on $\mathcal{E}(u,\Gamma) := \mathcal{W}(u) + \mathcal{H}^{n-1}(\Gamma)$ *there exists a map* $t \mapsto (u(t), \Gamma(t))$ *starting at* (u_0, Γ_0) *and satisfying the (smooth) boundary condition* $\psi(t)$, *such that*

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$$J_{u(t)} \subseteq \Gamma(t)$$

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$$\mathcal{E}(u(t), \Gamma(t)) \leq \mathcal{E}(v, \Gamma) \qquad \forall\, \Gamma \supseteq \Gamma(t) \qquad \forall\, v\big|_{\partial\Omega} = \psi(t)\,, \quad J_v \subseteq \Gamma$$

- *Energy balance:* $t \mapsto \mathcal{E}(u(t), \Gamma(t))$ *is AC and for a.e.* $t \in [0,T]$

$$\frac{d}{dt}\Big(\mathcal{E}(u(t), \Gamma(t))\Big) = \int_\Omega \nabla W(\nabla u(t)) : \nabla\left(\dot{\psi}(t) \circ \psi^{-1}(t) \circ u(t)\right)$$

Such a map is called quasistatic evolution

Quasistatic evolution

Remark

The process is “rate-independent” (see Mielke):
global stability and energy-dissipation balance hold

Quasistatic evolution

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Rate-independence

If $u(t)$ is a solution associated to the datum $\psi(t)$,
then $u \circ \alpha(t)$ is a solution associated to the datum $\psi \circ \alpha(t)$
for every reparametrization α of time

Quasistatic evolution

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Remark

Global stability means that at every $t \in [0,T]$ the pair $(u(t),\Gamma(t))$ is a solution to the unilateral minimum problem

$$\min\{\mathcal{E}(v,\Gamma): \quad \Gamma(t) \subseteq \Gamma, \quad v\big|_{\partial\Omega} = \psi(t), \quad J_v \subseteq \Gamma\}$$

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Under further regularity assumptions, the energy balance law

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathcal{E}(u(t), \Gamma(t))\Big) = \int_\Omega \nabla W(\nabla u(t)) : \nabla\left(\dot{\psi}(t) \circ \psi^{-1}(t) \circ u(t)\right) \mathrm{d}x$$

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Remark

Under further regularity assumptions, the energy balance law is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t} \Big(\mathcal{E}(u(t), \Gamma(t)) \Big) = \int_{\partial\Omega} \nabla W(\nabla u(t))\, \nu_\Omega \cdot \nabla \dot{\psi}(t) \, \mathrm{d}\mathcal{H}^{n-1}$$

Sketch of the proof of the existence result

1. Approximation with incremental minimum problems by time discretization

- $(u_k^0, \Gamma_k^0) := (u_0, \Gamma_0)$
- (u_k^i, Γ_k^i) a solution of: $\min \{\mathcal{E}(u,\Gamma) \colon \Gamma \supseteq \Gamma_k^{i-1}, \, u\big|_{\partial\Omega} = \psi(t_k^i), \, J_u \subseteq \Gamma\}$

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6. Reverse energy inequality: via the global stability. This proves energy balance

First difficulty: Global stability

Convergence of minimizers (idea)

- $\Gamma_k \longrightarrow \Gamma$
- $u_k \rightharpoonup u$
- u_k minimal:

$$\mathcal{E}(u_k) \leq \mathcal{E}(v) \qquad \forall v: \quad J_v \subseteq \Gamma_k$$

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In order to pass to the limit...

Given v jumping on Γ, we construct v_k jumping on Γ_k so that $\mathcal{E}(v_k) \longrightarrow \mathcal{E}(v)$

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In order to construct v_k, we have to transfer the jumps of v from Γ to Γ_k

Crack transfer

Transferring the jumps from Γ to Γ_k

Given v jumping on Γ, we construct $v_k := v \circ \varphi_k$ jumping on Γ_k

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Reflections are forbidden by non-interpenetration

Dal Maso-L., 2010

φ_k is defined using dilations close to identity

We can control $W(\nabla v_k)$ by $W(\nabla v)$:

$$\nabla v_k = \nabla v \cdot \mathbf{\Lambda}_k\,, \qquad \mathbf{\Lambda}_k \text{ coefficient matrix of the dilation}$$

$$\exists\, \gamma \in (0,1): \quad |\mathbf{\Lambda}_k - I| \le \gamma \quad \Rightarrow \quad W(\nabla v \cdot \mathbf{\Lambda}_k) \le C(W(\nabla v) + 1)$$

(consequence of the estimate on the Eshelby tensor $|P(F)| \le C\,(W(F) + 1)$)

Second difficulty: Energy estimates

Problem: Compare $\mathcal{W}(u(t))$ with $\mathcal{W}(u(s))$

Different boundary conditions $\psi(t)$, $\psi(s)$

Additive splitting

Compare $u(t)$ with $\psi(t) - \psi(s) + u(s)$

Not compatible with local irreversibility!

Second difficulty: Energy estimates

Problem: Compare $\mathcal{W}(u(t))$ with $\mathcal{W}(u(s))$

Different boundary conditions $\psi(t)$, $\psi(s)$

Multiplicative splitting (Francfort-Mielke, 2006)

Compare $u(t)$ with $\psi(t) \circ \psi(s)^{-1} \circ u(s)$

Then one can employ the estimates on the stress tensors (Eshelby/Kirchhoff)

Assumptions on $\psi(t,x)$:

- $\psi(t,\cdot)$ can be extended to a diffeomorphism of the container K into itself
- ψ and $\nabla\psi$ are of class C^1 in time

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Up to now we have assumed that...

- ψ is C^1 in space, while the deformations are only *SBV*
- $\psi, \nabla\psi$ are C^1 in time, so we can consider only C^1 reparametrizations

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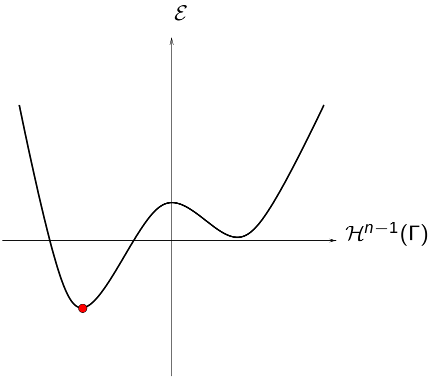
Weaker assumptions (L., 2011)

- ψ is Lipschitz in space
- the process is invariant under Lipschitz reparametrizations of time

Drawback of global minimization

Global minimizers $\Rightarrow$ Jumps between energy wells, overtaking barriers



$t_1 \geq 0$

(Here $n = 2$)

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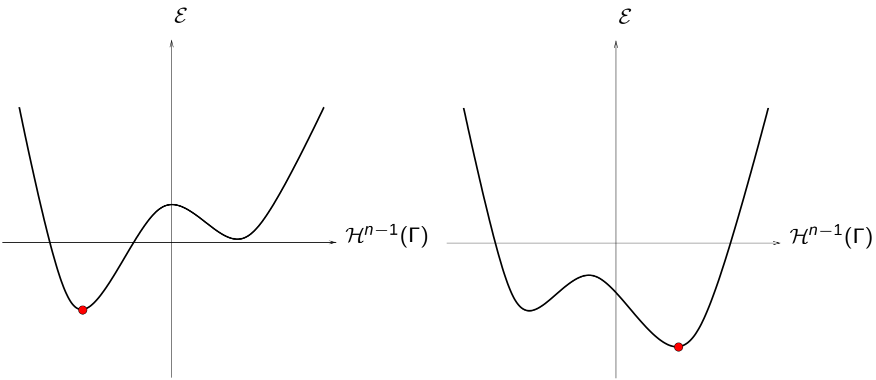

$t_1 \geq 0$ $\qquad$ $t_2 > t_1$

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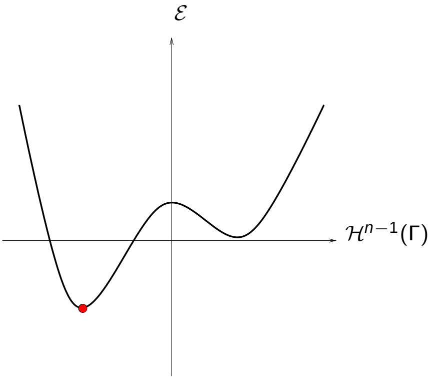
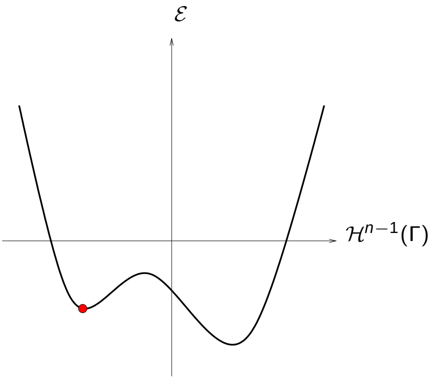
Local minimization

Local minimization $\Rightarrow$ Jumps happen later, without overtaking barriers





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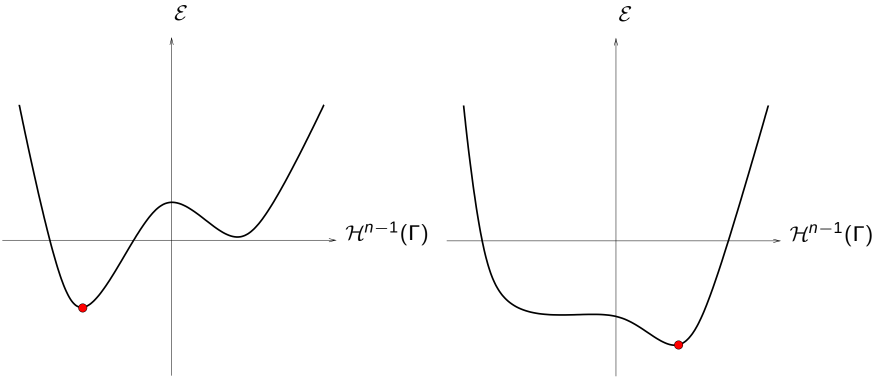

$t_1 \geq 0$ $\qquad$ $t_3 > t_2$

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Approaches based on local minimization

They provide a better behaviour when jumps in time occur

- Vanishing viscosity: brittle crack
  - ⋆ Prescribed crack path

    Toader-Zanini (2008), Negri-Ortner (2008), Knees-Mielke-Zanini (2008-10)
  - ⋆ Free crack path (with constraints)

    Dal Maso-Toader (2002), L.-Toader (2011)

- Vanishing viscosity: cohesive zone
  - ⋆ Prescribed crack path

    Dal Maso-Zanini (2007), Cagnetti (2008), Cagnetti-Toader (2008)

- ε-slides
  - ⋆ Free crack path

    Larsen (2010)

Conclusion

- We study problems of crack growth in the framework of finite elasticity

  The model is compatible with the requirement of non-interpenetration

- Equilibrium configurations are global minima of the total energy

  This allows us to consider a general class of possible cracks

- The evolution is approximated by solutions to discrete-time problems

  The incremental solutions converge to a quasistatic evolution

- Nonphysical phenomena of jump between energy wells are possible

  Jumps are better characterized by a local-minimality approach