Bounded vorticity, bounded velocity (Serfati) solutions to 2D Euler equations

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Euler equations

Euler Equations for incompressible, non-viscous (ideal) fluid flow in $2D$:

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla p, \\ \operatorname{div} u = 0, \\ u(t=0) = u^0. \end{cases}$$

- $u = (u^1, u^2)$ is a vector field.
- p is the scalar pressure.
- $(u \cdot \nabla) u$ is the directional derivative of u in its own direction.
- If the fluid domain, Ω, has a boundary, we enforce, $u \cdot \mathbf{n} = 0$ on the boundary.
- If the domain is unbounded, we impose conditions at infinity.

Vorticity equation

Vorticity is defined by

$$\omega = \operatorname{curl} u := \partial_1 u^2 - \partial_2 u^1,$$

the scalar curl of u.

Vorticity equation in 2D:

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0, \\ \operatorname{div} u = 0, \\ \operatorname{curl} u = \omega. \end{cases}$$

Vorticity formulation

The vorticity formulation of the Euler equations, then, is

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0, \\ u = K[\omega]. \end{cases}$$

Biot-Savart law: If the fluid domain is $\mathbb{R}^2$ then $K[\omega] = K * \omega$ with

$$K(x) = \frac{x^\perp}{2\pi |x|^2}.$$

Now, $K \in L^p_{loc}(\mathbb{R}^2)$, $1 \leq p < 2$ and K is q-th power integrable at infinity, with $q > 2$. Hence, to calculate $K * \omega$ we need $\omega \in L^{q'} \cap L^{p'}$ with $p' > 2$ and $q' < 2$; e.g. $\omega \in L^1 \cap L^\infty$.

Existence and uniqueness for vorticity in L^∞ for a bounded domain is due to Yudovich (1963), and extended to $L^1 \cap L^\infty$ for $\mathbb{R}^2$ by Majda.

Going beyond the Biot-Savart law

Classical existence and uniqueness results rely upon the integral,

$$u(x) = \int_{\mathbb{R}^2} K(x-y)\omega(y)\,dy = \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\omega(y)\,dy,$$

defining the Biot-Savart law being absolutely convergent.

In fact, Brunelli (2010) shows that the condition,

$$\int_{\mathbb{R}^2} \frac{\omega_0(x)}{|x|}\,dx < \infty,$$

is equivalent to the Biot-Savart law integral being absolutely convergent, and in this setting proves existence and uniqueness of (u, ω), with $|u|$ growing at most like $\sqrt{|x|}$ at infinity.

Brunelli's condition forces decay at infinity and excludes periodic flows.

Serfati

In 1995 Ph. Serfati published a four-page announcement in CRAS of existence and uniqueness for 2D Euler in $\mathbb{R}^2$ for $u_0 \in L^\infty$ such that $\omega_0 = \text{curl}\ u_0 \in L^\infty$. The proof was terse and incomplete, but brilliant.

With this type of initial data, the Biot-Savart law holds only as a distribution, as Brunelli showed.

(Serfati has another 1995 paper, in which he proved well-posedness for u_0 in C^{1+r}, $r > 0$, with u and ∇p in $C([0, \infty); C^{1+r})$.)

In this talk, we discuss Serfati's work, with an extension to a type of continuous dependence on initial data and to domains exterior to a connected, simply connected obstacle.

This is a report on work in progress.

Related results

- Taniuchi (2004) gives a complete, and very different proof of Serfati's result in the full plane with a slight generalization to allow slightly unbounded initial vorticity (as Yudovich 1995 did in a bounded domain). Uses Littlewood-Paley decomposition and Bony's paradifferential calculus, and rests strongly on Serfati's other 1995 paper. Does not generalize to exterior domains.
- Taniuchi, Tashiro, and Yoneda (2010) are concerned with *almost periodic* flows *in the full plane*. They prove continuous dependence (in $B^0_{\infty,1}$).
- Giga, Inui, and Matsui (1999) prove existence and uniqueness of solutions to the Navier-Stokes equations with velocity bounded and uniformly continuous (which includes Serfati initial data).
- Cozzi (2009, 2010) proves the vanishing viscosity limit of "viscous Serfati" solutions to inviscid ones in the full plane.

Motivation

Why study vorticity with no decay? Main uniqueness result is for vorticity in $L^1 \cap L^\infty$ (Yudovich 1963), but L^1 hypothesis is to make sense of Biot-Savart law. Uniqueness should be a local issue—behavior of vorticity at infinity should not be important.

In light of Taniuchi, Tashiro, and Yoneda's work, why revisit Serfati?

- Local versus non-local.
- Need new idea to substitute for the use of Biot-Savart law.
- Broader potential applications in Serfati's key idea ("Serfati identity").
- Extension to an exterior domain.

Why should bounded velocity solutions exist?

If initial velocity is in L^2, a simple energy argument shows that the L^2 norm of the velocity is conserved for all time.

L^∞-analog:

$$
\begin{aligned}
\partial_t u^j(x) &= \partial_t \int_\Omega K^j(x,y)\omega(y)\,dy = \int_\Omega K^j(x,y)\partial_t\omega(y)\,dy \\
&= -\int_\Omega K^j(x,y)(u\cdot\nabla\omega)(y)\,dy \\
&= -\int_\Omega K^j(x,y)\,\mathrm{curl}(u\cdot\nabla u)(y)\,dy \\
&= \int_\Omega K^j(x,y)\,\mathrm{div}\,[(u\cdot\nabla u)(y)]^\perp\,dy \\
&= -\int_\Omega [(u\cdot\nabla u)(y)]^\perp\cdot\nabla K^j(x,y)\,dy \\
&= \int_\Omega (u\cdot\nabla u)(y)\cdot\nabla^\perp K^j(x,y)\,dy.
\end{aligned}
$$

Using the identity,

$$\int_\Omega (u \cdot \nabla u) \cdot V = -\int_\Omega (u \cdot \nabla V) \cdot u$$

gives

$$\begin{aligned}\partial_t u^j(x) &= \int_\Omega (u \cdot \nabla u)(y) \cdot \nabla^\perp K^j(x, y)\, dy \\ &= -\int_\Omega (u(y) \cdot \nabla_y)\, \nabla_y^\perp K^j(x, y) \cdot u(y)\, dy.\end{aligned}$$

After integrating in time,

$$\boxed{u^j(t, x) = (u^0)^j(x) - \int_0^t \int_\Omega (u(s, y) \cdot \nabla_y)\, \nabla_y^\perp K^j(x, y) \cdot u(s, y)\, dy\, ds.}$$

But...

In the full plane, $\nabla_y \nabla_y K(x, y)$ is not in L^1.

Let a be a smooth cutoff of the origin and let $a_\epsilon(\cdot) = a(\cdot/\epsilon)$. Then Serfati obtained what we call the Serfati identity:

$$\boxed{\begin{aligned} u^j(t,x) &= (u^0)^j(x) + \int_\Omega a_\epsilon(x-y) K^j(x,y)(\omega(t,y) - \omega^0(y))\,dy \\ &\quad - \int_0^t \int_\Omega (u(s,y)\cdot\nabla_y)\,\nabla_y^\perp \left[(1 - a_\epsilon(x-y))K^j(x,y)\right] \\ &\qquad\qquad\qquad\qquad \cdot u(s,y)\,dy\,ds. \end{aligned}}$$

$$\|a_\varepsilon K\|_{L^1} \le C\varepsilon, \qquad \left\|\nabla_y^2 \left[(1 - a_\varepsilon)K\right]\right\|_{L^1} \le \frac{C}{\varepsilon}.$$

Thus,

$$\|u\|_{L^\infty} \le \|u^0\|_{L^\infty} + C\epsilon + \frac{C}{\epsilon}\int_0^t \|u(s)\|_{L_\infty}^2 \, ds.$$

Letting

$$\epsilon = \left(\int_0^t \|u(s)\|_{L_\infty}^2 \, ds\right)^{1/2}$$

gives

$$\|u\|_{L^\infty} \le \|u^0\|_{L^\infty} + C\left(\int_0^t \|u(s)\|_{L_\infty}^2 \, ds\right)^{1/2}.$$

L^∞-bound

Squaring both sides,

$$\|u\|_{L^\infty}^2 \le 2\|u^0\|_{L^\infty}^2 + C\int_0^t \|u(s)\|_{L^\infty}^2 \, ds.$$

Applying Gronwall's lemma,

$$\boxed{\|u(t)\|_{L^\infty} \le \sqrt{2}\|u^0\|_{L^\infty} e^{Ct}.}$$

Existence in the full plane

- What we have done so far is formal: we need to apply these estimates to a sequence, u_n, of smooth solutions to the Euler equations whose initial data converges, in the Serfati norm, to u^0.

- We depart from Serfati's approach in this regard, following more closely Majda's proof of existence of Yudovich solutions (vorticity in $L^1 \cap L^\infty$, velocity in L^2) that exploits the transport of the vorticity by the flow, establishing convergence of particle trajectories.

- Serfati's bound on the velocity, which is uniform over n, replaces the uniform bound, $\|u_n\|_{L^\infty} \le C\left[\|\omega^0\|_{L^1} + \|\omega^0\|_{L^\infty}\right]$.

- Many technical details are being suppressed here: in particular, smooth approximations of the initial velocity (in an exterior domain) is fairly involved.

Definition of weak solution

We require of our weak solutions those properties required to prove uniqueness (which are satisfied by the solutions we construct):

1. Velocity lies in $L^\infty_{loc}(\mathbb{R}; S)$, where S is the space of divergence-free vector fields tangent to the boundary with the norm,
$$\|u\|_S = \|u\|_{L^\infty} + \|\operatorname{curl} u\|_{L^\infty}.$$
2. Euler equations hold weakly against div-free test functions.
3. Serfati identity holds for at least one cutoff function, a.
4. Vorticity is transported by the flow.
5. Velocity has a spatial log-Lipschitz MOC uniformly over finite time.

(4) and (5) are redundant in that each implies the other.

Uniqueness

- Serfati's strategy: assume two solutions u_1, u_2, with the same initial data. Let X_1 and X_2 be their respective flow maps. Show $X_1 = X_2$. This implies $u_1 = u_2$.

- We follow this basic strategy, but depart from Serfati (primarily) in the first and last step.

- Define $\mu\colon [0, \infty) \to [0, \infty)$ strictly increasing with $\mu(0) = 0$ such that

$$\mu(h) = -Ch \log h$$

for h in $(0, 1/e)$.

Steps in proof of uniqueness

- Define

$$h = h(t) = \sup_{x \in \Omega} |X_1(t,x) - X_2(t,x)|.$$

- We will bound

$$\sup_{x \in \Omega} |u_1(t, X_1(t,x)) - u_2(t, X_2(t,x))|.$$

- By the triangle inequality, $|u_1(t, X_1) - u_2(t, X_2)|$

$$\begin{aligned} &\le |u_2(t, X_1) - u_2(t, X_2)| + |u_1(t, X_1) - u_2(t, X_1)| \\ &\le \mu(|X_1 - X_2|) + |u_1(t, X_1) - u_2(t, X_1)| \\ &\le \mu(h(t)) + |u_1(t, X_1) - u_2(t, X_1)|, \end{aligned}$$

where X_j is short for $X_j(t,x)$.

- Subtract Serfati identity for u_1 and u_2 to give

$$|u_1(t, X_1(t,x)) - u_2(t, X_1(t,x))| \le I_1 + I_2,$$

where

$$\begin{aligned} I_1 &= \left| \int_\Omega a(X_1(t,x) - y) K(X_1(t,x), y)(\omega^1(t,y) - \omega^2(t,y))\, dy \right|, \\ I_2 &= \int_0^t \int_\Omega |\nabla_y \nabla_y \left((1 - a(X_1(s,x) - y)) K(X_1(s,x), y)\right)| \\ &\qquad\qquad\qquad |u_1 \otimes u_1 - u_2 \otimes u_2|\,(s,y)\, dy\, ds. \end{aligned}$$

- Change variables Lagrangianly, use measure-preserving property of flow maps, and properties of K to show that

$$I_1 \le -C\|\omega^0\|_{L^\infty} h \log h = C\mu(h).$$

This is the most difficult step, particularly in an exterior domain.

- Bound

$$I_2 = \int_0^t \int_\Omega |\nabla_y \nabla_y ((1 - a(X_1(s,x) - y))K(X_1(s,x),y))|$$

by

$$C \int_0^t \mu(h(s))\, ds + C \int_0^t \|u_2(s, X_2(s,\cdot)) - u_1(s, X_1(s,\cdot))\|_{L^\infty}\, ds$$

again using properties of K.

- Putting these bounds together, what we have shown is that

$$\begin{aligned} &|u_1(t, X_1(t,x)) - u_2(t, X_2(t,x))| \\ &\qquad \le C \int_0^t \mu(h(s))\, ds + C(1 + h(t))\mu(h(t)) \\ &\qquad\qquad + C \int_0^t \|u_2(s, X_2(s,\cdot)) - u_1(s, X_1(s,\cdot))\|_{L^\infty}\, ds. \end{aligned}$$

- Letting

$$J(s) = \|u_1(s, X_1(s,\cdot)) - u_2(s, X_2(s,\cdot))\|_{L^\infty}$$

and taking the supremum over all x in Ω, we conclude that

$$J(t) \le C \int_0^t \mu(h(s))\, ds + C(1 + h(t))\mu(h(t)) + C \int_0^t J(s)\, ds.$$

- Letting

$$M(t) = \int_0^t J(s)\, ds,$$

$$J(t) \le C \int_0^t \mu(h(s))\, ds + C(1 + h(t))\mu(h(t)) + C \int_0^t J(s)\, ds$$

becomes (after first showing that $h(t) \le M(t)$)

$$M'(t) \le C(1 + t + M(t))\mu(M(t)) + CM(t) =: \nu(M(t)).$$

- As one can show, ν is Osgood-continuous ($\int_0^1 ds/\nu(s) = \infty$), so $M(t) \equiv 0$ by Osgood's lemma. Hence $J \equiv 0$, so that $X_1 \equiv X_2$.

“Continuous dependence on initial data”

Theorem

Suppose that the initial velocities, u_1^0 and u_2^0, having vorticities, ω_1^0 and ω_2^0, are such that $u_1^0 - u_2^0$ lies in the space,

$$S_p := \left\{ u \in (L^\infty(\Omega))^2 \colon \operatorname{div} u = 0, u \cdot \mathbf{n} = 0, \omega \in L^p(\Omega) \right\}$$

for some p in $(2, \infty]$, with $\|\cdot\|_{S^p} = \|\cdot\|_{L^\infty} + \|\omega(\cdot)\|_{L^p}$. Then for all sufficiently small $s_0 = \|u_1^0 - u_2^0\|_{S^p}$,

$$\|u_1(t) - u_2(t)\|_{L^\infty} \le s_0 e^{Ct} + C_t (s_0 t)^{e^{-Ct(2+t)}} \left[\log C_t + s_0 t e^{-Ct(2+t)} \right] \left[C(2+t) e^{Ct} + 1 \right],$$

where the constants, C and C_t, depend on the initial data and on p, with C_t a continuous function of time.

Controlling low frequencies gives smooth solutions

Theorem (Chemin 1996)

Let u^0 be a divergence-free vector field lying in the Zygmund space, $\mathcal{C}^r$, $r > 1$. There exists a unique $T^* > 0$ and a unique solution to the Euler equations with u lying in $L^\infty_{loc}([0, T^*); \mathcal{C}^r)$. Moreover,

$$T^* < \infty \implies \int_0^{T^*} \|u(t)\|_1 \, dt = \infty.$$

But,

$$\|u(t)\|_1 \le C \|u(t)\|_{L^\infty} + C \|\omega(u(t))\|_{L^\infty} \le C \|u(t)\|_S :$$

Theorem (Reproducing a result of Serfati 1995)

In $\mathbb{R}^2$, these $\mathcal{C}^r$-solutions are global in time.

Concluding remarks

- Uniqueness argument can be adapted to a domain exterior to a connected, simply connected domain.

- Existence argument in an exterior domain awaits a reformulation of Serfati's identity that allows a uniform bound on the L^∞ norms.

- One of the motivations for studying Serfati's result was to look at perturbations of periodic initial velocity.

- This line of study brings a natural question to mind: can we characterize those vorticities which are "Serfati" (bounded velocity with bounded vorticity) even in the full plane? Note that, if $\omega_0 \equiv 1$ then u_0 must at least grow linearly, hence it is *not* Serfati. All (doubly) periodic flows are Serfati, though, as are all compactly supported bounded vorticities.