Geometric evolution of interfaces in the functionalized Cahn-Hilliard equation

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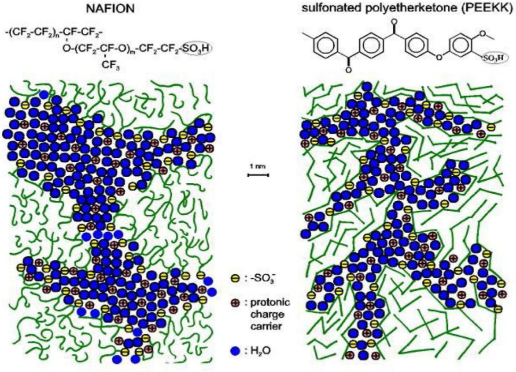
Outline

- Background
- Formal Results
- Rigorous Results

Functionalization

- In Chemistry functionalization involves attachment of short, acid tipped sidechains to the chains of polymer backbones to modify soluability properties of the original polymer.
- We wish to model phase separation of functionalized polymer in a solvent.



K.-D. Kreuer, J. Memb. Sci. 185 (2001) 29.

Energy Formulation

- The Canham-Helfrich free energy of an interface describes the most general energy that can be written as a quadratic symmetric polynomial of curvatures k_i. In $d = 3$, it is given in terms of the mean and Gaussian curvatures:

$$\mathcal{E}_{CH}(\Gamma) = \int_\Gamma \kappa_s K + \kappa_b (H - H_{\text{int}})^2 + \sigma dS$$

  - σ - energy density per unit surface area,
  - κ_b and κ_s - energy densities attributed to deformations,
  - $H = \frac{1}{2}(k_1 + k_2)$ - mean curvature, $K = k_1 k_2$ - Gaussian curvature,
  - H_{int} intrinsic (zero-energy) value of the mean curvature.
- Usefulness limited by difficulty in describing merging and pinch-off events.
- Does not easily couple to the physics outside of the interface.
- As an alternative, we consider a phase field model.

- Example: The Cahn-Hilliard energy

$$\mathcal{E}[u] = \int_\Omega \frac{\epsilon^2}{2}|\nabla u|^2 + W(u)dx,$$

where ε scales the interfacial width and W is a smooth, double-well potential with two equal minima at $u = \pm 1$.

- Interface evolution for gradient flows studied by Evans, Soner & Souganidis; de Mottoni & Schatzman; Pego; Alikakos, Bates & Chen, ...
- Γ-convergence is due to Modica and Mortola

$$\varepsilon^{-1}\mathcal{E} \to_\Gamma \int_\Gamma \sigma dS.$$

- Example: The modified De Giorgi functional

$$\mathcal{E}_G[u] = \int_\Omega \Big(-\epsilon^2 \Delta u + W'(u) \Big)^2 + \epsilon^2 \Big(\frac{\epsilon^2}{2} |\nabla u|^2 + W(u) \Big) dx.$$

The sharp interface limit was shown by M. Röger and R. Schätzle.

$$\varepsilon^{-3} \mathcal{E}_G \to_\Gamma \int_\Gamma \kappa_b H^2 + \sigma dS.$$

- Introduce the Functionalized Cahn-Hilliard energy as a model for interface evolution in functionalized polymer-solvent mixtures:

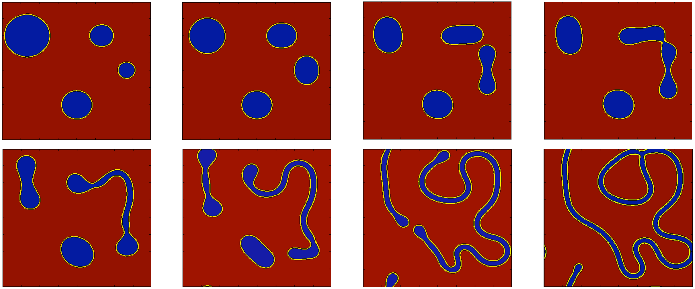
$$\mathcal{E}_F[u] = \int_\Omega \frac{1}{2} \Big(-\epsilon^2 \Delta u + W'(u) \Big)^2 - \epsilon^2 \eta \Big(\frac{\epsilon^2}{2} |\nabla u|^2 + W(u) \Big).$$

This energy balances the elastic energy required to bend the interface and the associated polymer backbone against the solvation and electrostatic energy released by formation of water-acid interface.

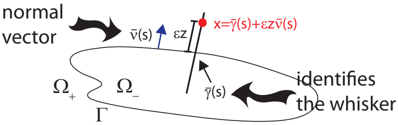
Simplest mass preserving gradient flow is:

$$u_t = -\Pi_0 \underbrace{\left((\epsilon^2\Delta - W''(u) + \epsilon^2\eta)(\epsilon^2\Delta u - W'(u))\right)}_{\frac{\delta\mathcal{E}_F}{\delta u}}.$$

where zero-mass projection subtracts the average value.



Question: What do steady-state solutions look like?
1) Heteroclinic front profiles seen in Cahn-Hilliard equation
2) New homoclinic solutions which correspond to bi-layer structures



To describe these solutions use a whiskered coordinate system connected to the interface Γ.

For a general change of coordinates: $x = \varphi(y)$ Laplace-Beltrami formulation for the Laplacian is given by

$$\Delta_x = \frac{1}{\sqrt{\det(\mathbf{G})}} \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial}{\partial y_i} G^{ij} \sqrt{\det(\mathbf{G})} \frac{\partial}{\partial y_j},$$

where $\mathbf{G}$ is the metric tensor

$$G_{ij} = \left\langle \frac{\partial x}{\partial y_i}, \frac{\partial x}{\partial y_j} \right\rangle_{\mathbb{R}^d}.$$

In the scaled whiskered coordinates $y = [s, z]$ and

$$\boxed{x = \varphi(y) = \gamma(s) + \varepsilon z \nu(s).}$$

In whiskered coordinates G takes the form

$$\mathbf{G} = \begin{pmatrix} \mathbf{G_0} & 0 \\ 0 & \varepsilon^2 \end{pmatrix},$$

Laplacian is

$$\boxed{\Delta_x = \varepsilon^{-2}\partial_z^2 + \varepsilon^{-1}\kappa(s,z)\partial_z + \Delta_{G_0},}$$

where κ and Δ_{G_0} are the extensions of mean curvature and Laplace-Beltrami operators off the interface Γ:

$$\kappa(s,z) = -\sum_{i=1}^{d-1} \frac{k_i}{1-\varepsilon z k_i} = -H(s) + O(\varepsilon z),$$

and

$$\Delta_{G_0} = \Delta_s + \varepsilon z D_{s,2},$$

where Δ_s is the Laplace-Beltrami operator on Γ and $D_{s,2}$ is a second order differential operator.

Heteroclinic Ansatz

Recall the FCH gradient flow

$$u_t = -\Pi_0 F(u).$$

At leading order, for u localized about a hypersurface Γ,

$$F(u) \equiv (\epsilon^2\Delta - W''(u) + \epsilon^2\eta)(\epsilon^2\Delta u - W'(u)) \sim (\partial_z^2 - W''(u))(\partial_z^2 u - W'(u)).$$

For a symmetric well the gradient flow has front critical points $\phi(z)$ satisfying

$$\partial_z^2\phi - W'(\phi) = 0.$$

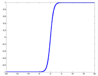
By incorporating curvature dependent terms into a perturbation ϕ_2 we build an ansatz

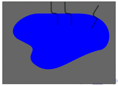
$$\Phi(x;\Gamma,b) = \Phi(s,z) = \phi(z) - \epsilon^2 \overbrace{\left(\frac{1}{2}H^2 + tr(A^2)\right)L_\phi^{-1}(z\phi')}^{\phi_2} + b,$$

where H is the mean curvature and A is the Weingarten map whose eigenvalues are the curvatures of Γ. For a smooth hypersurface Γ, we have

$$||F(\Phi)||_{C^4(\Omega)} = O(\epsilon^3).$$

We refer to this as 'dressing' the interface Γ with the 1D profile.





Formal Results (Physica D, 2011, with N. Gavish, K. Promislow and L. Yang)

A formal slow manifold reduction, projecting the residual $F(\Phi)$ onto the neutral modes of the linearization yields a coupled evolution for the constant background state, $b = \varepsilon^3 b_3$, and the normal velocity V_n,

$$V_n = -\varepsilon^3 \left[\left(\left(\Delta_s + \eta \right) H - \frac{H^3}{2} + Htr(A^2) \right) + \frac{2}{\sqrt{\det(\mathbf{g})} ||\phi'||^2_{L^2(\mathbb{R})}} \left(\mu_0^2 b_3 - \partial_t b_3 \right) \right],$$

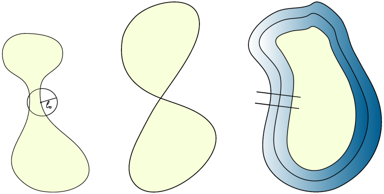
$$\partial_t b_3 + \varepsilon \frac{4\mu_0^2 |\Gamma|}{\sqrt{\det(\mathbf{g})} ||\phi'||^2_{L^2(\mathbb{R})} |\Omega|} b_3 = -\frac{2\varepsilon}{|\Omega|} \int_\Gamma \left(\Delta_s + \eta - \frac{H^2}{2} + tr(A^2) \right) Hds.$$

In the remainder of this presentation we address the rigorous derivation of these equations.

Rigorous Results - definition of admissible interfaces

Definition

Denote by $\mathcal{G}$ the family of smooth compact oriented (d-1)-dimensional manifolds without boundary embedded in $\mathbb{R}^d$. For $K > 0$ denote by $\mathcal{G}_K$ the set of manifolds $\Gamma \in \mathcal{G}$ satisfying the following assumptions:
(i) the principal curvatures and their derivatives up to the fourth order are bounded in $L^\infty(\Gamma)$ norm by K,
(ii) the whiskers of length $1/K$ (in the unscaled distance) do not intersect each-other,
(iii) the volume $vol(\Gamma)$ of Γ is bounded by K.



Definition

We associate to each $\Gamma \in \mathcal{G}_K$, $b \in \mathbb{R}$ the corresponding single-layer dressed ansatz,

$$\Phi(x;\Gamma,b) := \eta_1(z)(\phi(z) + \varepsilon^2\phi_2(s,z)) + b + \bar{\eta}_1(z).$$

Here $b = \varepsilon^3 b_3$ is an $O(\varepsilon^3)$ parameter that incorporates the small, spatially-constant variation of the background state of Φ away from the limiting values of ± 1, η_1 is a smooth cutoff function which is equal to one near the interface and is zero away from it. Similarly, $\bar{\eta}_1(x)$ takes the values ± 1 away from the interface and is zero near the interface.

Definition

For each $K, \bar{b} > 0$, we define the single-layer dressed manifold

$$\mathcal{M}_{K,\bar{b}} := \{\Phi(x;\Gamma,b) : \Gamma \in \mathcal{G}_K, b \in (-\bar{b}\varepsilon^3, \bar{b}\varepsilon^3)\}.$$

The reduction of the FCH to a curvature driven flow requires a detailed analysis of the linearization

$$\mathcal{L}_\phi := (-\varepsilon^2\Delta + W''(\Phi) - \varepsilon^2\eta)(-\varepsilon^2\Delta + W''(\Phi)) - W'''(\Phi)(\varepsilon^2\Delta\Phi - W'(\Phi)),$$

of F about the ansatz Φ.

- We utilize a decomposition $\mathcal{Z}_M \oplus \mathcal{Z}_M^\perp$ that breaks $L^2(\Omega)$ into two approximately-$\mathcal{L}_\phi$ invariant subspaces, with the bilinear form associated with $\mathcal{L}_\phi$ uniformly coercive when restricted to $\mathcal{Z}_M^\perp$.
- We show that a solution that starts in a sufficiently small neighborhood of $\mathcal{M}_{K,\bar{b}}$ may be decomposed as

$$u(x,t) = \Phi(x;\Gamma_t, b) + w(x,t),$$

where $w \in \mathcal{Z}_M^\perp$ and Γ_t denotes the interface at time t.

Reduction of slow flow to a normal velocity

- We project the residual $F(\Phi)$ onto the slow space of $\mathcal{L}_\phi$.
- **Assuming** the continued smoothness of the interface, we show that there exists a choice of normal velocity

$$V_n = \varepsilon^4 V_n^0(h) + V_n^c(h, w),$$

with V_n^0 depending only upon the second fundamental form, h, of the interface, such that the remainder w will remain small, in an appropriate norm, for all $t \in [0, T_f \varepsilon^{-4}]$.
- In particular, after an $O(\varepsilon^{-1})$ transient associated to the relaxation of the background state, the normal velocity reduces to

$$V_n^0 = -\Pi_{0,\Gamma}\left((\Delta_s + \eta)H - \frac{H^3}{2} + H\,\mathrm{tr}(A^2) \right),$$

where H is the mean curvature, A is the Weingarten map whose eigenvalues are the curvatures of Γ, and $\Pi_{0,\Gamma}$ is the zero-mass projection associated to the surface integral over Γ.

Key Steps - Laplace-Beltrami Spectrum

- Consider the Laplace-Beltrami operator $\Delta_s : H^2(\Gamma) \to L^2(\Gamma)$ given locally on Γ by

$$\Delta_s = \frac{1}{\sqrt{\det \mathbf{g}}} \sum_{i=1}^{d-1} \sum_{j=1}^{d-1} \frac{\partial}{\partial s_i} g^{ij} \sqrt{\det \mathbf{g}} \frac{\partial}{\partial s_j},$$

and denote the (nonnegative) eigenvalues of $-\Delta_s$ by $\{\beta_{j,\Gamma}\}$ and the corresponding eigenfunctions by $\{\Theta_{j,\Gamma}\}$,

$$-\Delta_s \Theta_{j,\Gamma} = \beta_{j,\Gamma} \Theta_{j,\Gamma}.$$

- The following is a key relation on the asymptotics of the large eigenvalues of the Laplace-Beltrami operator, first proved by Weyl

$$(\beta_m)^{(d-1)/2} \sim \frac{(2\pi)^{d-1}/\omega_d}{vol\Gamma} m, \quad \text{as } m \to \infty.$$

- Weyl asymptotic formula implies that for $M = [M_1 \varepsilon^{-(d-1)}]$

$$\beta_M \sim \frac{(2\pi)^2 M_1^{2/(d-1)}}{(vol\Gamma \omega_d)^{2/(d-1)}} \varepsilon^{-2}.$$

Key Step - Analysis of Linearization

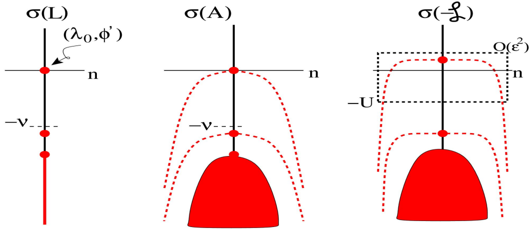
- In 'whiskered' variables, acting on functions localized about $\Gamma \in \mathcal{G}_K$,

$$\mathcal{L} = L^2 + \epsilon^4 \Delta_s^2 + \epsilon P_1(\partial_z) + \epsilon^2 P_2(\partial_z, \partial_{s_i}),$$

  where L is a $1D$ operator on each whisker

$$L = \partial_z^2 - W''(\phi).$$

- L is a Sturm-Liouville operator, so the spectrum on $\mathbb{R}$ is easy to establish.



$L = \partial_z^2 - W''(\phi)$ $\quad A = \varepsilon^2 \Delta - W''(\phi)$ $\quad \mathcal{L} = A^2 + O(\varepsilon^2).$

  Question: How does the 1D structure impact the spectrum of the full linearization on $\mathbb{R}^d$?

- For Allen-Cahn/Cahn-Hilliard linearizations about fronts shown by Alikakos and Fusco in 2D and Chen in higher dimensions.

Definition

For each $M > 0$, the $M+1$ dimensional slow space associated to $\mathcal{L}_\phi$ is defined by

$$\mathcal{Z}_M := span\left(\{Z_i\}_{i=1\dots M} \cup \{1\}\right),$$

where

$$Z_i(x) := \eta_1 \Theta_{i,\Gamma}(s)\phi'(z);$$

$\mathcal{Z}_M^\perp$ is the orthogonal complement of $\mathcal{Z}_M$ in $L^2(\Omega)$.

Remark: We will show that $\mathcal{Z}_M^\perp$ corresponds to the fast eigenspace of $\mathcal{L}_\phi$, in the sense that $\mathcal{L}_\phi$ is coercive on $\mathcal{Z}_M^\perp$.

Theorem: Spectrum of the Full Linear Operator

Coercivity Estimates for the Linear Operator. *Fix* $K, \bar{b} > 0$ *and* $\varepsilon > 0$ *sufficiently small. There exists* $M_{1-} > 0$ *independent of* ε*, such that for all* $M_1 \geq M_{1-}$ *the following bounds hold for all* w *from the associated fast space,* $\mathcal{Z}_M^{\perp}$*, where* $M = M_1 \varepsilon^{-(d-1)}$

$$(\mathcal{L}w, w)_{L^2(\Omega)} \geq \frac{1}{32}\varepsilon^4 \beta_{M+1}^2 ||w||_{L^2(\Omega)},$$

$$||\mathcal{A}_\phi^2 w||_{L^2(\Omega)} \geq \frac{1}{32}\varepsilon^4 \beta_{M+1}^2 ||w||_{L^2(\Omega)},$$

$$||\mathcal{A}_\phi w||_{L^2(\Omega)} \geq \frac{1}{8}\varepsilon^2 \beta_{M+1} ||w|_{L^2(\Omega)},$$

$$||\mathcal{A}_\phi w||_{L^2(\Omega)} \geq C\varepsilon^4 \beta_{M+1} ||w||_{H^2(\Omega)},$$

where

$$\mathcal{A}_\phi := \varepsilon^2 \Delta - W''(\Phi) + \varepsilon^2 \eta,$$

Remark: This result is also extended to bi-layer dressings, when the linearization of the $1D$ operator has two asymptotically small eigenvalues.

Key Steps - Initial Value Decomposition

Any initial data u_0 which is close to the manifold $\mathcal{M}_{K,\bar{b}}$ can be (trivially) written as

$$u_0 = \Phi(x; \Gamma_*, b_*) + w_*(x),$$

for some $\Gamma_* \in \mathcal{G}_K$ with w_* small in an appropriate norm. To obtain decay of w, we look for an orthogonal decomposition of the form

$$u_0 = \Phi(x; \Gamma_0, b) + w_0(x, t),$$

with the additional condition that w_0 lie in the fast space, that is $w_0 \in \mathcal{Z}_M^{\perp}(\Gamma_0)$. We search for Γ_0 among interfaces Γ_p near Γ_* in the form

$$\gamma_p(s) = \gamma_*(s) + \nu_*(s)R(s),$$

where $\nu_*(s)$ is the normal to Γ_* at $\gamma_*(s)$. We take candidates for R as among the Galerkin sums on the first M Laplace-Beltrami eigenmodes, $\{\Theta_i\}_{i=1}^{M}$ of Γ_*,

$$R = \sum_{i=1}^{M} p_i \Theta_i(s),$$

and determine the parameters $p = (p_1, \ldots, p_M)$ to impose the orthogonality condition

$$w_0 := u_0 - \Phi_0 \in \mathcal{Z}_M^{\perp}(\Gamma_0). \tag{3.1}$$

Main Difficulties

- Loss of one derivative of spatial regularity when decomposing the initial data -
$$\gamma_p(s) = \gamma_*(s) + \nu_*(s)R(s).$$
- Smoothness of the interfaces under the normal velocity flow - the analysis of interface evolution equation for Γ which is expressed as a curvature driven flow on the first and second fundamental forms of Γ is nontrivial.
- The higher order corrections V_n^c to the normal velocity enter into the curvature flow. We have control of this correction term in only relatively weak norms, such as L^2 and L^∞. We require estimates on higher-order derivatives of w to better control V_n^c.
- Even for $d = 2$, when the fundamental forms are scalar, the flow reduces, at leading order, to a generalized Kuramoto-Sivashinsky equation,
$$\partial_\tau H - (\partial_s H) \int_0^s V_n^0 H ds = -\left(\partial_s^2 + H^2\right) \Pi_{0,\Gamma} \left(\partial_s^2 + \eta + \frac{H^2}{2}\right) H,$$
whose regularity is (just) outside the known results. Here $\Pi_{0,\Gamma}$ is the zero-mass projection associated to the surface integral over Γ.

Main Assumption

The scaled normal velocity

$$\bar{V}_n := \varepsilon^{-4} V_n = V_n^0(h) + \varepsilon^{-4} V_n^c(h, w)$$

is equivalent (up to rigid body motions) to the curvature driven flow on the fundamental forms of the hypersurface Γ,

$$\begin{aligned} \partial_\tau g_{ij} &= -2\bar{V}_n h_{ij}, \\ \partial_\tau h_{ij} &= -\nabla_i \nabla_j \bar{V}_n + \sum_{l,m} \bar{V}_n h_{il} g^{lm} h_{mj}. \end{aligned} \tag{3.2}$$

Assumption (*)

We assume that the higher-order curvature driven flow (3.2) is well-posed in $H^4(\mathbf{S})$ *for all scaled time* $\tau \in [0, T_f]$, *where* $\mathbf{S} \subset \mathbf{R}^{d-1}$ *is the reference set for the interface parameterization* γ.

Interface Evolution Reduction

We recall that the dynamics are governed by the FCH gradient flow

$$\begin{aligned} \partial_t u(x,t) &= -\Pi_0 F(u), \\ u(x,0) &= u_0(x), \end{aligned}$$

Time-derivative of the ansatz Φ is

$$\partial_t \Phi(x;\Gamma_t;b) = \partial_z \Phi(z(x;t), s(x;t)) \frac{\partial z(x;t)}{\partial t} + \varepsilon^2 D_s \phi_2 \cdot \frac{\partial s(x;t)}{\partial t} + \partial_t b,$$

where

$$\frac{\partial z(x;t)}{\partial t} = -\frac{1}{\varepsilon} V_n(s(x;t)).$$

Substituting the decomposition $u(x,t) = \Phi(x;\Gamma_t;b(t)) + w(x,t)$ into the gradient flow, we obtain

$$\begin{aligned} -\frac{1}{\varepsilon} V_n(s,t)\partial_z \Phi + \varepsilon^2 D_s \phi_2 \cdot \frac{\partial s(x;t)}{\partial t} + \partial_t b + \partial_t w &= -\Pi_0 F(\Phi + w) \\ &= -\Pi_0 F(\Phi) - \Pi_0 \mathcal{L}_\phi w - \Pi_0 \mathcal{N}(w) \quad (**), \end{aligned}$$

where

$$\mathcal{N}(w) = F(\Phi + w) - F(\Phi) - \mathcal{L}_\phi w,$$

represents the nonlinear terms in w.

A priori estimates on normal velocity

We define the $\mathcal{A}$-norm of $u \in H^2(\Omega)$ as

$$||u||_{\mathcal{A}} := ||\Delta u||_{L^2(\Omega)} + \varepsilon^{-2}||u||_{L^2(\Omega)}.$$

Projecting (**) onto the space $\mathcal{Z}_M$ and balancing the first terms on the left and right-hand sides, we obtain bounds on the normal velocity.

Proposition

The following estimates hold for the normal velocity

$$||V_n||_{L^\infty(\Gamma)} \le M_1 C(K) \left(\varepsilon^{5-(d-1)} + \varepsilon^{7/2-(d-1)} ||w||_{\mathcal{A}} \right),$$

$$\left\| V_n^c \right\|_{L^2(\Gamma)} = \left\| V_n - \varepsilon^4 V_n^0 \right\|_{L^2(\Gamma)} \le C(K)\sqrt{M_1} \left(\varepsilon^{(11-d)/2} + \varepsilon^{(8-d)/2} ||w||_{\mathcal{A}} \right),$$

where

$$V_n^0 = -\left[\left(\Delta_s + \eta \right) H - \frac{H^3}{2} + Htr(A^2) \right] - \frac{2}{||\phi'||^2_{L^2(\mathbb{R})}} \left(\mu_0^2 b_3 - \partial_t b_3 \right).$$

and M_1 is the slow-space dimension parameter.

Main Theorem - Part 1

Let the space dimension $d = 2$ *or* 3 *and assume that Assumption (*) is satisfied. Fix the admissible interface parameters* $\bar{b}, K > 0$ *and* $\bar{b}_0 < \bar{b}$, $K_0 < K$. *Fix the slow-space dimension parameter* $M_1 > 0$. *Then there exist* $\frac{1}{2}\bar{B}_0 > \bar{B} > 0$ *and* $U, \varepsilon_0, T_f > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$ *and for all* u_0 *satisfying*

$$\inf_{\substack{\Gamma \in \mathcal{G}_{K_0} \\ b \in (-\bar{b}_0 \varepsilon^3, \bar{b}_0 \varepsilon^3)}} \|u_0 - \Phi(\cdot, \Gamma, b)\|_{\mathcal{A}} < \bar{B}_0 \varepsilon^{d-3/2},$$

then for $t < T_f \varepsilon^{-4}$ *we have a decomposition of the solution* $u(x,t)$

$$u(x,t) = \Phi(x, \Gamma_t, b(t)) + w(x,t)$$

where $b(t) = \varepsilon^3 b_3(t)$, w *lies in the* L^2 *orthogonal complement of the slow space,* $w \in \mathcal{Z}_M^{\perp}$, *with the slow dimension* $M = [M_1 \varepsilon^{-(d-1)}]$ *and*

$$||w||_{\mathcal{A}} \leq \bar{B}_0 \varepsilon^{d-3/2} e^{-\frac{U}{2} t} + \bar{B} \varepsilon^{3/2}.$$

Main Theorem - Part 2

Moreover, the remainder term in the normal velocity $V_n = \varepsilon^4 V_n^0 + V_n^c$ of the interface Γ_t satisfies

$$\|V_n^c\|_{L^2(\Gamma)} = \left\|V_n - \varepsilon^4 V_n^0\right\|_{L^2(\Gamma)} \leq C(K)\sqrt{M_1}\left(\varepsilon^{(11-d)/2} + ||w||_{\mathcal{A}}\varepsilon^{(8-d)/2}\right)$$

where V_n^0 takes the following form

$$V_n^0 = -\left[\left(\Delta_s + \eta\right)H - \frac{H^3}{2} + Htr(A^2)\right] - \frac{2}{\sqrt{\det(\mathbf{g})}||\phi'||^2_{L^2(\mathbb{R})}}\left(\mu_0^2 b_3 - \partial_t b_3\right),$$

and the background state evolves according to

$$\partial_t b_3 + \varepsilon \frac{4\mu_0^2|\Gamma|}{\sqrt{\det(\mathbf{g})}||\phi'||^2_{L^2(\mathbb{R})}|\Omega|} b_3 = -\frac{2\varepsilon}{|\Omega|}\int_\Gamma \left(\Delta_s + \eta - \frac{H^2}{2} + tr(A^2)\right)Hds$$

$$+ O\left(\sqrt{M_1}\varepsilon^{(8-d)/2}||w||_{\mathcal{A}}, \sqrt{M_1}\varepsilon^{(11-d)/2}\right).$$

Main Theorem - Part 3

The evolution equations result in a two time-scale system, which, after an $O(\varepsilon^{-1})$ time drives the background to its quasi-equilibrium, and V_n^0 relaxes to

$$V_n^0 = -\Pi_{0,\Gamma}\left(\Delta_s + \eta - \frac{H^2}{2} + tr(A^2)\right) H.$$