# Coherent structures, turbulent cascades and regularity theory

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*Energy cascades and flux locality in physical scales of the 3D NSE* (with R. Dascaliuc), Commun. Math. Phys. **305** (2011), 199–220.

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*A geometric measure-type regularity criterion for solutions to the 3D Navier-Stokes equations* (submitted to Arch. Rational Mech. Anal.) http://arxiv.org/abs/1111.0217

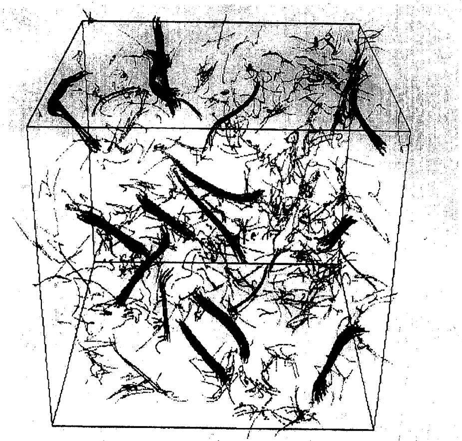

Figure 11: Intermittent vortex filaments in a 3D computational simulation of equilibrium homogeneous turbulence. (She *et al.*, 1991)

"Half a century after Kolmogorov's work on the statistical theory of fully developed turbulence, we still wonder how his work can be reconciled with Leonardo's half a millennium old drawings of eddy motion in the study for the elimination of rapids in the river Arno."

– U. Frisch, *Turbulence*, The Legacy of A.N. Kolmogorov, 1994

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a more specific question: what is the role of *coherent vortex structures* in the mathematical theory of *turbulent cascades*?

zero-step = finding a mathematical framework suitable for *encoding geometric information on the flow* in the theory of turbulent cascades

$R_0$ = the *macro scale* associated with the flow; $B(0, 2R_0)$ contained in $\Omega$ (the global spatial domain)

$x_0$ in $B(0, R_0)$

$0 < R \leq R_0$

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$f =$ a locally integrable function (density) on $B(x_0, 2R)$

a *local physical scale* $R$ – associated to the point $x_0$ – is realized via bounds on distributional derivatives of $f$,

$$|(D^\alpha f, \psi)| \leq \int_{B(x_0,2R)} |f||D^\alpha \psi| \leq \left(c(\alpha)\frac{1}{R^{|\alpha|}}|f|, \psi^{\delta(\alpha)}\right)$$

for some $c(\alpha) > 0$ and $\delta(\alpha)$ in $(0, 1)$

spatiotemporal cut-offs

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$\phi = \phi_{x_0,R,T} = \psi\,\eta$ on $B(x_0, 2R) \times (0, T)$

where $\eta = \eta_T(t) \in C^\infty(0, T)$ and $\psi = \psi_{x_0,R}(x) \in \mathcal{D}(B(x_0, 2R))$ satisfying

$$0 \le \eta \le 1, \quad \eta = 0 \text{ on } (0, T/3), \quad \eta = 1 \text{ on } (2T/3, T), \quad \frac{|\eta'|}{\eta^{\rho_1}} \le \frac{C}{T} \tag{1}$$

and

$$0 \le \psi \le 1, \quad \psi = 1 \text{ on } B(x_0, R), \quad \frac{|\nabla\psi|}{\psi^{\rho_2}} \le \frac{C}{R}, \quad \frac{|\triangle\psi|}{\psi^{2\rho_2 - 1}} \le \frac{C}{R^2}\,, \tag{2}$$

for some $\frac{1}{2} < \rho_1, \rho_2 < 1$

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for some $\frac{1}{2} < \rho_1, \rho_2 < 1$

the case $x_0 = 0$ and $R = R_0$ corresponds to the macro scale domain cut-off $\phi_0$, $\phi_0 = \eta\psi_0$

for $x_0$ near the boundary of the macro scale domain, $S(0, R_0)$, assume additional conditions,

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and, if $B(x_0, R)$ is not included in $B(0, R_0)$, then $\psi \in \mathcal{D}(B(0, 2R_0))$ with $\psi = 1$ on $B(x_0, R) \cap B(0, R_0)$ satisfying, in addition to (2), the following:

$$\begin{aligned}&\psi = \psi_0 \text{ on the part of the cone centered at zero and passing through}\\ &S(0, R_0) \cap B(x_0, R) \text{ between } S(0, R_0) \text{ and } S(0, 2R_0)\end{aligned} \tag{4}$$

and

$$\begin{aligned}&\psi = 0 \text{ on } B(0, R_0) \setminus B(x_0, 2R) \text{ and outside the part of the cone}\\ &\text{centered at zero and passing through } S(0, R_0) \cap B(x_0, 2R)\\ &\text{between } S(0, R_0) \text{ and } S(0, 2R_0)\end{aligned} \tag{5}$$

# ensemble averages; physical scales

a *physical scale* $R$ – associated to the macro scale domain $B(0, R_0)$ – is realized via suitable ensemble-averaging of the localized quantities with respect to '$(K_1, K_2)$-covers at scale $R$'

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let $K_1$ and $K_2$ be two positive integers, and $0 < R \leq R_0$. a cover $\{B(x_i, R)\}_{i=1}^n$ of $B(0, R_0)$ is a $(K_1, K_2)$-*cover at scale* $R$ if

$$\left(\frac{R_0}{R}\right)^3 \leq n \leq K_1 \left(\frac{R_0}{R}\right)^3,$$

and any point $x$ in $B(0, R_0)$ is covered by at most $K_2$ balls $B(x_i, 2R)$

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the parameters $K_1$ and $K_2$ represent the maximal *global* and *local multiplicities*, respectively for a physical density of interest $f$, consider time-averaged, per unit mass – spatially localized to the cover elements $B(x_i, R)$ – local quantities $\hat{f}_{x_i,R,T}$,

$$\hat{f}_{x_i,R,T} = \frac{1}{T}\int_0^T \frac{1}{R^3}\int_{B(x_i,2R)} f(x,t)\phi^{\delta}_{x_i,R,T}(x,t)\,dx\,dt$$

for some $0 < \delta \le 1$, and denote by $\langle F\rangle_R$ the *ensemble average* given by

$$\langle F\rangle_R = \frac{1}{n}\sum_{i=1}^{n} \hat{f}_{x_i,R,T}$$

the key feature of the ensemble averages $\{\langle F\rangle_R\}_{0<R\le R_0}$ is that $\langle F\rangle_R$ being *stable*, i.e., nearly independent on a particular choice of the cover – with the fixed parameters $K_1$ and $K_2$ – indicates there are no significant sign-fluctuations of the density $f$ at scales comparable or greater than $R$

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on the other hand, if $f$ does exhibit significant sign-fluctuations at scales comparable or greater than $R$, suitable rearrangements of the cover elements up to the maximal multiplicities will result in the range of $\langle F\rangle_R$ containing an interval of the form $(-M, M)$ for a large $M$

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consequently, for an *a priori* sign-varying density, the ensemble averaging process acts as a *coarse detector of the sign-fluctuations at scale* $R$ (the larger the maximal multiplicities $K_1$ and $K_2$, the finer detection) as expected, for a non-negative density $f$, all the averages are comparable to each other throughout the full range of scales $R$, $0 < R \leq R_0$; in particular, they are all comparable to the simple average over the macro scale domain as expected, for a non-negative density $f$, all the averages are comparable to each other throughout the full range of scales $R$, $0 < R \leq R_0$; in particular, they are all comparable to the simple average over the macro scale domain

$$\frac{1}{K_*} F_0 \leq \langle F \rangle_R \leq K_* F_0 \tag{6}$$

for all $0 < R \leq R_0$, where

$$F_0 = \frac{1}{T} \int \frac{1}{R_0^3} \int f(x,t) \phi_0^\delta(x,t)\, dx\, dt,$$

and $K_* = K_*(K_1, K_2) > 1$

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more generally, the bound (6) holds for any non-negative distribution (integration replaced by duality); in fact, this was a key technical ingredient in the proof of existence of 3D inviscid cascade presented in [Dascaliuc and G., 2011]

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$$(\partial_t + u \cdot \nabla - \triangle)|\omega|^2 + |\nabla\omega|^2 = \alpha|\omega|^2$$

$$\alpha(x) = \frac{3}{4\pi} P.V. \int D\left(\hat{y}, \xi(x+y), \xi(x)\right) |\omega(x+y)| \frac{1}{|y|^3}\, dy,$$

where $\hat{y}$ is the unit vector in the $y$-direction, $\xi$ is the vorticity direction and the *geometric* kernel $D$ is defined by

$$D(e_1, e_2, e_3) = (e_1 \cdot e_3)\left(e_1 \cdot \ (e_2 \times e_3)\right)$$

for any triple of unit vectors $e_1, e_2$ and $e_3$

note that

$$|D\,(\hat{y},\xi(x+y),\xi(x))\,|\leq|\sin\varphi\big(\xi(x+y),\xi(x)\big)|$$

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and later in [Beirao da Veiga and Berselli, 2002] where the Lipschitz condition was replaced by $\frac{1}{2}$-Hölder

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$$
\begin{aligned}
\int \frac{1}{2}|\omega(x,t)|^2\psi(x)\,dx &+ \int_0^t \int |\nabla\omega|^2\phi\,dx\,ds \\
&= \int_0^t \int \frac{1}{2}|\omega|^2(\phi_t + \triangle\phi)\,dx\,ds \\
&+ \int_0^t \int \frac{1}{2}|\omega|^2(u\cdot\nabla\phi)\,dx\,ds + \int_0^t \int (\omega\cdot\nabla)u\cdot\phi\omega\,dx\,ds
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(cf. [G., 2009])

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(\omega\cdot\nabla)u\cdot\phi\omega\,(x) &= \phi^{\frac{1}{2}}(x)\,\frac{\partial}{\partial x_i}u_j(x)\,\phi^{\frac{1}{2}}(x)\,\omega_i(x)\,\omega_j(x)\\
&= -c\,P.V.\int_{B(x_0,2r)}\epsilon_{jkl}\,\frac{\partial^2}{\partial x_i\partial y_k}\,\frac{1}{|x-y|}\,\phi^{\frac{1}{2}}\,\omega_l\,dy\;\phi^{\frac{1}{2}}(x)\,\omega_i(x)\,\omega_j(x)+\text{ LOT}\\
&= -c\,P.V.\int_{B_(x_0,2r)}\big(\omega(x)\times\omega(y)\big)\cdot G_\omega(x,y)\,\phi^{\frac{1}{2}}(y)\,\phi^{\frac{1}{2}}(x)\,dy+\text{ LOT}\\
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where $\epsilon_{jkl}$ is the Levi-Civita symbol,

$$
\big(G_\omega(x,y)\big)_k=\frac{\partial^2}{\partial x_i\partial y_k}\,\frac{1}{|x-y|}\,\omega_i(x)
$$

and LOT denotes the lower order terms

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and later in [G. and Guberović, 2010] to introduce a family of *scaling-invariant* regularity classes featuring a balance between coherence of the vorticity direction and the vorticity magnitude

the following regularity class – a scaling-invariant improvement of $\frac{1}{2}$-Hölder coherence – is included,

$$\int_{t_0-(2R)^2}^{t_0} \int_{B(x_0,2R)} |\omega(x,t)|^2 \ \rho^2_{\frac{1}{2},2R}(x,t) dx\, dt < \infty; \tag{9}$$

$$\rho_{\gamma,r}(x,t) = \sup_{y\in B(x,r), y\neq x} \frac{|\sin\varphi\big(\xi(x,t),\xi(y,t)\big)|}{|x-y|^\gamma}$$

let $\mathcal{R}$ be a region contained in the global spatial domain $\Omega$. the inward enstrophy flux through the boundary of the region is given by

$$-\int_{\partial\mathcal{R}} \frac{1}{2}|\omega|^2 (u \cdot n)\, d\sigma = -\int_{\mathcal{R}} (u \cdot \nabla)\omega \cdot \omega \, dx$$

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localization of evolution of the enstrophy to cylinder $B(x_0, 2R) \times (0, T)$ leads to the following version of the enstrophy flux,

$$\int \frac{1}{2}|\omega|^2 (u \cdot \nabla\phi)\, dx = -\int (u \cdot \nabla)\omega \cdot \phi\omega\, dx \tag{10}$$

# 3D enstrophy cascade

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since $\nabla\phi = (\nabla\psi)\eta$, and $\psi$ can be constructed such that $\nabla\psi$ points inward, (10) represents *local inward enstrophy flux, at scale* $R$ (more precisely, through the layer $S(x_0, R, 2R)$) *around the point* $x_0$

considering a $(K_1, K_2)$-cover $\{B(x_i, R)\}_{i=1}^n$ at scale $R$, for some $0 < R \leq R_0$, local inward enstrophy fluxes at scale $R$ – associated to the cover elements $B(x_i, R)$ – are then given by

$$\int \frac{1}{2}|\omega|^2 (u \cdot \nabla\phi_i)\, dx, \tag{11}$$

for $1 \leq i \leq n$

assuming smoothness on $(0,T)$, the identity (7) written for $B(x_i,R)$ yields the following expression for time-integrated local fluxes,

$$\begin{aligned}\int_0^t\int\frac{1}{2}|\omega|^2(u\cdot\nabla\phi_i)\,dx\,ds &= \int\frac{1}{2}|\omega(x,t)|^2\psi_i(x)\,dx+\int_0^t\int|\nabla\omega|^2\phi_i\,dx\,ds\\ &\quad-\int_0^t\int\frac{1}{2}|\omega|^2\big((\phi_i)_s+\triangle\phi_i\big)\,dx\,ds\\ &\quad-\int_0^t\int(\omega\cdot\nabla)u\cdot\phi_i\,\omega\,dx\,ds, \qquad (12)\end{aligned}$$

for any $t$ in $(2T/3,T)$ and $1\le i\le n$

denoting the time-averaged local fluxes per unit mass associated to the cover element $B(x_i, R)$ by $\hat{\Phi}_{x_i,R}$,

$$\hat{\Phi}_{x_i,R} = \frac{1}{t}\int_0^t \frac{1}{R^3}\int \frac{1}{2}|\omega|^2(u\cdot\nabla\phi_i)\,dx, \tag{13}$$

the main quantity of interest is the ensemble average of $\{\hat{\Phi}_{x_i,R}\}_{i=1}^n$,

$$\langle\Phi\rangle_R = \frac{1}{n}\sum_{i=1}^n \hat{\Phi}_{x_i,R} \tag{14}$$

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the goal is to formulate a set of physically reasonable conditions on the flow in $B(0,2R_0)\times(0,T)$ implying the strict positivity and stability of $\langle\Phi\rangle_R$ across a suitable range of scales – *existence of the enstrophy cascade*

**(A1) Coherence Assumption**

let $M > 0$ (large). assume that there exists a positive constant $C_1$ such that

$$|\sin\varphi\big(\xi(x,t),\xi(y,t)\big)| \le C_1|x-y|^{\frac{1}{2}}$$

for any $(x,y,t)$ in $\big(B(0,2R_0)\times B(0,2R_0+R_0^{\frac{2}{3}})\times(0,T)\big)\cap\{|\nabla u|>M\}$

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for any $(x,y,t)$ in $\big(B(0,2R_0)\times B(0,2R_0+R_0^{\frac{2}{3}})\times(0,T)\big)\cap\{|\nabla u|>M\}$

note that the previous local regularity results [G. and Zhang, 2006, G., 2009] imply that – under (A1) – the *a priori* weak solution in view is in fact smooth inside $B(0,2R_0)\times(0,T)$ and can, moreover, be smoothly continued (locally-in-space) past $t=T$; in particular, we can write (12) with $t=T$

**(A2) Modified Kraichnan Scale**

denote by $E_0$ time-averaged enstrophy per unit mass associated with the integral domain $B(0, 2R_0) \times (0, T)$,

$$E_0 = \frac{1}{T} \int \frac{1}{R_0^3} \int \frac{1}{2} |\omega|^2 \phi_0^{2\rho-1} \, dx \, dt,$$

by $P_0$ a modified time-averaged palinstrophy per unit mass,

$$P_0 = \frac{1}{T} \int \frac{1}{R_0^3} \int |\nabla \omega|^2 \phi_0 \, dx \, dt + \frac{1}{T} \frac{1}{R_0^3} \int \frac{1}{2} |\omega(x, T)|^2 \psi_0(x) \, dx$$

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(A2) is a requirement that the modified Kraichnan scale associated with the integral domain $B(0, 2R_0) \times (0, T)$ be dominated by the integral scale,

$$\sigma_0 < \beta R_0,$$

for a suitable constant $\beta = \beta(\rho, K_1, K_2, M, B_T)$, $0 < \beta < 1$, where $B_T = \sup_{t \in (0,T)} \|\omega(t)\|_{L^1}$; this is finite provided, e.g., $\omega_0$ in $L^1$ [Constantin, 1990]

**(A3) Localization of the Integral Domain and Modulation**

the general set up considered is one of the Leray solutions satisfying (A1). as already mentioned, (A1) implies smoothness; however, the control on regularity-type norms is only local

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$$\int_0^T \int_{B(0,2R_0+R_0^{\frac{2}{3}})} |\omega|^2 \, dx \, dt \le \frac{1}{C_2},$$

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the modulation assumption on the evolution of local enstrophy on $(0,T)$ – consistent with the choice of the temporal cut-off $\eta$ – reads

$$\int |\omega(x,T)|^2 \psi_0(x) \, dx \geq \frac{1}{2} \sup_{t \in (0,T)} \int |\omega(x,t)|^2 \psi_0(x) \, dx$$

**Theorem**

*Let $u$ be a Leray solution on $\mathbb{R}^3 \times (0,T)$ satisfying (A1)-(A3) on the spatiotemporal integral domain $B(0, 2R_0 + R_0^{\frac{2}{3}}) \times (0,T)$, and suppose that $\omega_0$ is in $L^1(\mathbb{R}^3)$. Then,*

$$\frac{1}{4K_*} P_0 \le \langle \Phi \rangle_R \le 4K_* \; P_0$$

*for all $R$, $\frac{1}{\beta}\sigma_0 \le R \le R_0$.*

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denoting the time-averaged local fluxes associated to the cover element $B(x_i, R)$ by $\hat{\Psi}_{x_i,R}$,

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$$\langle \Psi \rangle_R = \frac{1}{n}\sum_{i=1}^{n} \hat{\Psi}_{x_i,R} = R^3 \, \langle \Phi \rangle_R \tag{16}$$

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the following locality result is a simple consequence of the universality of the cascade of the time and ensemble-averaged local fluxes *per unit mass* $\langle \Phi \rangle_R$ presented in the previous theorem

**Theorem**

*Let* $u$ *be a Leray solution on* $\mathbb{R}^3 \times (0,T)$ *satisfying (A1)-(A3) on the spatiotemporal integral domain* $B(0, 2R_0 + R_0^{\frac{2}{3}}) \times (0,T)$, *and suppose that* $\omega_0$ *is in* $L^1(\mathbb{R}^3)$. *Let* $R$ *and* $r$ *be two scales within the inertial range delineated in the previous theorem. Then*

$$\frac{1}{16K_*^2}\left(\frac{r}{R}\right)^3 \le \frac{\langle\Psi\rangle_r}{\langle\Psi\rangle_R} \le 16K_*^2\left(\frac{r}{R}\right)^3.$$

*In particular, if* $r = 2^k R$ *for some integer* $k$, *i.e., through the dyadic scale,*

$$\frac{1}{16K_*^2}\;2^{3k} \le \frac{\langle\Psi\rangle_{2^k R}}{\langle\Psi\rangle_R} \le 16K_*^2\;2^{3k}.$$

**Theorem**

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previous locality results include locality of the filtered flux – via coarse graining approach – presented in [Eyink, 2005] and [Eyink and Aluie, 2009], and locality of the flux in the Littlewood-Paley setting obtained in [Cheskidov, Constantin, Friedlander and Shvydkoy, 2008]

let $\Omega$ and $K$ be an open and a closed set in the complex plane, respectively

$\omega = \omega(\cdot, \Omega, K) =$ a unique bounded harmonic function on $\Omega \setminus K$ such that $\omega = 1$ on $K$ and $\omega = 0$ on $\partial\Omega \setminus K$

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(i) the harmonic measure is increasing (as a measure) with respect to both $K$ and $\Omega$

(ii) let $f$ be analytic in $\Omega \setminus K$, $|f| \leq M$, and $|f| \leq m$ on $K$. then

$$|f(z)| \leq m^{\theta} \, M^{1-\theta}$$

for any $z$ in $\Omega \setminus K$, where $\theta = \omega(z, \Omega, K)$ [this a refined form of the maximum modulus principle for analytic functions in $\Omega \setminus K$ (the log-convexity of the modulus of $f$ – sometimes referred to as "two-constants theorem")]

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(ii) a result on extremal properties of the harmonic measure in the unit disk $\mathbb{D}$ obtained in [Solynin, 1999]

**Theorem**

*Let $K$ be a closed subset of $[-1,1]$ such that $|K| = 2\lambda$ for some $\lambda$, $0 < \lambda < 1$, and suppose that $0 \in \mathbb{D} \setminus K$. Then*

$$\omega(0, \mathbb{D}, K) \geq \omega(0, \mathbb{D}, K_\lambda) = \frac{2}{\pi} \arcsin \frac{1-(1-\lambda)^2}{1+(1-\lambda)^2}$$

*where $K_\lambda = [-1, -1+\lambda] \cup [1-\lambda, 1]$.*

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the above theorem provides a generalization of the classical Beurling's result from 1933 in which $K$ is a finite union of intervals lying on one side of the origin. this was conjectured in [Segawa, 1988], and the symmetric version was previously resolved in [Essen and Haliste, 1989]

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**Theorem**

*Let $u_0$ be in $L^\infty(\mathbb{R}^3)$. Then, there exists an absolute constant $c_0 > 1$ such that setting $T = \dfrac{1}{c_0^2 \|u_0\|_\infty^2}$, a unique mild solution $u = u(t)$ on $[0, T]$ has the analytic extension $U = U(t)$ to the region*

$$\mathcal{R}_t = \{x + iy \in \mathbb{C}^3 : |y| \leq \frac{1}{c_0}\sqrt{t}\}$$

*for any $t$ in $(0, T]$. In addition,*

$$\|U(t)\|_{L^\infty(\mathcal{R}_t)} \leq c_0 \|u_0\|_\infty$$

*for all $t$ in $[0, T]$.*

the vorticity version of the above theorem is as follows (the proof is analogous; utilizing the Biot-Savart law to close each iteration) the vorticity version of the above theorem is as follows (the proof is analogous; utilizing the Biot-Savart law to close each iteration)

**Theorem**

*Let $\omega_0$ be in $L^\infty(\mathbb{R}^3)$. Then, there exists an absolute constant $d_0 > 1$ such that setting $T = \dfrac{1}{d_0^2 \|\omega_0\|_\infty}$, a unique mild solution $\omega = \omega(t)$ on $[0, T]$ has the analytic extension $\Omega = \Omega(t)$ to the region*

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*for any $t$ in $(0, T]$. In addition,*

$$\|\Omega(t)\|_{L^\infty(\mathcal{R}_t)} \le d_0 \|\omega_0\|_\infty$$

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since our main tool in the proof is the harmonic measure maximum principle, the spatial regularity is recorded in $L^\infty = L^\infty(\mathbb{R}^3)$; starting from an initial value in $L^\infty$, we consider the corresponding unique mild solution $u$ on the maximal interval of regularity $(0, T^*)$ (there is always 'the first singular time')

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note that – due to the local-in-time well-posedness of the equations in $L^\infty$ – for an arbitrary large $L$, there exists an 'escape time' $t_L$; i.e., for any $L > 0$, there exists a time $t_L$ in $(0, T^*)$, such that $\|u(t)\|_\infty > L$ for all $t$ in $(t_L, T^*)$

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for $M > 0$, denote by $\Omega_M(t)$ the super-level set at time $t$; more precisely,

$$\Omega_M(t) = \{x \in \mathbb{R}^3 : |u(x,t)| > M\}$$

## Definition

*Let $x_0$ be a point in $\mathbb{R}^3$, $r > 0$, $S$ an open subset of $\mathbb{R}^3$ and $\delta$ in $(0,1)$.*

*The set $S$ is linearly $\delta$-sparse around $x_0$ at scale $r$ in weak sense if there exists a unit vector $d$ in $S^2$ such that*

$$\frac{|S \cap (x_0 - rd, x_0 + rd)|}{2r} \leq \delta.$$

.

the main result (for the velocity formulation) reads as follows

**Theorem**

*Let $u_0$ be in $L^\infty$, and consider the corresponding unique mild solution $u$ on the maximal interval of regularity $[0, T^*)$.*
*Let $\delta$ be in $(0,1)$, $h = h(\delta) = \frac{2}{\pi} \arcsin \frac{1-\delta^2}{1+\delta^2}$, $\alpha = \alpha(\delta) \geq \frac{1-h}{h}$, and $M = M(\delta) = \frac{1}{c_0^\alpha} \|u(t^e)\|_\infty$ where $c_0$ is the constant in the local-in-time analyticity estimate. Assume that there exists an escape time $t^e$ and a time $t$ in $\left[t^e + \frac{1}{4c_0^2 \|u(t^e)\|_\infty^2}, t^e + \frac{1}{c_0^2 \|u(t^e)\|_\infty^2}\right]$ such that for any spatial point $x_0$, there exists a scale $r$, $0 < r \leq \frac{1}{2c_0^2 \|u(t^e)\|_\infty}$, with the property that the super-level set $\Omega_t(M)$ is linearly $\delta$-sparse around $x_0$ at scale $r$ in weak sense.*

*Then, $T^*$ is in fact not a singular time, and the solution $u$ can be smoothly continued past $T^*$.*

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**Lemma**

*Let* $x_0$ *be a point in* $\mathbb{R}^3$, $\rho > 0$, $S$ *an open subset of* $\mathbb{R}^3$ *and* $\delta$ *in* $(0,1)$.

*Suppose that*

$$\frac{|S \cap B(x_0,\rho)|}{|B(x_0,\rho)|} \le (1-\delta)^3. \tag{17}$$

*Then, there exists a scale* $r$, $0 < r \le \rho$, *such that* $S$ *is linearly* $\delta$*-sparse around* $x_0$ *at scale* $r$ *in weak sense.*

recall that the distribution function of a function $f$ – encoding virtually all information on the size of $f$ – is given by

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an *amorphous* regularity criterion – obtained from the previous theorem via the lemma as a geometrically worst case scenario – is a requirement on the decrease of the distribution function of the solution, uniformly in $(T^* - \epsilon, T^*)$

**Theorem**

*Let $u_0$ be in $L^\infty$, and consider the corresponding unique mild solution $u$ on the maximal interval of regularity $[0, T^*)$.*

*Assume that there exists $\epsilon$, $0 < \epsilon < T^*$, such that*

$$\lambda_{u(s)}(\beta) = o\Big(\frac{1}{\beta^3}\Big), \quad \beta \to \infty,$$

*uniformly in $(T^* - \epsilon, T^*)$ (the same rate for all $s$ in the interval).*

*Then, $T^*$ is in fact not a singular time, and the solution $u$ can be smoothly continued past $T^*$.*

it is instructive to compare this result with the well-known uniform-in-time boundedness of the $L^3$-norm of $u$-regularity criterion given in [Escauriaza, Seregin and Sverak, 2003]

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however, since the inclusion is not continuous, i.e., the size of the $L^3$ norm provides no information on the rate at which $\alpha^3 \lambda_f(\alpha)$ decreases to $0$, as $\alpha \to \infty$, the two regularity criteria are in fact not directly comparable the vorticity version is completely analogous

**Theorem**

*Let $\omega_0$ be in $L^\infty$, and consider the corresponding unique mild solution $\omega$ on the maximal interval of regularity $[0, T^*)$.*
*Let $\delta$ be in $(0,1)$, $h = h(\delta) = \frac{2}{\pi} \arcsin \frac{1-\delta^2}{1+\delta^2}$, $\alpha = \alpha(\delta) \geq \frac{1-h}{h}$, and $M = M(\delta) = \frac{1}{d_0^\alpha} \|\omega(t^e)\|_\infty$ where $d_0$ is the constant in the local-in-time analyticity estimate. Assume that there exists an escape time $t^e$ and a time $t$ in $\left[t^e + \frac{1}{4d_0^2 \|\omega(t^e)\|_\infty}, t^e + \frac{1}{d_0^2 \|\omega(t^e)\|_\infty}\right]$ such that for any spatial point $x_0$, there exists a scale $r$, $0 < r \leq \frac{1}{2d_0^2 \|\omega(t^e)\|_\infty^{\frac{1}{2}}}$, with the property that the super-level set $\Omega_t(M)$ is linearly $\delta$-sparse around $x_0$ at scale $r$ in weak sense.*

*Then, $T^*$ is in fact not a singular time, and the solution $\omega$ can be smoothly continued past $T^*$.*

# geometric scenario leading to closing the scaling gap

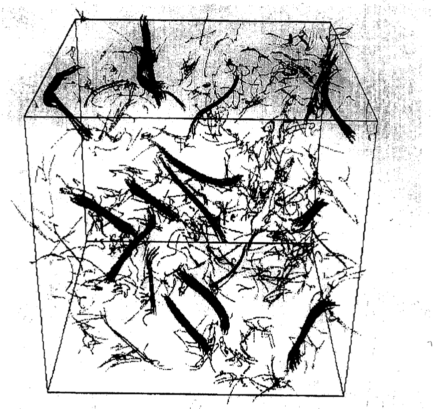

Figure 11: Intermittent vortex filaments in a 3D computational simulation of equilibrium homogeneous turbulence. (She *et al.*, 1991)

theorem $\Longrightarrow$

as long as the diameters of the tubes are dominated by $\dfrac{1}{C_1}\dfrac{1}{\|\omega(t)\|_\infty^{\frac{1}{2}}}$, for a suitable constant $C_1 > 1$, no blow up can occur

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assuming that the length of a tube is comparable with the macro scale, this implies the decrease of the tubes' diameters of at least $C_3\dfrac{1}{\|\omega(t)\|_\infty^{\frac{1}{2}}}$, which is exactly the scale needed for the application of the theorem

assuming that the 'shape', i.e., the general geometry is correct, the *big* assumption is that the length of a vortex tube is comparable to the macro scale; this was simply borrowed from the picture painted by the numerical simulations

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back to $(K_1, K_2)$-covers and ensemble averages

$$\begin{aligned}\int_0^t \int (\omega \cdot \nabla)u \cdot \phi_i \, \omega \, dx \, ds = & \int \frac{1}{2}|\omega(x,t)|^2 \psi_i(x) \, dx + \int_0^t \int |\nabla \omega|^2 \phi_i \, dx \, ds \\ & - \int_0^t \int \frac{1}{2}|\omega|^2 \big((\phi_i)_s + \triangle \phi_i\big) \, dx \, ds \\ & - \int_0^t \int \frac{1}{2}|\omega|^2 (u \cdot \nabla \phi_i) \, dx \, ds, \end{aligned} \tag{18}$$

for any $t$ in $(2T/3, T)$, and $1 \leq i \leq n$

denote the time-averaged local vortex-stretching terms per unit mass associated to the cover element $B(x_i, R)$ by $VST_{x_i,R,t}$,

$$VST_{x_i,R,t} = \frac{1}{t}\int_0^t \frac{1}{R^3}\int (\omega\cdot\nabla)u\cdot\phi_i\,\omega\,dx\,ds \tag{19}$$

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the quantity of interest is the ensemble average of $\{VST_{x_i,R,t}\}_{i=1}^n$; namely,

$$\langle VST\rangle_{R,t} = \frac{1}{n}\sum_{i=1}^{n} VST_{x_i,R,t} \tag{20}$$

,

denote by $E_{0,t}$ time-averaged enstrophy per unit mass associated with the integral domain $B(0, 2R_0) \times (0, t)$,

$$E_{0,t} = \frac{1}{t} \int_0^t \frac{1}{R_0^3} \int \frac{1}{2} |\omega|^2 \phi_0^{2\rho-1} \, dx \, ds,$$

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by $P_{0,t}$ a modified time-averaged palinstrophy per unit mass,

$$P_{0,t} = \frac{1}{t} \int_0^t \frac{1}{R_0^3} \int |\nabla \omega|^2 \phi_0 \, dx \, ds + \frac{1}{t} \frac{1}{R_0^3} \int \frac{1}{2} |\omega(x,t)|^2 \psi_0(x) \, dx$$

(the modification is due to the shape of the temporal cut-off $\eta$), denote by $E_{0,t}$ time-averaged enstrophy per unit mass associated with the integral domain $B(0, 2R_0) \times (0, t)$,

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(the modification is due to the shape of the temporal cut-off $\eta$),

and by $\sigma_{0,t}$ a corresponding modified Kraichnan scale,

$$\sigma_{0,t} = \left( \frac{E_{0,t}}{P_{0,t}} \right)^{\frac{1}{2}}$$

ensemble-averaging the local enstrophy equality yields the following result on the strict positivity and stability of the averaged vortex-stretching term at time $t$

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A DYNAMIC ESTIMATE ON $VST$ ACROSS A RANGE OF SCALES

let $u$ be a Leray-Hopf solution on a global spatiotemporal domain $\Omega \times (0,T)$, smooth on $B(0,2R_0) \times (0,T)$, and let $0 < \gamma < 1$. assume that $\frac{1}{c_* \gamma}\sigma_{0,t} < R_0$ for a suitable constant $c_*$. then, there exist $c_1(\gamma)$ and $c_2(\gamma)$ such that $c_1(\gamma) \to 1^-$, as $\gamma \to 0^+$ and $c_2(\gamma) \to 1+$, as $\gamma \to 0^+$, and

$$c_1(\gamma)\frac{1}{K_1}\, P_{0,t} \le \langle VST \rangle_{R,t} \le c_2(\gamma) K_2\, P_{0,t} \tag{21}$$

for all $R$ within the range $\frac{1}{c_* \gamma}\sigma_{0,t} \le R \le R_0$