Isoperimetric sets inside almost-convex cones

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Outline

1) Formulation of the problem.

2) A result of Lions-Pacella (1990) on the isoperimetric inequality in convex cones:

   Under suitable hypotheses, cone convex $\Rightarrow$ isoperimetric regions are balls centered at the origin.

3) Stability forms of isoperimetric results.

4) Statement of our main theorem:

   The characterization of isoperimetric regions as balls centered at the origin persists for some almost-convex cones.

5) Some ideas of the proof: Compactness arguments, Stability for isoperimetric inequalities, Sharp Poincaré estimate.

Formulation of the problem

$$A \text{ open subset of the sphere } S^{N-1}(\subset \mathbb{R}^N)$$

$$\mathcal{C}_A = \{tx : x \in A, t > 0\} \text{...cone over } A.$$

Fix $m > 0$. We look for measurable $E \subset \mathcal{C}_A$ minimizing the relative perimeter

$$P(E; \mathcal{C}_A)$$

among all such sets satisfying the volume constraint $|E| = m$.

These sets are called isoperimetric sets inside the cone.

Theorem: (P.L. Lions and F. Pacella '90) $N \geq 2$, A open $\subset S^{N-1} \subset \mathbb{R}^N$, $\mathcal{C}_A$ cone over A. Suppose $\mathcal{C}_A$ is convex. Then

$$\frac{P(E;\mathcal{C}_A)}{|E|^{\frac{N-1}{N}}} \geq \frac{P(B;\mathcal{C}_A)}{|B_1 \cap \mathcal{C}_A|^{\frac{N-1}{N}}}$$

for all $E \subset \mathcal{C}_A$ measurable with $|E| < \infty$.

Moreover, if A is smooth, equality holds if and only if $E = B_R \cap \mathcal{C}_A$ for suitable $R > 0$.

Note: $P(B_1;\mathcal{C}_A) = N|B_1 \cap \mathcal{C}_A|$, so the right-hand side is $N|B_1 \cap \mathcal{C}_A|^{1/N}$.

- Lions-Pacella:
  Via Brunn-Minkowski inequality $|A + B|^{1/N} \geq |A|^{1/N} + |B|^{1/N}$.
  They also show: without convexity, the result fails in general.

- M. Ritoré, C. Rosales ('03):
  Existence of isoperimetric regions in smooth cones strictly contained in the upper hemisphere; characterization of isoperimetric regions in convex cones via second variation arguments.

- Figalli-E. Indrei ('12):
  Stability form of the Lions-Pacella result, characterization also valid for non-smooth convex cones.

> Define $\alpha(E) = \frac{P(E;\mathcal{C}_A)}{N|B_1 \cap \mathcal{C}_A|^{1/N}|E|^{(N-1)/N}} - 1$. Then $\mathcal{C}_A$ open convex cone containing no lines, $K = B_1 \cap \mathcal{C}_A$, $0 < |E| < \infty$ implies
>
> $$\frac{|E \Delta sK|}{|E|} \lesssim \sqrt{\alpha(E)} \text{ with } s \text{ chosen s.t. } |E| = |sK|.$$

- X. Cabré, X. Ros-Oton, J. Serra: "ABP method."

We discuss an extension of the Lions-Pacella characterization to a certain class of "almost-convex" cones. Let $d_{L^\infty}(\partial X, \partial Y)$ denote Hausdorff distance between ∂X and ∂Y in the sphere.

Definition: $\eta > 0$, $r > 0$, $S_+(\eta) := \{\xi \in S^{N-1} : \xi_N > \eta\}$,

$$\Pi_+(\eta, r) := \Big\{ A \text{ open} \subset\subset S_+(\eta) : \text{for } x \in \partial A \text{ there exist balls } B_r^+, B_r^- \text{ of radius } r \text{ with } B_r^+ \subset A,\ B_r^- \subset S^{N-1} \setminus A, \text{ and } x \in \partial B_r^+ \cap \partial B_r^- \Big\}.$$

Theorem: (B., Figalli) Fix $N \geq 3$, $\eta > 0$, $r > 0$. Then there exists $\epsilon > 0$ such that if

$$A, A' \in \Pi_+(\eta, r) \text{ are s.t. } \mathcal{C}_A \text{ is convex, } d_{L^\infty}(\partial A', \partial A) < \epsilon,$$

then for all $m > 0$ the unique minimizer E'_* of $E \mapsto P(E; \mathcal{C}_{A'})$ among sets of finite perimeter with $|E'_*| = m$ is given by

$$E'_* = B \cap \mathcal{C}_{A'}, \quad B \text{ ball centered at the origin, suitable radius.}$$

Strategy: Compactness arguments, inspired by results on stability for isoperimetric problems.

Some approaches for stability results for isoperimetric inequalities on $\mathbb{R}^N$ (estimating isop. deficit from below by notions of asymmetry):

(i) Symmetrization (N. Fusco, F. Maggi, A. Pratelli, Annals '06)

(ii) Optimal transport (Figalli, Maggi, Pratelli '07)

(iii) Regularity theory of "almost minimizers" for perimeter / "selection principle" (M. Cicalesi, G.P. Leonardi '10)

(Note: The ABP method also leads to a stability form of the Lions-Pacella result.)

An important step in (iii) is a "restricted" stability result due to Fuglede, which applies to nearly spherical sets.

Theorem: (Fuglede, '89) $D \subset \mathbb{R}^N$. Set

$$\mu(D) = \frac{P(D) - P(B)}{P(B)}.$$

Then D nearly spherical (star-shaped wrt. barycenter($=0$) so $\partial D = \{(1+u(\xi), \xi)$ in polar coordinates: $\xi \in S^{N-1}\}$ with $\|u\|_{L^\infty}$, $\|\nabla u\|_{L^\infty} \le \epsilon$) implies

$$\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \lesssim \mu(D).$$

Remarks:

- Fuglede estimate is a sharpening of Bonnesen type inequalities

$$D \subset \mathbb{R}^2, B_{r_1} \subset D \subset B_{r_2} \quad \rightsquigarrow \quad (r_2 - r_1)^2 \le \frac{|\partial D|^2}{4\pi} - |D|$$

  Paper of Fuglede also has an estimate for convex sets.

- A few words about the proof: Taylor expansion of the "area integral" and expansion by spherical harmonics.

Fuglede, continued:

Set $d\sigma = d\mathcal{H}^{N-1}(S^{N-1})$. Without loss of generality, assume $|D| = |B| = \omega_N$, and that the barycenter of D is at the origin, i.e.

$$\int (1+u)^{N+1} x d\sigma = 0.$$

Then

$$\begin{aligned}
\frac{P(D)}{P(B)} &= \int (1+u)^{N-1} \sqrt{1 + \frac{|\nabla u|^2}{(1+u)^2}} d\sigma \\
&= \int 1 + \frac{|\nabla u|^2}{2} + (N-1)u + \frac{(N-1)(N-2)}{2} u^2 d\sigma \\
&\quad + O(\epsilon)(\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2)
\end{aligned}$$

Now, $|D| = |B|$ implies $\int u d\sigma = -(\frac{N-1}{2} + O(\epsilon))\|u\|^2_{L^2}$, so

$$\frac{P(D) - P(B)}{P(B)} = \frac{1}{2} \int |\nabla u|^2 - (N-1)u^2 d\sigma + O(\epsilon)(|u\|^2_{L^2} + \|\nabla u\|^2_{L^2}),$$

and it is enough to show that for $\eta > 0$, if $\epsilon > 0$ is small enough,

$$(1-\eta)\|u\|^2_{L^2} + \frac{1}{2}\|\nabla u\|^2_{L^2} \leq \|\nabla u\|^2_{L^2} - (N-1)\|u\|^2_{L^2}.$$

For this, expand by spherical harmonics and use the spectral gap (uses that barycenter is the origin!).

Our approach is inspired by this, combined with compactness properties of minimizers.

Step 1: Suppose the claim fails. Then there exist sequences $(A_{*,n})$, $(A_n) \subset \Pi_+(\eta, r)$ s.t. $\mathcal{C}_{A_{*,n}}$ convex,

$$d(\partial A_n, \partial A_{*,n}) \to 0$$

and the cone over A_n has a minimizer which is not a section of a ball.

By rescaling and compactness: there is a limiting convex cone $\mathcal{C}_{A_*}$ and a limiting set E_* with (after extracting a subsequence) $|E_n \Delta E_*| \to 0$ as $n \to \infty$ and E_* is a minimizer for $\mathcal{C}_{A_*}$.

Now, the Lions-Pacella result implies $E_* = \mathcal{C}_{A_*} \cap B$.

By the regularity theory for minimizers/almost minimizers of perimeter (in the interior and up to the boundary, c.f. G. De Philippis and F. Maggi '15), for n sufficiently large, ∂E_n can be written as a graph over A_n:

$$u_n(\xi) = \sup\{t > -1 : (1+t)\xi \in E_n\}$$

$$\rightsquigarrow \quad \partial E_n \cap \mathcal{C}_{A_n} = \Big\{(1 + u_n(\xi))\xi : \xi \in A_n\Big\}.$$

A first variation argument now implies that $(\nabla u_n) \cdot \nu = 0$ on ∂A_n, so that $|E_n \Delta E| \to 0$ and the almost minimizing property imply

$$\|u_n\|_{L^\infty} \to 0, \quad \|\nabla u_n\|_{L^\infty} \to 0$$

as $n \to \infty$.

This gives

$$\begin{aligned} P(E_n; \mathcal{C}_{A_n}) \geq \mathcal{H}^{N-1}(A_n) & \\ &+ \frac{1-\delta}{2}(\|\nabla u_n\|_{L^2}^2 - (N-1)\|u_n\|_{L^2}^2) \\ &- \frac{\delta}{2}\|u_n\|L^2 \end{aligned}$$

for fixed $\delta > 0$, when n is sufficiently large.

Step 2: Uniform Poincaré inequality on $\Pi_+(\eta, r)$.

Proposition: For fix $N \geq 3$, $\eta > 0$, $r > 0$, there exists $\epsilon > 0$ and $C_1 > \sqrt{N-1}$ s.t. for $A, A' \in \Pi_+(\eta, r)$ with $\mathcal{C}_A$ convex, $d_{L^\infty}(\partial A, \partial A') < \epsilon$,

$$\|u - \overline{u}\|_{L^2(A')} \leq \frac{1}{C_1} \|\nabla u\|_{L^2(A')},$$

with

$$\overline{u} = \frac{1}{\mathcal{H}^{N-1}(A')} \int_{A'} u d\mathcal{H}^{N-1}.$$

Key idea for the uniform Poincaré inequality:

The sharp constant is determined by

$$\mu_1(A) = \inf\left\{\frac{\|\nabla u\|_{L^2(A)}}{\|u\|_{L^2(A)}} : u \in W^{1,2}(A), \int_A u d\mathcal{H}^{N-1} = 0, u \not\equiv 0\right\},$$

which is stable under d_{L^∞} convergence for sets in $\Pi_+(\eta, r)$ (this is related to the existence of a uniformly bounded extension operator; D. Chenais). Moreover, $\Pi_+(\eta, r)$ has "good" compactness properties.

Claim: For suitable $\epsilon > 0$, there exists $c_0 > \sqrt{N-1}$ s.t. $\mu_1(A) \geq c_0$ for all $A \in \Pi_+^{\text{conv}}(\eta, r) = \{A \in \Pi_+(\eta, r) : \mathcal{C}_A \text{ is convex}\}$.

If not, there is a sequence (A_n) in Π_+^{conv} s.t.

$$\limsup_{n\to\infty} \mu_1(A_n) \leq \sqrt{N-1}.$$

Now, compactness in the class of convex sets implies we can find

$$A_* \in \Pi_+^{\text{conv}}$$

s.t. $\mu_1(A_*) \leq \sqrt{N-1}$.

But: results of J.F. Escobar ('90) and Y. Alkhutov and V.G. Maz'ya ('09, '13) imply

> for convex domains A_* in the sphere, $\mu_1(A_*) \geq \sqrt{N-1}$, with equality for smooth cones if and only if A_* is the hemisphere!

In particular, $A_* \in \Pi_+^{\text{conv}} \Rightarrow \mu_1(A_*) > \sqrt{N-1}$. This gives the desired contradiction.

Step 3: Recalling from the area integral computation as in Fuglede, for fixed $\delta > 0$, $\epsilon > 0$,

$$
\begin{aligned}
&P(E_n; \mathcal{C}_{A_n}) - \mathcal{H}^{N-1}(A_n) \\
&\quad \geq \frac{1-\delta}{2} \|\nabla u_n\|_{L^2}^2 - \left(\frac{(N-1)(1-\delta)}{2} + \frac{\delta}{2} \right) \\
&\qquad \cdot \left(\|u_n - \overline{u_n}\|_{L^2} + |\overline{u_n}| \mathcal{H}^{N-1}(A_n)^{1/2} \right)^2 .
\end{aligned}
$$

By Cauchy-Schwarz, volume normalization, and our smallness assumptions,

$$
|\overline{u_n}| \leq \frac{C\epsilon}{(1 - C\epsilon)\mathcal{H}^{N-1}(A_n)^{1/2}} \|u_n - \overline{u_n}\|_{L^2}
$$

for all n sufficiently large (depending on ϵ).

This gives, for n sufficiently large (depending on ϵ and δ),

$$
\begin{aligned}
&P(E_n; \mathcal{C}_{A_n}) - \mathcal{H}^{N-1}(A_n) \\
&\quad \geq \frac{1-\delta}{2} \|\nabla u_n\|_{L^2}^2 - \left(\frac{(N-1)(1-\delta)}{2} + \frac{\delta}{2} \right) \\
&\qquad \cdot \left(1 + \frac{C\epsilon}{1 - C\epsilon}\right)^2 \|u_n - \overline{u_n}\|_{L^2}^2 .
\end{aligned}
$$

Now, note that the L^2 norm of $u_n - \overline{u_n}$ is bounded by $(1/c_1)\|\nabla u\|_{L^2}$.

Thus, choosing $\delta > 0$ and $\epsilon > 0$ small enough, we obtain

$$P(E_n; \mathcal{C}_{A_n}) - \mathcal{H}^{N-1}(A_n) \geq C(N,\eta)\|\nabla u_n\|^2_{L^2(A_n)}.$$

On the other hand, E_n minimizing $E \mapsto P(E; \mathcal{C}_{A_n})$ among finite perimeter sets with $|E| = \mathcal{H}^{N-1}(A_n)/N$ implies

$$P(E_n; \mathcal{C}_{A_n}) - \mathcal{H}^{n-1}(A_n) \leq P(B; \mathcal{C}_{A_n}) - \mathcal{H}^{n-1}(A_n) = 0$$

where B is the unit ball centered at the origin.

Thus, for n sufficiently large, $\|\nabla u_n\|_{L^2} = 0$, and u_n is constant. This contradicts the original choice of the sequences (the sets E_n were non-ball minimizers!).

Thank you!