The typical area-minimizing Current is unique and has Multiplicity One

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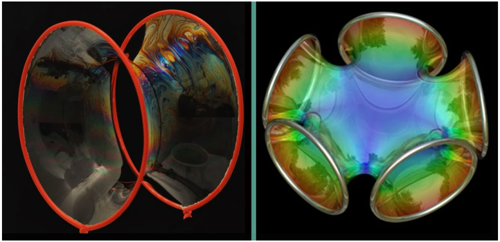
Outline

Plateau Problem

- Joseph Plateau (1801-1883)
- Experiments with soap bubbles/films
- Minimal surface



Problem

For a fixed boundary Γ, find the surface Σ of least area among those with $\partial\Sigma = \Gamma$.

For graph(u) minimizer and $\phi : D^2 \to \mathbb{R}$ with compact support, we have

$$0 = \frac{d}{dt}\bigg|_{t=0} \mathcal{H}^2(\mathrm{graph}(u + t\phi)) = -\int_{D^2} \underbrace{\mathrm{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right)}_{=0 \text{ Minimal Surface Equation}} \phi \, \mathrm{d}\mathcal{L}^2.$$

Theorem (Douglas, Radó)

Fix $b : \mathbb{S}^1 \to \mathbb{R}$. Then there exists a minimizer u with $u|_{\mathbb{S}^1} = b$.

- Douglas showed equivalence to minimizing the Dirichlet functional, only works in $\mathbb{R}^3$
- Fields medal 1936
- the set of graphs is not compact

Currents - "Sobolev surfaces"

Definition

An m-dim current is a continuous linear functional $T : \mathcal{D}^m(\mathbb{R}^{m+n}) \to \mathbb{R}$.

- 0-dimensional currents are distributions.
- For Σ an oriented surface,

$$T_\Sigma(\omega) = \int_\Sigma \omega.$$

- Stokes Theorem

$$T_{\partial\Sigma}(\omega) = \int_{\partial\Sigma} \omega = \int_\Sigma \mathrm{d}\omega = T_\Sigma(\mathrm{d}\omega).$$

Definition (Boundary)

The boundary of a current T is the $(m-1)$-current

$$\partial T : \mathcal{D}^{m-1}(\mathbb{R}^{m+n}) \to \mathbb{R},\ \omega \mapsto T(\mathrm{d}\omega).$$

Integral Currents

Definition (Integer Rectifiable Current)

An integer rectifiable m-current T is given by

$$T(\omega) = \int_E \theta(x)\langle \omega(x), \tau(x)\rangle \mathsf{d}\mathcal{H}^m(x),$$

where E is m-rectifiable, τ is an orientation of E and $\theta(x) \in \mathbb{Z}$ for a.e. x (multiplicity). T is denoted by $[E, \tau, \theta]$.

Definition (Mass)

The mass of a current T is given by

$$\mathbf{M}(T) := \sup_{|\omega| \leq 1, \omega \in \mathcal{D}^m} T(\omega).$$

Compactness

Theorem (Federer-Fleming (1960))

Let $\{T_i\}_{i\in\mathbb{N}}$ be a sequence of integral currents with $\mathbf{M}(T_i) + \mathbf{M}(\partial T_i) \leq C$ for all i, then there is a subsequence converging to an integer rectifiable current T.

Definition (Area-minimzing)

An integral current T is called area-minimzing if for all integral currents S with $\partial S = \partial T$, we have

$$\mathbf{M}(T) \leq \mathbf{M}(S).$$

Regularity Theorems

Theorem (Almgren, De Lellis-Spadaro)

Let $T \in \mathbf{I}_m(\mathbb{R}^{n+m})$ be area-minimizing with $\partial T = T_\Gamma$ regular. Then the set of interior points $p \in \text{supp}(T) \setminus \text{supp}(\partial T)$ for which $B_r(p) \cap \text{supp}(T)$ is *not* a smooth mfld for any $r > 0$, is a set of $\mathcal{H}$-dimension $\leq m - 2$.

Theorem (De Lellis-De Philippis-Hirsch-Massaccesi)

Let $T \in \mathbf{I}_m(\mathbb{R}^{n+m})$ be area-minimizing with $\partial T = T_\Gamma$ regular. Then the set of boundary points $p \in \text{supp}(\partial T)$ for which there is an $r > 0$ such that $B_r(p) \cap \text{supp}(T)$ is a regular mfld with boundary, is open dense in $\text{supp}(\partial T)$.

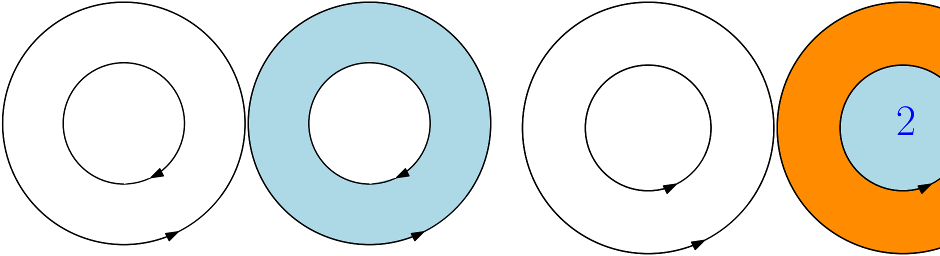
Two-sided boundaries

Definition

A boundary point $p \in \Gamma$ is called two-sided if

$$\liminf_{r \to 0} \frac{\|T\|(B_r(p))}{\omega_m r^m} \geq \frac{3}{2}.$$

Otherwise p is called one-sided.



Main Theorem

Theorem (Caldini-Marchese-Merlo-S.)

The typical $(m-1)$-dim $C^{3,\alpha}$-boundary $\Gamma \subset \mathbb{R}^{n+m}$ has a unique area-minimzing current T with $\partial T = T_\Gamma$ and moreover, T has multiplicity $\theta \equiv 1$.

Definition (Space of boundaries)

Let Γ be an oriented, closed submanifold of dimension $m-1$. Let $U \subset \mathbb{R}^{n+m}$ be open and $f : \Omega \subset \mathbb{R}^{m-1} \to \mathbb{R}^{n+1}$ a $C^{3,\alpha}$-map s.t. $\Gamma \cap U = \text{graph}(f)$. Then we define the space of boundaries as

$$X_\varepsilon := \{u \in C^{3,\alpha}(\Omega) : f - u \equiv 0 \text{ on } \Omega \setminus \Omega', \|f - u\|_{C^{3,\alpha}} < \varepsilon\}.$$

Claim: The set of $u \in X_\varepsilon$ such that graph$(u) \cap \mathcal{M}$ has empty interior (relative to graph(u)) for any m-dim mfld $\mathcal{M}$ of class $C^{3,\beta}$, $\beta > \alpha$, is residual (wrt $C^{3,\alpha}$-norm).

Uniqueness: Dense implies residual

Lemma

Let $NU := \{b \in X_\varepsilon :$ there are $T^1 \neq T^2$ area-min with boundary $b\}$. Then we can decompose this set as

$$\bigcup_{k\geq 1} \left\{ b \in X_\varepsilon : \exists T^1, T^2 \text{ area-min with boundary } b,\, \mathbb{F}(T^1 - T^2) \geq \frac{1}{k} \right\}$$

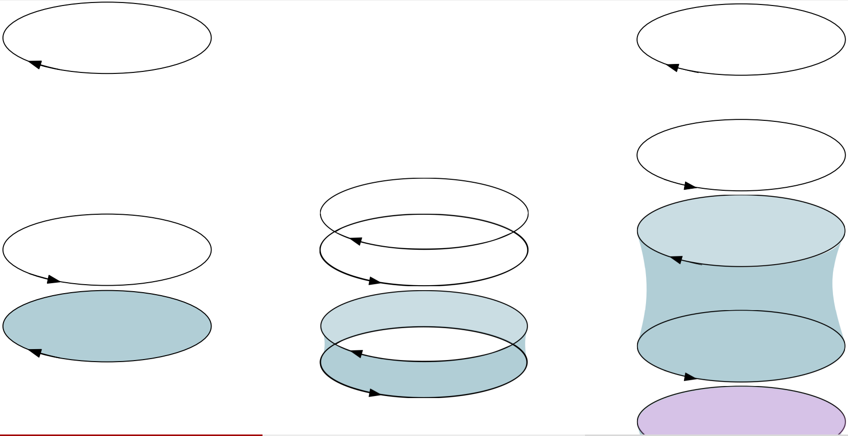
and moreover if $X_\varepsilon \setminus NU$ is dense, then it is residual.

Proof: We know NU_k has empty interior (as NU has), need to show: closed. Observe that $\mathbb{F}(T^1 - T^2) \geq \frac{1}{k}$ is stable under $\mathbb{F}$-convergence and limit of minimizers is minimizing.

Proof Strategy I: One-sided case

Morgan's argument

Fix a minimizer T for Γ. If $p \in \Gamma$ is one-sided, then T is regular at p. Perturb Γ into the interior of T to favorize it. Moreover, if every boundary point is one-sided, the minimizer has multiplicity one.



Proof Strategy II: Two-sided case

Structure Theorem (De Lellis-De Philippis-Hirsch-Massaccesi)

Assume T is area-minimizing with boundary $\Gamma = \sqcup \Gamma_i$. Then $T = \sum_{j=1}^{N} Q_j T_j$ with $Q_j \in \mathbb{N}$, $\partial T_j = \sum_i \pm [\![\Gamma_i]\!]$ and

- either Γ_i is one-sided and all regular points of T at Γ_i have mult one,
- or Γ_i is two-sided, exactly two of the T_j's touch it and all regular points of T at Γ_i have mult $Q_j - \frac{1}{2}$. Moreover the sum of those two T_j's is again area-minimizing (for its boundary).

Assume for $p \in \Gamma$, there is an area-minimizing T that is two-sided and regular at p. Then $S = (T_j + T_{j'}) \llcorner B_r(p)$ is area-minimizing for which $\Gamma \cap B_r(p)$ are regular *interior* points. Thus $\mathcal{M} := \mathrm{supp}(S)$ is a smooth m-dim mfld containing and open set of Γ $\lightning$ (happens only to a meager set of boundaries)

Proof Strategy III: Banach-Mazur Game

Goal: Those $u \in X_\varepsilon = \{u \in C^{3,\alpha} : f - u \equiv 0 \text{ on } \Omega \setminus \Omega', \|f-u\|_{C^{3,\alpha}} < \varepsilon\}$ which lie in a more regular mfld, form a meager set.

Theorem (The Game)

Let X be a complete metric space and $A \subset X$. There are two players P1 and P2:

- P1 chooses $U_1 \subset X$ nonempty and open,
- P2 chooses $V_1 \subset U_1$ nonempty and open,
- P1 chooses $U_2 \subset V_1$ nonempty and open,
- ...

Then there is a strategy for P2 that forces $\bigcap_{i \in \mathbb{N}} U_i \cap A = \emptyset$ iff A is meager in X, i.e. $A = \bigcup_{j \in \mathbb{N}} A_j$ with A_j nowhere dense (i.e. $\overset{\circ}{\overline{A_j}} = \emptyset$).

For us: $X = X_\varepsilon$ is the space of perturbed boundaries,
$A = \{u : \text{graph}(u) \text{ intersects some } \mathcal{M} \in C^{3,\beta} \text{ in a relative open set}\}$.

Game strategy

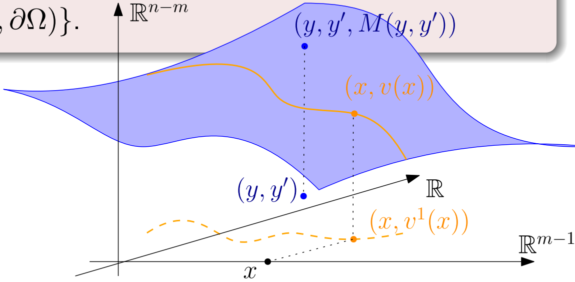
Proposition

Given a boundary $\omega \in X_\varepsilon$, $x \in \Omega$ and $\bar\rho, j > 0$, we find $u \in X_\varepsilon$, $\rho > 0$ s.t.

1. $\mathscr{B}_\rho(u) \subset \mathscr{B}_{\bar\rho}(\omega)$,
2. for every $v \in \mathscr{B}_\rho(u)$, $M \in C^{3,\beta}(\Omega \times \mathbb{R})$ with $\|M\|_{C^{3,\beta}} < j$, the projection misses a point, i.e.

$$\pi_\Omega\big(\text{graph}(M) \cap \text{graph}(v) \cap C_r(x)\big) \neq B_r(x),$$

where $r := \min\{\frac{1}{j}, \text{dist}(x, \partial\Omega)\}$.



Game strategy: proof

1. Assume (by contradiction) that for all $u \in \mathscr{B}_{\bar\rho}(\omega)$ there is a sequence $\rho_i \downarrow 0$, M_i, $v_i \in \mathscr{B}_{\rho_i}(u)$ s.t.
$$\pi_\Omega\big(\mathrm{graph}(M_i) \cap \mathrm{graph}(v_i) \cap C_r(x)\big) = B_r(x).$$
2. Let $\alpha < \gamma < \beta$ and choose
$$u(z) := \varphi_\delta * \omega(z) + (0, \ldots, 0, \delta\eta(z)|z_1 - x_1|^{3+\gamma}),$$
for some η cut-off and φ_δ standard mollifier.
3. Observe $u \in X_\varepsilon$ and for $\delta > 0$ small $u \in B_{\bar\rho}(\omega)$.
4. By (1), we have for all $y \in B_r(x)$
$$\big(y, v_i^1(y), v_i^{2\cdots(n-m+1)}(y)\big) = \big(y, y', M_i(y, y')\big).$$
5. Compare components, Taylor everything, find contradiction to $\alpha < \gamma < \beta$.

Winning the game

1. Choose a dense family $\{x_\ell\} \subset \Omega$.
2. Iteratively choose balls as in the proposition with the additional constraint $\rho_k < \frac{1}{k+1}$.
3. Find $\omega_\infty \in \bigcap_{k\geq 1} \mathscr{B}_{\rho_k}(u_k)$. Then for every k and $M \in C^{3,\beta}$, we have
$$\pi_\Omega\big(\text{graph}(M) \cap \text{graph}(\omega_\infty) \cap C_{1/k}(x_k)\big) \neq B_{1/k}(x_k).$$
4. Assume (by contradiction) that graph(ω_∞) intersects any $C^{3,\beta}$-mfld in a relatively open set $U \subset \Omega$. As $\{x_\ell\} \subset \Omega$ are dense, it contains one of the $B_{1/k}(x_k)$. $\lightning$
5. As the only element in the intersection does not lie in our set A, we conclude that A is meager.

Thank you!