Carleson ε^2 conjecture in higher dimensions

CNA Seminar - Carnegie Mellon University

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Joint work with Xavier Tolsa and Michele Villa

Slides include drawings of Michele Villa

Quantitative rectifiability

Definition:

A set $E \subset \mathbb{R}^m$ is k-rectifiable if there exist Lipschitz functions $f_i : \mathbb{R}^k \to \mathbb{R}^m$ such that

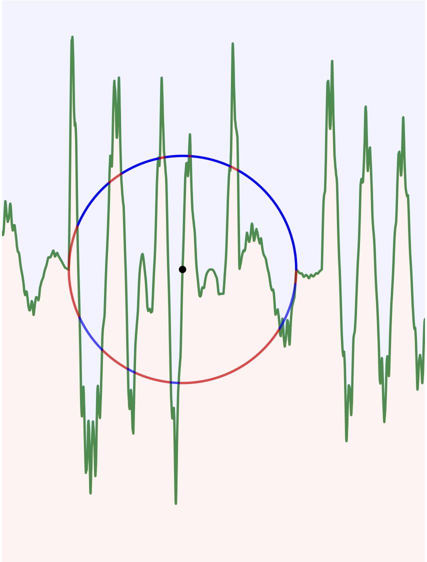
$$\mathcal{H}^k\left(E \setminus \bigcup_{i=1}^{\infty} f_i(\mathbb{R}^k)\right) = 0.$$

$$\beta^k_{2,E}(x,r) := \left(\inf_{P \text{ plane}} \frac{1}{r^{k+2}} \int_{B(x,r)\cap E} \text{dist}(y,P)^2 d\mathcal{H}^k(y)\right)^{1/2}$$

Theorem (Azzam - Tolsa 2015/Edelen-Naber-Valtorta 2016)

$$\int_0^1 \beta^k_{2,E}(x,r)^2 \frac{dr}{r} < \infty \text{ a.e in } E \iff E \text{ is } k \text{ rectifiable.}$$

Carleson ε^2 conjecture

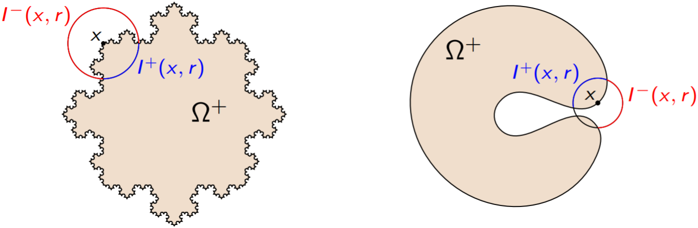


Coefficients $\varepsilon(x, r)$

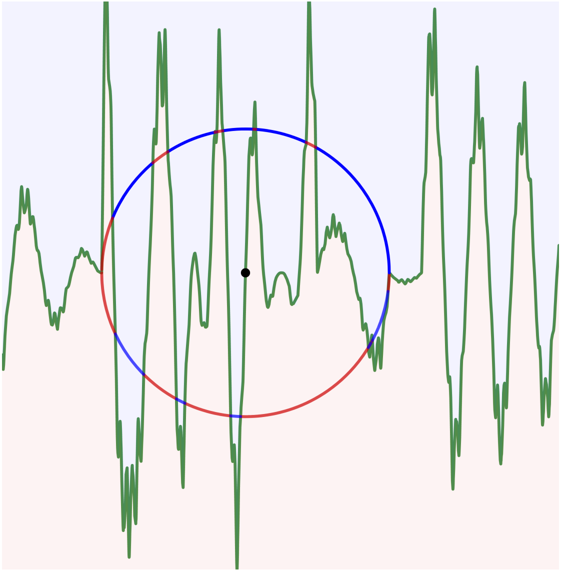
Definition:

Γ Jordan curve, Ω^+ inner domain and Ω^- exterior domain.
$I^{\pm}(x, r)$: connected component of $\partial B(x, r) \cap \Omega^{\pm}$ of greatest length.

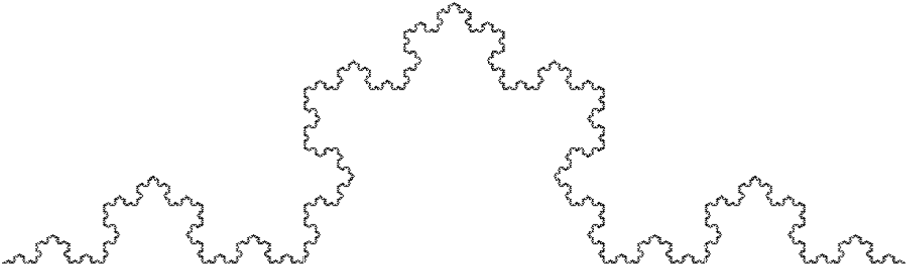
$$\varepsilon(x, r) = \frac{1}{r} \max \left\{ \left| \pi r - \text{length}(I^+(x, r)) \right|, \left| \pi r - \text{length}(I^-(x, r)) \right| \right\}.$$



Coefficients $\varepsilon(x, r)$



Coefficients $\varepsilon(x, r)$



Carleson's conjecture in 2d

Theorem (Jaye - Tolsa - Villa 2019):

Γ Jordan curve. Up to $\mathcal{H}^1$ measure zero

$$\int_0^1 \varepsilon(x,r)^2 \frac{dr}{r} < \infty \iff \Gamma \text{ admits a tangent in } x.$$

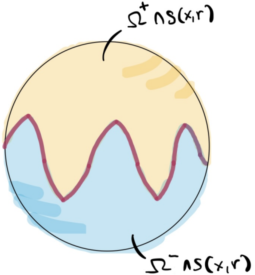
Two problems

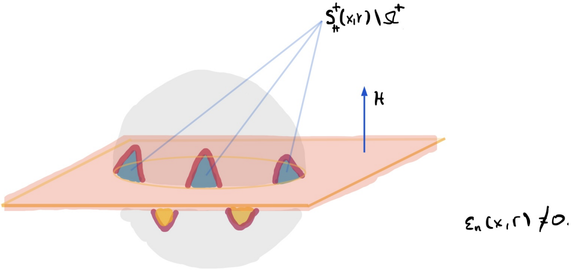
1. What are the right coefficients?
2. What kind of domains should we consider? Can we consider domains split by continuous images of spheres?

We need that if $\varepsilon(x, r) = 0$ then the set agrees with a sphere in $\partial B(x, r)$.

New coefficients ε

Ω^+, Ω^- open subsets of $\mathbb{R}^{n+1}$. $x \in \partial\Omega^+ \cap \partial\Omega^-$. S_H^+ halfspace of the sphere.





Definition:

$$\varepsilon_n(x,r) = \frac{1}{r^n} \inf_{H \text{plane}} \max_{\pm} (\mathrm{Vol}_{\mathbb{S}^n}(S_H^{\pm} \setminus \Omega^{\pm}))$$

Rectifiability

Theorem A (F.- Tolsa - Villa 2023)

The set

$$E := \{x \in \partial\Omega^+ \cap \partial\Omega^- : \int_0^1 \varepsilon_n(x,r)^2 \frac{dr}{r} < \infty\}$$

is n-rectifiable.

Reminder: This means there exist Lipschitz functions $f_i : \mathbb{R}^n \to \mathbb{R}^{n+1}$ such that

$$\mathcal{H}^n\left(E \setminus \bigcup_{i=1}^{\infty} f_i(\mathbb{R}^n)\right) = 0.$$

Observation:

Rectifiability of E is equivalent to E having approximate tangents (with respect to itself). Theorem A does not imply that $\partial\Omega^\pm$ admits a classical tangent plane.

Small coefficient case

At small scales the coefficient $\varepsilon_n(x, r)$ should be typically small in a uniform way. The first step is understanding the following question:

Question

Suppose on a ball B_0 we have:

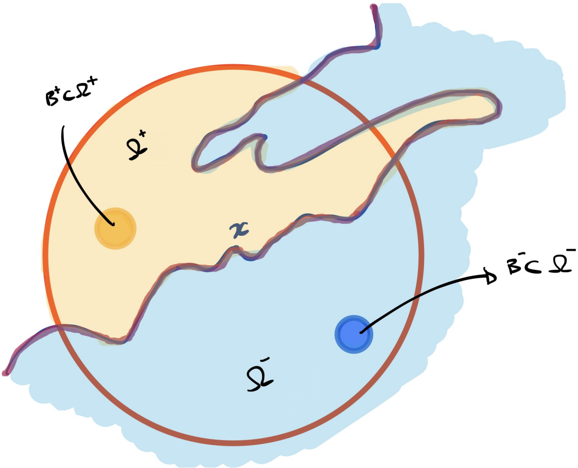
1. Mass on the unit ball: $\mathcal{H}^n(E \cap B_0) \gtrsim r(B_0)^n$
2. Uniformly small coefficients $\int_0^{10r(B_0)} \varepsilon_n(x, r)^2 \frac{dr}{r} < \delta \, \forall x \in E \cap 10B_0$

what can we say about the structure of E and $\Omega^{\pm}$ on B_0?

This step is not only natural, but also useful. We can then paste this local information and improve it.

Corkscrews

We say that B_0 admits corkscrew balls if there are balls $B^{\pm} \subseteq \Omega^{\pm} \cap B_0$ with $\text{rad}(B^{\pm}) \simeq \text{rad}(B_0)$.



Almost corkscrews

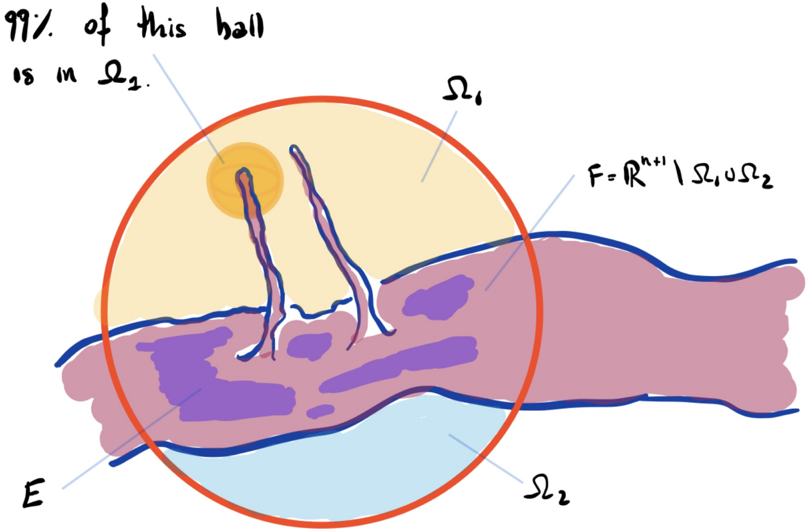
We can't exactly get corkscrew balls but we can get almost corkscrew balls.

Lemma

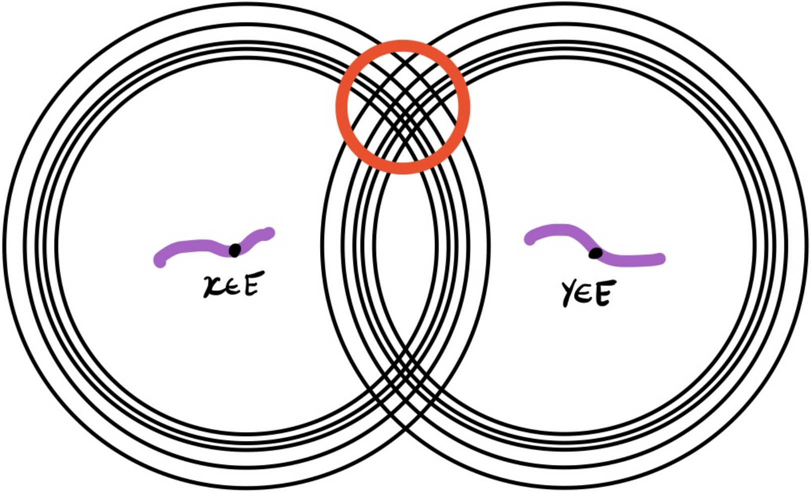
Suppose $|x-y| = R$ and $\int_0^{8R} \varepsilon_n(x,r)^2 + \varepsilon_n(y,r)^2 \frac{dr}{r} < \delta$ then there exist $B^{\pm}$ almost corkscrew balls:

1. $B^{\pm} \subseteq B(x, 16R)$ y $\mathrm{rad}(B^{\pm}) \simeq R$.
2. $$\mathrm{Vol}^{n+1}(B^{\pm} \setminus \Omega^{\pm}) \lesssim \delta^{1/4} R^{n+1}.$$

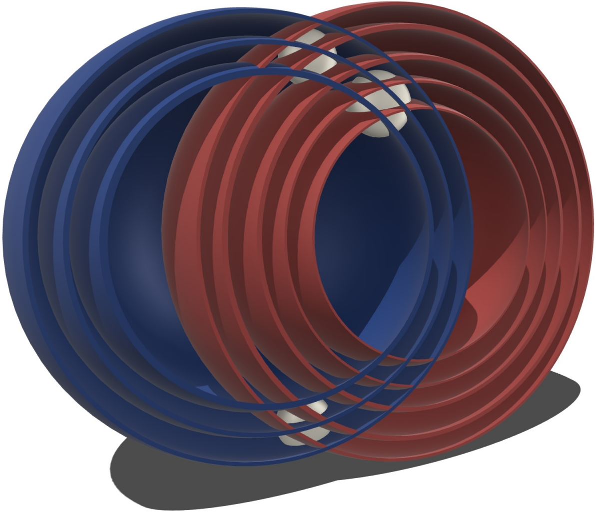
Almost corkscrew



Almost corkscrew - sketch

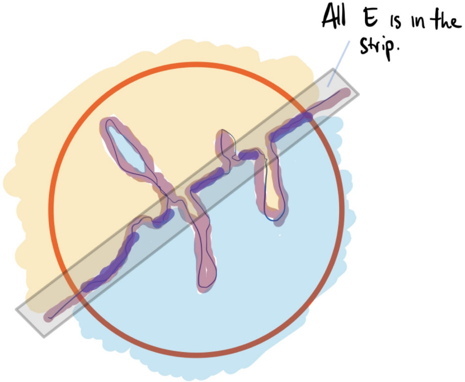


Almost corkscrew



Flatness of E

Under the small coefficient hypothesis:



This means, if δ is small enough then there exists some plane P such that

$$\sup_{E \cap B_0} \text{dist}(x, E) < \varepsilon r(B_0).$$

Flatness sketch

The proof goes by contradiction. One can smoothen the coefficients ε_n to the coefficients

$$\alpha^{\pm}(x,r) = \left| \frac{1}{r^{n+1}} \int_{\Omega^{\pm}} e^{-\frac{|x-y|^2}{r^2}} dy - \frac{\pi^{\frac{n+1}{2}}}{2} \right|$$

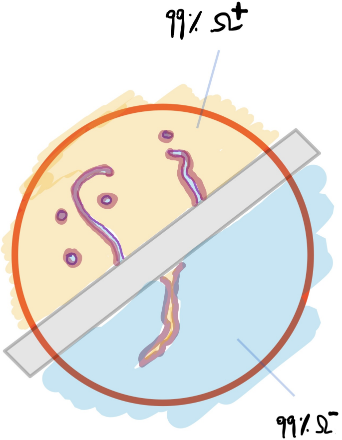
Notice $\frac{\pi^{\frac{n+1}{2}}}{2}$ is the number obtained if Ω^+ was a half-space.

1. The coefficients ε_n control the coefficients α.
2. In particular in the blowup limit $\alpha^{\pm}(x,r) = 0 \; \forall x \in E$, $\forall r \in (0,8)$.
3. α are real analytic $E \subseteq Z$ then Z is a real analytic variety.
4. Z is a plane. Contradiction! This step uses $\exists$ almost corkscrew.

Domain splitting

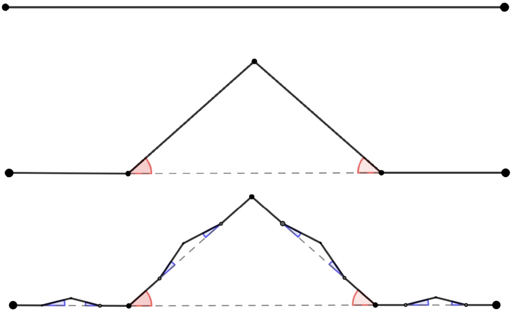
In this step we improved the almost-corkscrew condition using the new information.

For many centers and scales, the best plane for ε_n is (up to a small error) the plane we obtained on the previous step.



An example

Variant of the Koch Snowflake with angles α_n.



What we proved is morally that $\alpha_n \to 0$. The limit E is a Lipschitz graph if and only if

$$\sum \alpha_n^2 < \infty.$$

An example

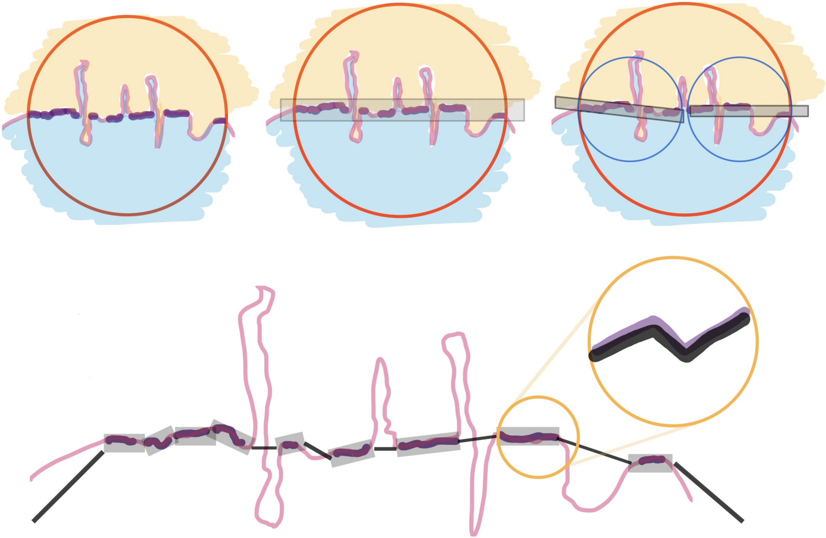
What we proved is morally that $\alpha_n \to 0$. The limit E is a Lipschitz graph if and only if

$$\sum \alpha_n^2 < \infty.$$

Moreover when $\sum \alpha_n^2 = \infty$, E intersects every Lipschitz graph Γ (or curve of finite length) on a measure zero set. This is

$$\mathcal{H}^1(E \cap \Gamma) = 0 \, \forall \Gamma \text{ Lipschitz graph}$$

Construction of a Lipschitz graph



Rectifiability

Theorem A (F.- Tolsa - Villa 2023)

The set

$$E := \{x \in \partial\Omega^+ \cap \partial\Omega^- : \int_0^1 \varepsilon_n^2(x,r)\frac{dr}{r} < \infty\}$$

is n-rectifiable. Moreover $\mathcal{H}^n$ almost every point on E, $\Omega^\pm$ admit a tangent in L^1:

$$\frac{\Omega^\pm - x}{r} \xrightarrow{L^1_{loc}} H^\pm$$

$H^\pm$ are the halfspaces determined by a plane.

Finer coefficients

Definition:

$$\varepsilon_{n-1}(x,r) = \frac{1}{r^n} \inf_{H \text{plane}} \max_{\pm} \left(\int_{\partial\Omega^{\pm} \cap \partial B(x,r)} \mathsf{dist}(y,H) d\mathcal{H}^{n-1}_{\infty}(y) \right)$$

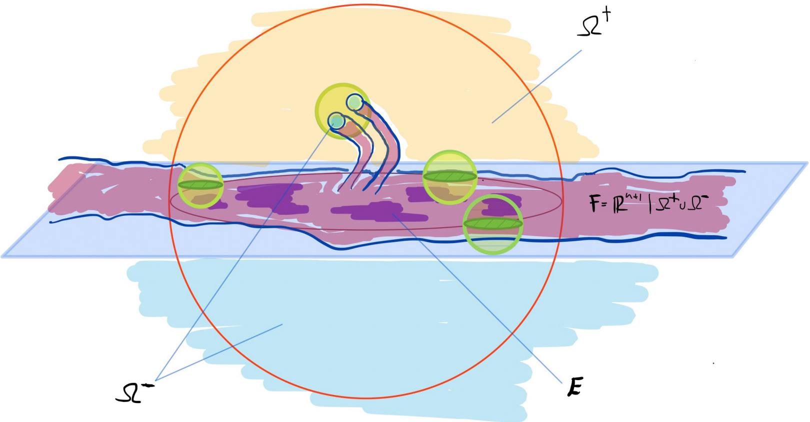
We note that the integral with respect to Hausdorff content *is not a measure*.

Definition of Hausdorff content and Choquet integral:

$$\mathcal{H}^s_{\infty}(E) = \inf\{\sum \mathsf{diam}(A_i)^s : E \subseteq \cup A_i\}$$

$$\int |f|^p d\mathcal{H}^s_{\infty} := \int_0^{\infty} \mathcal{H}^s_{\infty}(E : |f(x)| > t) t^{p-1} dt$$

True tangents - sketch



Its enough to know that thet set has mas everywhere (which is detected by the Hausdorff content). Not only on Euclidean space but also on spheres!

Slicing

Question

If E is compact in $B(0,1)$, when is the following true?

$$\mathcal{H}^n_\infty(E) \simeq \int_0^1 \mathcal{H}^{n-1}_\infty(E \cap \partial B(x,r))dt \, \forall x \in E$$

Theorem B and Slicing

Definition:

$$\varepsilon_s(x,r) = \frac{1}{r^n} \inf_{H\text{plane}} \max_{\pm} \left(\int_{\partial\Omega^{\pm}\cap\partial B(x,r)} \text{dist}(y,H)^{n-s} d\mathcal{H}^s_\infty(y) \right)$$

Theorem B (F., Tolsa, Villa 2023)

Assume $\Omega^{\pm}$ have the s-capacity density condition $s \in (n, n+1)$. Up to measure zero

$$\int_0^1 \varepsilon_s(x,r)^2 + \varepsilon_n(x,r)^2 \frac{dr}{r} < \infty \iff \partial\Omega^{\pm} \text{ admits a tangent at } x.$$

Ω^+ satisfies the capacity density condition (CDC) if $\exists t > s$ such that for every ball centered on $\partial\Omega^+$

$$\mathcal{H}^t_\infty(\partial\Omega^+ \cap B) \gtrsim r(B)^s$$

Quantitative stability of Faber-Krahn

Let $\lambda_1(\Omega)$ be the first eigenvalue of the Laplacian.

Theorem (Brasco-De Phillipis- Velichkov 2013)

Assume $|\Omega| = |B|$, with B the unit ball. There exists a translation of B such that:

$$\lambda_1(\Omega) - \lambda_1(B) \gtrsim |\Omega \Delta B|^2.$$

Theorem (Allen-Kriventsov-Neumayer 2021)

Assume that $|\Omega| = |B|$, with B the unit ball. Assume B has the same barycenter as Ω. We take u_Ω, u_B first eigenfunctions of the Laplacian extended by zero:

$$\lambda_1(\Omega) - \lambda_1(B) \gtrsim |\Omega \Delta B|^2 + \int |u_\Omega - u_B|^2.$$

Theorem (F.- Tolsa - Villa 2023)

Assume that $|\Omega| = |B|$, with B the unit ball and Ω has the s-capacity density condition. If B has the same barycenter as Ω, then:

$$\lambda_1(\Omega) - \lambda_1(B) \gtrsim |\Omega \Delta B|^2 + \int_{\partial\Omega \cap B} \mathsf{dist}(x, \partial B)^{n-s} d\mathcal{H}^s_\infty.$$

Relationship with Carleson ε^2 conjecture

If $\Sigma \subseteq \mathbf{S}^n$, first eigenfunction of the Laplacian u_Σ has homogeneous harmonic extension, whose degree we call $\alpha(\Sigma)$.

$$\lambda_1 = \alpha\,(\alpha + n - 1)$$

Alt-Cafarelli-Friedman monotonicity formula

Given two positive harmonic functions $u^\pm$ with zero boundary value over disjoint domains $\Omega^\pm \subseteq \mathbb{R}^{n+1}$ for every $x \in \partial\Omega^+ \cap \partial\Omega^-$

$$J(x,r) := \left(\frac{1}{r^2}\int_{B(x,r)\cap\Omega^+} \frac{|\nabla u^+(y)|^2}{|y-x|^{n-1}}dy\right)\cdot\left(\frac{1}{r^2}\int_{B(x,r)\cap\Omega^-} \frac{|\nabla u^-(y)|^2}{|y-x|^{n-1}}dy\right)$$

The function $J(x,r)$ is monotone and it satisfies:

$$\frac{\partial_r J(x,r)}{J(x,r)} \geq \frac{2}{r}\left(\alpha^+(x,r) + \alpha^-(x,r) - 2\right) \geq 0$$

where $\alpha^\pm(x,r)$ are the α for $\Omega^\pm \cap \partial B(x,r)$

Source of inspiration

Another result of Allen-Kriventsov-Neumayer:

Theorem (Allen-Kriventsov-Neumayer 2022)

We take sets $\Omega^\pm$ and two sub-harmonic functions $u^\pm$. The set

$$E := \{x : \lim_{r\to 0} J(x,r) > 0\}$$

is n-rectifiable. The blowups for almost every point in E are unique and linear.

Theorem B recharged

Theorem (F. - Tolsa - Villa 2023)

Given domains $\Omega^{\pm}$ with the Capacity density condition we have:

$$\varepsilon_s(x,r)^2 \lesssim \alpha^+(x,r) + \alpha^-(x,r) - 2$$

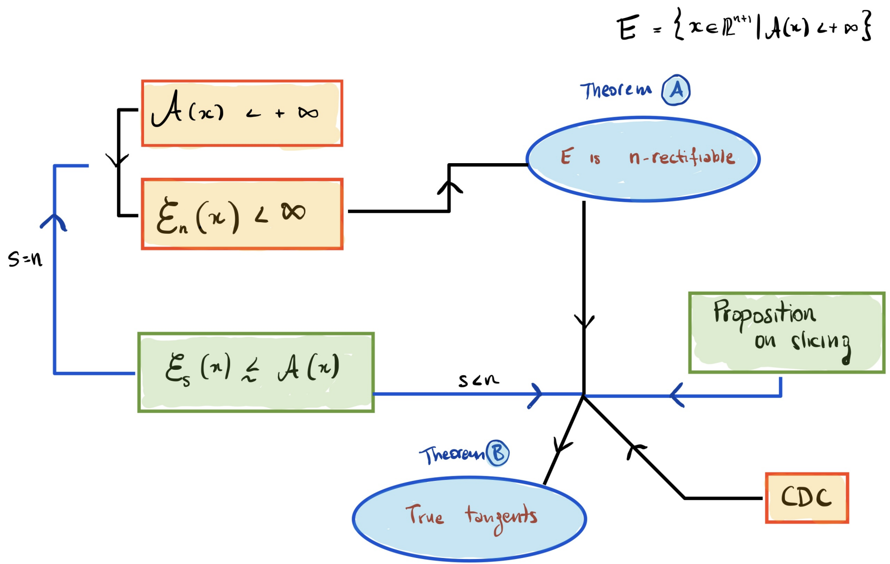
.

Theorem B Recharged (F.- Tolsa - Villa 2023)

Assume that $\Omega^{\pm}$ have the n-capacity density condition. Up to $\mathcal{H}^n$ measure zero $\partial\Omega^{\pm}$ admits a tangent at x if and only if

$$\int_0^1 \min\left(1, \alpha^+(x,r) + \alpha^-(x,r) - 2\right) \frac{dr}{r} < \infty$$

Review



E = {x ∈ ℝ^{n+1} | A(x) < +∞}
Theorem A
A(x) < +∞
E is n-rectifiable
E_n(x) < ∞
S=n
Proposition on slicing
E_s(x) ≲ A(x)
s<n
Theorem B
True tangents
CDC

Thank you for coming!

Some ideas of the proof

The extension u_ω

$$\widetilde{u}_\Omega(x,t) = u_\Omega(x)e^{-\sqrt{\lambda_1(\Omega)}t}$$

is harmonic in the cylinder $\widetilde{\Omega} := \Omega \times [0,1]$.

- The proof uses the maximum principle for harmonic functions at the level of integrals with respect to harmonic measure at the cylinder $\widetilde{\Omega}$.
- The dominant term is controlled by an integral over a nicer domain with Lipschitz interior. Errors are controlled either by $\lambda_1(\Omega) - \lambda_1(B)$ or by $\int_{\Omega\cap B} |u_\Omega - u_B|^2$ (which is enough by AKN).
- Over nice Lipschitz domains harmonic measure behaves like surface measure.

Refences

- Fleschler, Tolsa, Villa: Faber-Krahn inequalities, the Alt-Caffarelli-Friedman formula, and Carleson's ε^2 conjecture in higher dimensions, preprint 2023.
- Fleschler, Tolsa, Villa: Carleson's ε^2 conjecture in higher dimensions, preprint 2023.