Equilibrium configurations for epitaxially strained crystalline films and material voids

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A general framework

Stress Driven Rearrangement Instabilities (SDRI): morphological instabilities of interfaces between elastic phases generated by the competition between elastic and (hyper)surface energies of perimeter type

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- capillary droplets;
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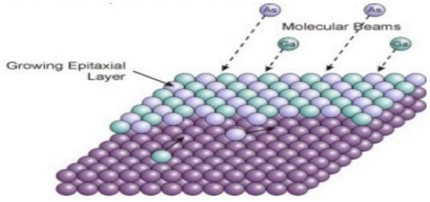
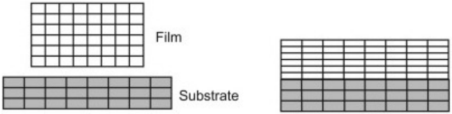
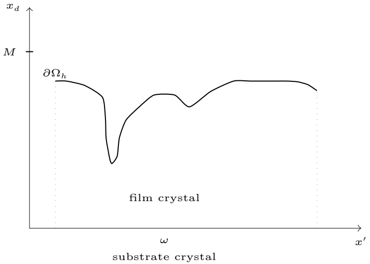
Here: minimization and approximation for static energies in any space dimension, in the context of linearized elasticity. Available results either in 2d for 1d 'discontinuity curves' with a bounded number of connected components or in simplified settings for elastic energy.

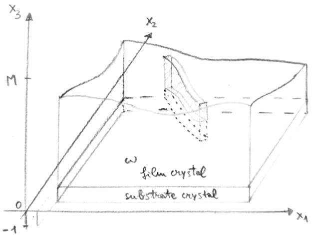
Epitaxially strained crystalline films





Etero-epitaxy: different lattice parameters between substrate and film crystals





Competition between elastic bulk energy and surface tension energy

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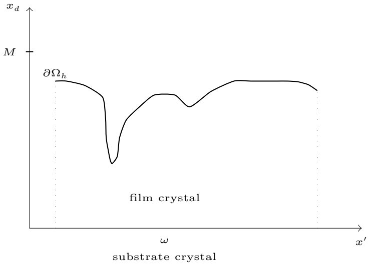
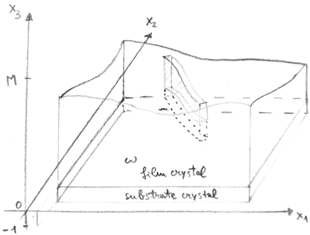




Competition between elastic bulk energy and surface tension energy

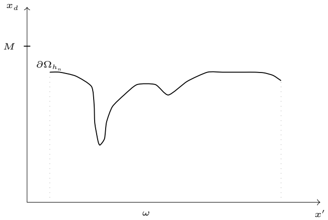
Bonnetier-Chambolle '02:

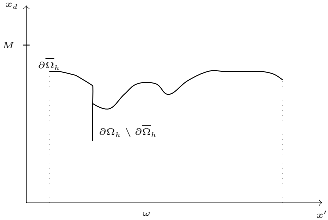
$$G(u,h) = \int_{\Omega_h^+} \mathbb{C}\, e(u) : e(u)\, \mathrm{d}x + \int_\omega \sqrt{1 + |\nabla h(x_1,x_2)|^2}\, \mathrm{d}(x_1,x_2)\,.$$

where $e(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ and

$$h \in C^1(\omega;[0,M]),\ \|h\|_1 = m,\ \omega \subset \mathbb{R}^2,\ \Omega_h^+ := \{x \in \omega \times \mathbb{R} : 0 < x_3 < h(x_1,x_2)\},$$

$$u|_{\Omega_h^+} \in H^1(\Omega_h^+),\ \ u = 0 \text{ in } (\omega\times(0,M+1)) \setminus \Omega_h^+,\ \ u = u_0 \text{ in } \omega\times(0,1)$$





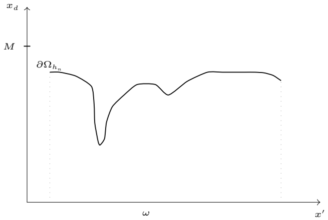
In 2D (so x_1 instead of (x_1, x_2) in G): for $\Omega = \omega \times (-1, M+1)$

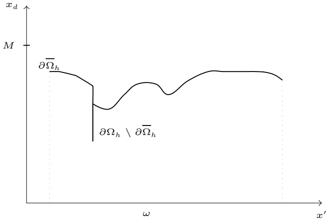
$\Omega \setminus \Omega_{h_n} \xrightarrow{\mathcal{H}} \Omega \setminus \Omega_h$ for some $h\colon \omega \to \mathbb{R}^+$ lsc

$\partial\Omega_{h_n} \xrightarrow{\mathcal{H}} K \supset \partial\Omega_h = \partial\overline{\Omega}_h \cup (\partial\Omega_h \setminus \partial\overline{\Omega}_h)$

Gołab Theorem (since $\partial\Omega_{h_n}$ connected in 1d): $\mathcal{H}^1(K) \leq \liminf_n \mathcal{H}^1(\partial\Omega_{h_n})$

Equilibrium configurations for epitaxially stained crystalline films in 2D





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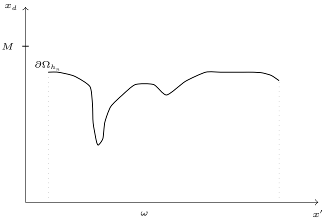

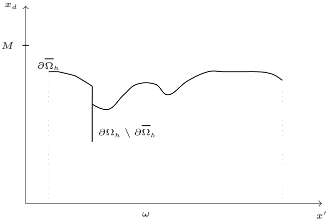

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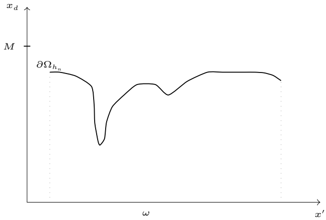
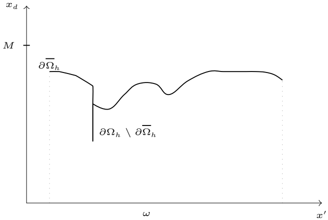
$e(u_n) \rightharpoonup \mathcal{E}$ in L^2 (up to a subsequence) +
Korn Inequality in any $A \subset \Omega_h$ Lipschitz $\Longrightarrow u_n \rightharpoonup u$ in $H^1_{\mathrm{loc}}(\Omega_h; \mathbb{R}^2)$ and
$e(u) = \mathcal{E}$ (in particular u is independent of further subsequences)





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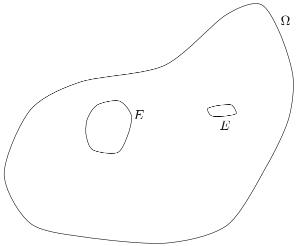
Therefore COMPACTNESS for (u_n, h_n) with $G(u_n, h_n) \leq M$ and

$$\widetilde{G}(u,h) := \int_{\Omega_h^+} \mathbb{C}\, e(u) : e(u)\, \mathrm{d}x + \mathcal{H}^1(\partial\overline{\Omega}_h) + 2\mathcal{H}^1(\partial\Omega_h \setminus \partial\overline{\Omega}_h) \leq \liminf_n G(u_n, h_n)$$

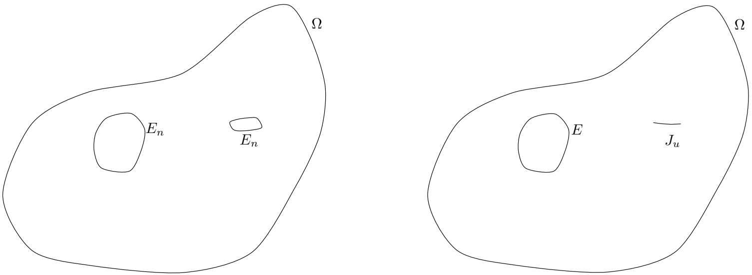


Braides-Chambolle-Solci '07:

$$F(u, E) = \int_{\Omega \setminus E} \mathbb{C}\, e(u) : e(u)\, \mathrm{d}x + \int_{\Omega \cap \partial E} \varphi(\nu_E)\, \mathrm{d}\mathcal{H}^{d-1}\,.$$

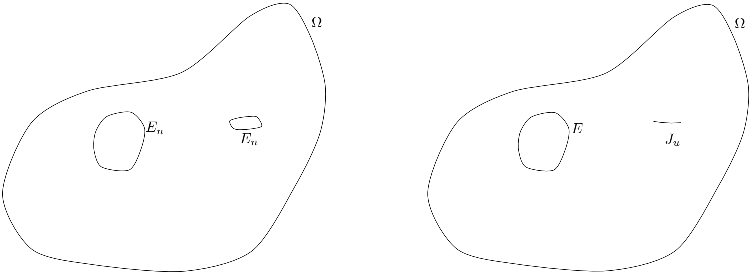
where φ is a norm and represents a possibly anisotropic density.
Minimization under Dirichlet b.c. on $\partial_D \Omega \subset \partial\Omega$ and a volume constraint on E.

Equilibrium configurations for material voids



In 2D: assuming an equibounded number of connected components for voids, one may still use Gołab Theorem to say

$$\widetilde{F}(u,E) := \int_{\Omega\setminus E} \mathbb{C}\, e(u) : e(u)\,\mathrm{d}x + \int_{\Omega\cap\partial E} \varphi(\nu_E)\,\mathrm{d}\mathcal{H}^1 + \int_{J_u\cap\Omega\setminus E} 2\varphi(\nu_u)\mathrm{d}\mathcal{H}^1 \leq \liminf_n F(u_n, E_n)$$



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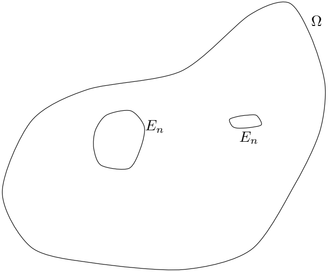
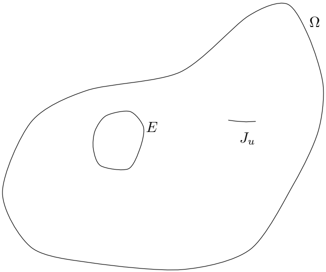
Without this restriction, one could even have $\partial E_n \xrightarrow{\mathcal{H}} \Omega$!





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The situation may be reproduced for connected surfaces (like graphs) in 3D

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- Fonseca, Fusco, Leoni, Morini '07–'15 / Chambolle, Larsen / Goldman, Zwicknagl, Bella '14-'15 / Davoli-Piovano '18: regularity for equilibria for epitaxially strained films (Finite number of vertical points and cuts; analytic profile outside, for $x_2 > 0$; study of wetting);

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Relaxation of functionals F and G

We look for the biggest lower semicontinuous functionals $\overline{F}$ and $\overline{G}$ with $\overline{F} \leq F$ and $\overline{G} \leq G$, wrt convergence in measure for u and in L^1 for χ_E or h.

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To prove that

- if $u_n \to u$ a.e. in Ω, $\chi_{E_n} \to \chi_E$ in $L^1(\Omega)$ then

$$\overline{F}(u, E) \leq \liminf_{n\to\infty} F(u_n, E_n)\,;$$

- for all (u, E) with $\overline{F}(u, E) < +\infty$, there are $(u_n, E_n) \to (u, E)$ with

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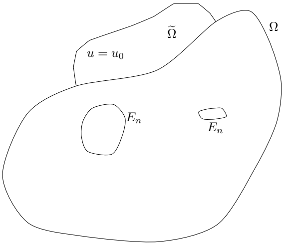
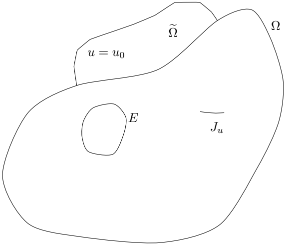
Bon-Cha '02: $d = 2$ for G; Cha-Sol '07: any d but ∇u in place of $e(u)$ for G;
Bra-Cha-Sol '07: any d but ∇u in place of $e(u)$ for F.

Form of $\overline{F}$ and minimization





Theorem (C.-Friedrich)

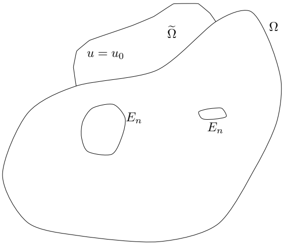
$$F_{\mathrm{Dir}}(u, E) = \int_{\Omega \setminus E} \mathbb{C}\, e(u) : e(u)\, \mathrm{d}x + \int_{(\Omega \cup \partial_D \Omega) \cap \partial E} \varphi(\nu_E)\, \mathrm{d}\mathcal{H}^{d-1}$$

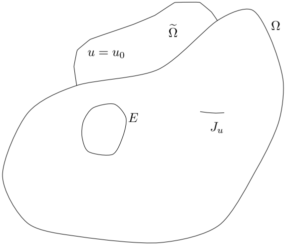
if E Lipschitz, $|E| = m \in (0, |\Omega|)$, $u|_{\Omega \setminus \overline{E}} \in H^1(\Omega \setminus E; \mathbb{R}^d)$, $u|_E = 0$, $u|_{\widetilde{\Omega} \setminus \Omega} = u_0$

relaxes into

$$\overline{F}_{\mathrm{Dir}}(u, E) = \int_{\Omega \setminus E} \mathbb{C}\, e(u) : e(u)\, \mathrm{d}x + \int_{(\Omega \cup \partial_D \Omega) \cap \partial^* E} \varphi(\nu_E)\, \mathrm{d}\mathcal{H}^{d-1} + \int_{J_{u'} \cap (\Omega \cup \partial_D \Omega) \setminus E^1} 2\varphi(\nu_u) \mathrm{d}\mathcal{H}^{d-1}$$

if $|E| = m$, $\mathcal{H}^{d-1}(\partial^* E) < +\infty$, $u = u\chi_{E_0}$, for $u' := u_0$ on $\widetilde{\Omega} \setminus \Omega$.





Theorem (C.-Friedrich)

$$F_{\mathrm{Dir}}(u, E) = \int_{\Omega \setminus E} \mathbb{C}\, e(u) : e(u)\, \mathrm{d}x + \int_{(\Omega \cup \partial_D \Omega) \cap \partial E} \varphi(\nu_E)\, \mathrm{d}\mathcal{H}^{d-1}$$

if E Lipschitz, $|E| = m \in (0, |\Omega|)$, $u|_{\Omega \setminus \overline{E}} \in H^1(\Omega \setminus E; \mathbb{R}^d)$, $u|_E = 0$, $u|_{\widetilde{\Omega} \setminus \Omega} = u_0$

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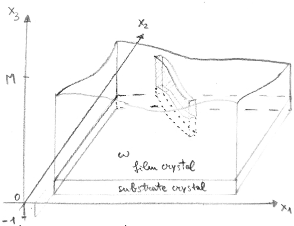
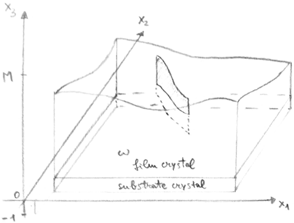
Moreover, $\overline{F}_{\mathrm{Dir}}$ admits minimizers

Form of $\overline{G}$ and compactness





$\Omega = \omega \times (-1, M+1)$

Theorem (C.-Friedrich)

$$G(u,h) = \int_{\Omega_h^+} \mathbb{C}\, e(u) : e(u)\, \mathrm{d}x + \int_\omega \sqrt{1 + |\nabla h(x')|^2}\, \mathrm{d}x'$$

$h \in C^1_m(\omega;[0,M])$, $u|_{\Omega_h^+} \in H^1(\Omega_h^+)$, $u=0$ in $(\omega \times \mathbb{R}^+)\setminus\Omega_h^+$, $u=u_0$ in $\omega \times (0,1)$

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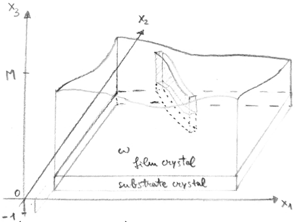
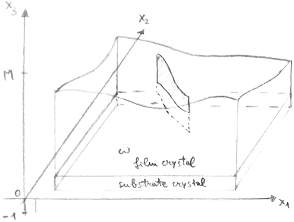
$$\overline{G}(u,h) = \int_{\Omega_h^+} \mathbb{C}\, e(u) : e(u)\, \mathrm{d}x + \mathcal{H}^{d-1}(\partial^*\Omega_h \cap \Omega) + 2\mathcal{H}^{d-1}(J'_u \cap \Omega_h^1)$$

if $u = u\chi_{\Omega_h}$, $u = u_0$ in $\omega \times (-1,0)$, $h \in BV_m(\omega;[0,M])$, $J'_u := J_u + \mathbb{R}^+ e_d$,

Form of $\overline{G}$ and compactness





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Moreover, any (u_n, h_n) with $\overline{G}(u_n, h_n) < M$ has a converging subsequence

The functional framework

E set of finite perimeter ($\chi_E \in BV(\Omega)$)
$E^s = \{x \in \mathbb{R}^d \colon \lim_{\varrho \to 0} \frac{|E \cap B_\varrho(x)|}{|B_\varrho(x)|} = s\}$
$\partial^* E = \mathbb{R}^d \setminus (E^0 \cup E^1)$ essential boundary

The functional framework

$$\overline{F}(u, E),\, \overline{G}(u, h) \sim \int_{\Omega} \mathbb{C}e(u) : e(u)\,\mathrm{d}x + \mathcal{H}^{d-1}(J_u) =: \mathrm{Gr}(u)$$

(recall $u = u\chi_{E_0}$ or $u = u\chi_{\Omega_h}$)

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- Control of $e(u) \rightsquigarrow$

  $BD(\Omega) := \{u \in L^1(\Omega;\mathbb{R}^d)\colon\ \mathrm{E}u \in \mathcal{M}_b(\Omega;\mathbb{M}^{d\times d}_{sym})\} = \{u\colon \sup_{|\varphi|\le 1} \int_\Omega u\cdot \mathrm{E}\varphi < \infty\}$

  $$u \in BD(\Omega) \Longrightarrow \mathrm{E}u = e(u)\mathcal{L}^d + (u^+ - u^-)\odot \nu_u \mathcal{H}^{d-1}\lfloor J_u + \mathrm{E}^c u$$

  $SBD^2(\Omega) := \{u \in BD(\Omega)\colon\ \mathrm{E}^c u = 0,\ e(u) \in L^2,\ \mathcal{H}^{d-1}(J_u) < \infty\}$

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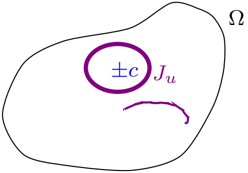
- But no L^1 control of $u \rightsquigarrow GSBD(\Omega)$ Dal Maso '12
  ($u \in GSBV$ if truncations in SBV, now they destroy control on $\mathrm{E}u$)



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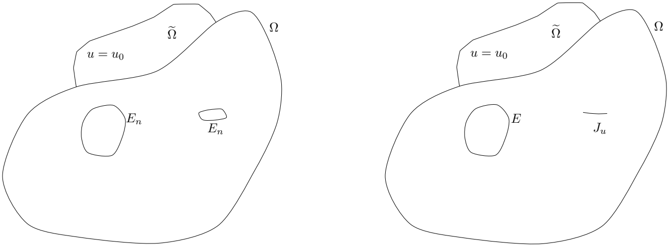
  J_u still $(d{-}1)$-countably rectifiable and for $\mathcal{L}^d$-a.e. x, $e(u)(x)$ s.t.

  $$\operatorname*{ap\,lim}_{y\to x} \frac{\big(u(y) - u(x) - e(u)(x)(y-x)\big)\cdot(y-x)}{|y-x|^2} = 0\,,$$

Relaxation for F



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Proven similarly to Ambrosio-Tortorelli approximation for Griffith with Dirichlet b.c. in Chambolle-C. '19

- $\liminf$ by slicing argument (separately for bulk and surface part)
- $\limsup$ by an adaptation of density of Chambolle-C. '19 + BraChaSol'07

Density in $GSBD$ (Chambolle-C. '19)

$$\forall u \text{ t.q. } \mathrm{Gr}_{\mathrm{Dir}}(u)<\infty \quad \exists u_k \text{ t.q. } J_{u_k} \text{ of class } C^1,\ u_k\in C^\infty(\Omega\setminus J_{u_k}),\text{ and}$$

$$u_k\longrightarrow u \text{ a.e.},\qquad \mathrm{Gr}_{\mathrm{Dir}}(u_k)\longrightarrow\mathrm{Gr}_{\mathrm{Dir}}(u)$$

Compactness for F

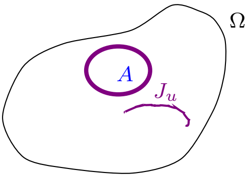
Compactness (Chambolle-C.)

Let $u_n \in GSBD^2(\Omega)$ with

$$\int_\Omega |e(u_n)|^2 \, \mathrm{d}x + \mathcal{H}^{d-1}(J_{u_n}) < M\,.$$

Then, up to a subsequence, for $A := \{x \colon |u_n(x)| \to \infty\}$, there is $u \in GSBD^2(\Omega)$, $u = 0$ in A s.t.

$$\begin{aligned} u_n &\to u && \text{a.e. in } \Omega \setminus A\,,\\ e(u_n) &\rightharpoonup e(u) && \text{in } L^2(\Omega \setminus A; \mathbb{M}^{d\times d}_{sym})\,,\\ \mathcal{H}^{d-1}(J_u \cup \partial^* A) &\le \liminf_{n\to\infty} \mathcal{H}^{d-1}(J_{u_n})\,. \end{aligned}$$



Relaxation for G

Much more difficult both in $\liminf$ and in $\limsup$, due to constraints on the jump set
To prove the verticality we use a variant of the

σ^2-convergence (Dal Maso-Francfort-Toader '05, Giacomini-Ponsiglione '06)

$\Gamma_n \subset U$, $\sup_n \mathcal{H}^{d-1}(\Gamma_n) < \infty$. Then $\Gamma_n \xrightarrow{\sigma^2} \Gamma$ iff

(i) $\Big((v_n)_n \subset SBV^2(U),\ J_{v_n} \subset \Gamma_n,\ v_n \to v\Big) \Longrightarrow J_v \subset \Gamma$

(ii) $\exists v$, $(v_n)_n \subset SBV^2(U)$ with $v_n \to v$, $J_{v_n} \subset \Gamma_n$, $J_v = \Gamma$

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We introduce an analogous notion of sets convergence for the "Griffith energy space" $GSBD^2$ in place of SBV^2, that we call σ^2_{sym}-convergence.
New technical point: consider also a limit set G_∞ with finite perimeter and $\partial^* G_\infty \subset \Gamma$, inside which we do not control anything

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σ^2_{sym}-convergence (C.-Friedrich)

$\Gamma_n \subset U$, $\sup_n \mathcal{H}^{d-1}(\Gamma_n) < \infty$, $U \subset U'$, $|U' \setminus U| > 0$. Then $\Gamma_n \xrightarrow{\sigma^2_{\mathrm{sym}}} (\Gamma, G_\infty)$ with

$$\partial^* G_\infty \cap U' \subset \Gamma, \quad \Gamma \cap (G_\infty)^1 = \emptyset \quad \text{iff}$$

(i) $\Big((v_n)_n \subset GSBD^2(U'),\ J_{v_n} \subset \Gamma_n,\ v_n = 0 \text{ in } U' \setminus U,\ v_n \to v\Big)$
$\Longrightarrow (J_v \setminus \Gamma) \subset (G_\infty)^1$

(ii) $\exists v$, $(v_n)_n \subset GSBD^2(U')$ with $v_n \to v$, $J_{v_n} \subset \Gamma_n$, $v_n = 0$ in $U' \setminus U$, and $J_v = \Gamma$, $\{v = \infty\} \subset G_\infty$

Compactness for σ^2_{sym}-convergence

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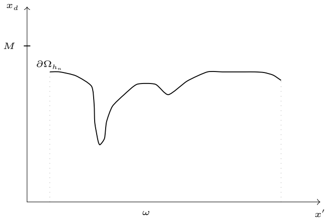
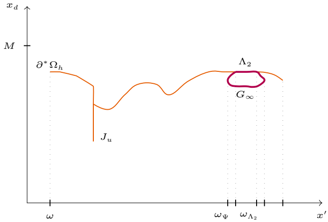
Compactness (C.-Friedrich)

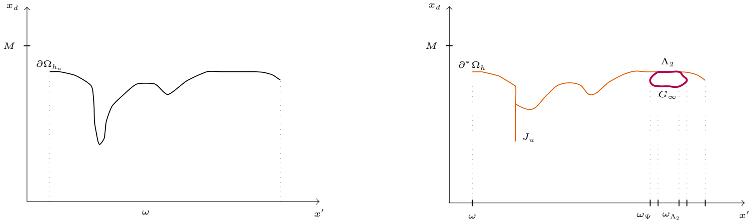
Every $(\Gamma_n)_n$ with $\sup_n \mathcal{H}^{d-1}(\Gamma_n) < \infty$, σ^2_{sym} converges, up to a subsequence, to (Γ, G_∞) with

$$\mathcal{H}^{d-1}(\Gamma) \leq \liminf_{n\to\infty} \mathcal{H}^{d-1}(\Gamma_n)\,.$$





1st step: $\partial\Omega_{h_n} \xrightarrow{\sigma^2_{\text{sym}}} (\Gamma, \emptyset)$

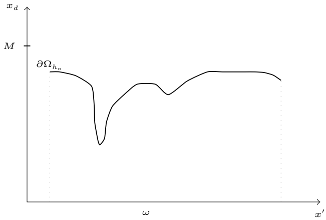


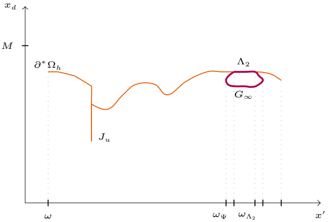
1st step: $\partial\Omega_{h_n} \xrightarrow{\sigma^2_{\rm sym}} (\Gamma, \emptyset)$

$U' := \omega\times(-1, M+1)$, $U := \omega\times(-\frac{1}{2}, M)$, $\Gamma_n := \partial^*\Omega_{h_n} \cap U' \xrightarrow{\sigma^2_{\rm sym}} \Gamma$, $U \prec \psi \prec U'$

1 Area Formula: $\int_\omega \#(\Lambda_y^{e_d})\,\mathrm{d}\mathcal{H}^{d-1}(y) = \int_\Lambda |\nu_\Lambda \cdot e_d|\,\mathrm{d}\mathcal{H}^{d-1} =: \mathrm{I}_V(\Lambda)$

Lower limit and compactness for G





1st step: $\partial\Omega_{h_n} \xrightarrow{\sigma^2_{\mathrm{sym}}} (\Gamma, \emptyset)$

$U' := \omega\times(-1, M+1)$, $U := \omega\times(-\frac{1}{2}, M)$, $\Gamma_n := \partial^*\Omega_{h_n} \cap U' \xrightarrow{\sigma^2_{\mathrm{sym}}} \Gamma$, $U \prec \psi \prec U'$

1. Area Formula: $\int_\omega \#(\Lambda^{e_d}_y)\,\mathrm{d}\mathcal{H}^{d-1}(y) = \int_\Lambda |\nu_\Lambda \cdot e_d|\,\mathrm{d}\mathcal{H}^{d-1} =: \mathrm{I}_V(\Lambda)$
2. $\mathrm{I}_V(\Gamma) = \mathrm{I}_V(J_v) \leq \liminf_n \mathrm{I}_V(J_{v_n}) \leq \liminf_n \mathrm{I}_V(\Gamma_n) = \mathcal{H}^{d-1}(\omega)$ for $(v_n)_n$, v as in (ii) σ^2_{sym}
3. $v_n = \psi\chi_{\Omega_{h_n}} \to \psi\chi_{\Omega_h}$ in (i) $\sigma^2_{\mathrm{sym}} \Longrightarrow \Gamma \supset (\partial^* G_\infty \cap U') \cup (\partial^*\Omega_h \cap U' \cap (G_\infty)^0)$
4. $\mathrm{I}_V(\Gamma) \geq 2\mathcal{H}^{d-1}(\omega_\Psi) + \mathcal{H}^{d-1}(\omega) - \mathcal{H}^{d-1}(\omega_{\Lambda_2}) \geq \mathcal{H}^{d-1}(\omega) + \mathcal{H}^{d-1}(\omega_\psi)$

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For $(v_n)_n$, v as in (ii) σ^2_{sym}, also

$$v_n'(x) := v_n(x', x_d - t)\chi_{\Omega_{h_n}}(x) \to v'(x) := v(x', x_d - t)\chi_{\Omega_h}(x)$$

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$$v'_n(x) := v_n(x', x_d - t)\chi_{\Omega_{h_n}}(x) \to v'(x) := v(x', x_d - t)\chi_{\Omega_h}(x)$$

((i) in σ^2_{sym}) + ($J_{v'_n} \subset \Gamma_n$) $\Longrightarrow J_{v'} \subset \Gamma$, so

$$(\Sigma + te_d) \cap \Omega_h^1 = (\Gamma + te_d) \cap \Omega_h^1 = (J_v + te_d) \cap \Omega_h^1 = J_{v'} \cap \Omega_h^1 \subset \Gamma \cap \Omega_h^1 = \Sigma$$

2nd step: $\Sigma := \Gamma \cap \Omega_h^1$ is vertical in Ω_h^1, i.e. $(\Sigma + te_d) \cap \Omega_h^1 \subset \Sigma$

For $(v_n)_n$, v as in (ii) σ^2_{sym}, also

$$v_n'(x) := v_n(x', x_d - t)\chi_{\Omega_{h_n}}(x) \to v'(x) := v(x', x_d - t)\chi_{\Omega_h}(x)$$

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3rd step: For $(v_n)_n$, v as in (ii) σ^2_{sym}, we apply lsc for voids (surface part) to $E_n = \Omega \setminus \Omega_{h_n}$, $E = \Omega \setminus \Omega_h$, $v_n \chi_{\Omega \setminus E_n}$, $v\chi_{\Omega \setminus E}$. Then

$$\mathcal{H}^{d-1}(\partial^* \Omega_h \cap \Omega) + 2\mathcal{H}^{d-1}(\Sigma) \leq \liminf_{n\to\infty} \mathcal{H}^{d-1}(\partial \Omega_{h_n})$$

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Final step: Lsc for volume part from that for voids +
(i) in σ^2_{sym} for u_n, $u \Longrightarrow J'_u \cap \Omega_h^1 := (J_u + \mathbb{R}e_d) \cap \Omega_h^1 \subset \Sigma$

Upper limit for G

Theorem (C.-Friedrich)

$$G(u,h)=\int_{\Omega_h^+}\mathbb{C}\,e(u):e(u)\,\mathrm{d}x+\int_\omega\sqrt{1+|\nabla h(x')|^2}\,\mathrm{d}x'$$

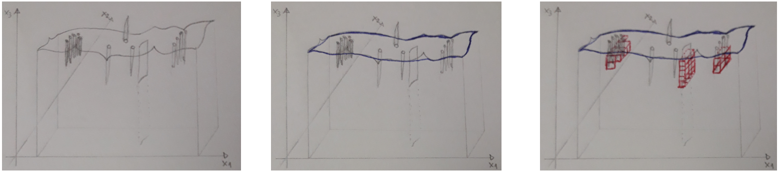
$h\in C^1_m(\omega;[0,M])$, $u|_{\Omega_h^+}\in H^1(\Omega_h^+)$, $u=0$ in $(\omega\times\mathbb{R}^+)\setminus\Omega_h^+$, $u=u_0$ in $\omega\times(0,1)$

relaxes into

$$\overline{G}(u,h)=\int_{\Omega_h^+}\mathbb{C}\,e(u):e(u)\,\mathrm{d}x+\mathcal{H}^{d-1}(\partial^*\Omega_h\cap\Omega)+2\mathcal{H}^{d-1}(J_u'\cap\Omega_h^1)$$

if $u=u\chi_{\Omega_h}$, $u=u_0$ in $\omega\times(-1,0)$, $h\in BV_m(\omega;[0,M])$, $J_u':=J_u+\mathbb{R}^+e_d$,

Upper limit for G

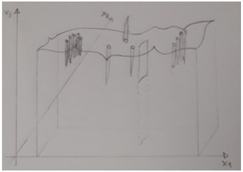
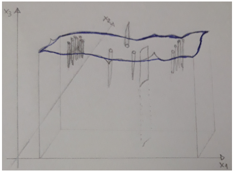
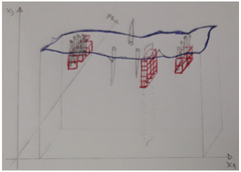


Step 1: smooth function g approximating h (in L^1-norm) –> possible set with small diffuse measure in Ω_g

For Step 2 clean up the jump set 'remaining below': careful use of 'quantitative' Poincaré-Korn inequality

$$\int_{Q_r} |u - a_r|^2 \, \mathrm{d}x \leq Cr \int_{Q_r} |e(u)|^2 \, \mathrm{d}x,$$

Upper limit for G

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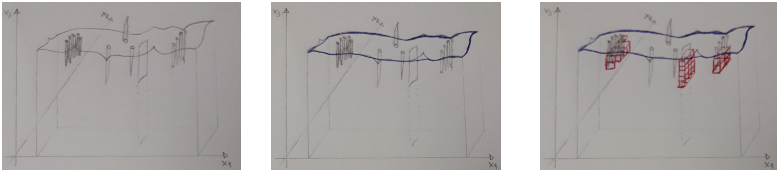

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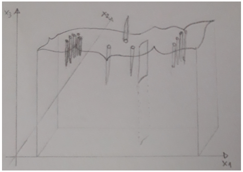
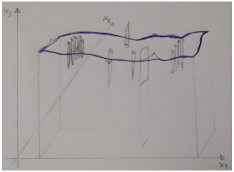
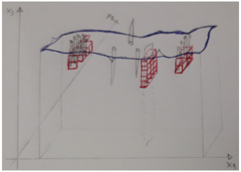
$$\int_{Q''_r} |e((u\chi_{Q'_r \setminus \omega} + a\chi_\omega) * \varphi_r) - e(u) * \varphi_r|^2 \,\mathrm{d}x \le c \left(\frac{\mathcal{H}^{d-1}(J_u)}{r^{d-1}} \right)^q \int_{Q_r} |e(u)|^2 \,\mathrm{d}x \,,$$

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Put 0 in cubes with 'large' jump set and with 'small' jump set and 'small' $|\Omega^1_h|$ for verticality

Generalized compactness (Chambolle-C.'20 Preprint)

Let $u_n \in GSBD^2(\Omega)$ with $\int_\Omega |e(u_n)|^2 \,\mathrm{d}x + \mathcal{H}^{d-1}(J_{u_n}) < M$. Then there are $\mathcal{P} = (P_j)_j$ a Caccioppoli partition, $(a_n)_n$ with $a_n = \sum_{j\in\mathbb{N}} a_n^j \chi_{P_j}$ and

$$|a_n^j(x) - a_n^i(x)| \to +\infty \quad \text{for -a.e. } x \in \Omega, \text{ for all } i \neq j\,,$$

pw infinitesimal rigid motions, $u \in GSBD^2(\Omega)$ s.t. (up to a subsequence)

$$\begin{aligned} u_n - a_n &\to u && \text{a.e. in } \Omega\,, \\ e(u_n) &\rightharpoonup e(u) && \text{in } L^2(\Omega;\mathbb{M}^{d\times d}_{sym})\,, \end{aligned}$$

$$\mathcal{H}^{d-1}(J_u \cup \partial^*\mathcal{P}) \leq \liminf_{n\to\infty} \mathcal{H}^{d-1}(J_{u_n})\,.$$

- This permits to deal with non-homogeneous bulk energies.
  Refined σ^2_{sym}-convergence in couples $(u,\mathcal{P})$? Problems in the topology, since we pass to modifications

Remarks and other applications

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- This permits to deal with non-homogeneous bulk energies. Refined σ^2_{sym}-convergence in couples $(u, \mathcal{P})$? Problems in the topology, since we pass to modifications
- First application of σ^2-convergence: existence of quasi-static evolution for non convex elastic energies in brittle fracture
- Regularity issues in 3d: even prove that the graph of equilibria is closed is hard

THANK YOU!

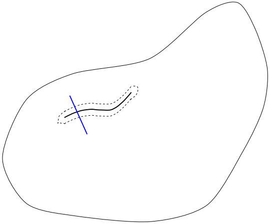
Phase-field approximation for Griffith

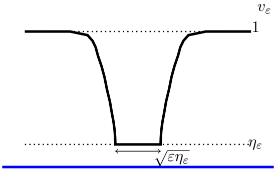
$$D_\varepsilon(u,v) := \int_\Omega \Big(v\,\mathbb{C}e(u) : e(u) + \frac{(1-v)^2}{4\,\varepsilon} + \varepsilon|\nabla v|^2 \Big)\,\mathrm{d}x \quad \text{in } H^1_{u_0}(\Omega;\mathbb{R}^d) \times V^1_{\eta_\varepsilon}$$

with $V^1_{\eta_\varepsilon} = \{\eta_\varepsilon \leq v \leq 1,\ \mathrm{tr}\,v = 1 \text{ on } \partial_D\Omega\}$, $H^1_{u_0} = \{u \in H^1 \colon \mathrm{tr}\,u = \mathrm{tr}\,u_0 \text{ on } \partial_D\Omega\}$,

Γ-converge as $\varepsilon \to 0$ (with $\frac{\eta_\varepsilon}{\varepsilon} \to 0$) to

$$D(u,1) := \int_\Omega \mathbb{C}e(u) : e(u)\,\mathrm{d}x + \mathcal{H}^{d-1}(J_u \cup (\partial_D\Omega \cap \{\mathrm{tr}\,u \neq \mathrm{tr}\,u_0\}))$$





Phase-field approximation for $\overline{G}$

$W(0)=W(1)=0$, $W>0$ in $(0,1)$, $\frac{1}{c_W}=\int_0^1\sqrt{2W(s)}\,\mathrm{d}s$, $\Omega=\omega\times(-1,M+1)$

$$G_\varepsilon(u,v):=\int_\Omega\Big((v^2+\eta_\varepsilon)f(e(u))+c_W\Big(\frac{W(v)}{\varepsilon}+\frac{\varepsilon}{2}|\nabla v|^2\Big)\Big)\,\mathrm{d}x$$

$$u\in H^1(\Omega;\mathbb{R}^d)\,,\quad u=u_0 \text{ in } \omega\times(-1,0)\,,$$

$$v\in H^1(\Omega;[0,1])\,,\quad v=1 \text{ in } \omega\times(-1,0)\,,\ v=0 \text{ in } \omega\times(M,M+1)\quad \partial_d v\le 0 \text{ a.e. in } \Omega\,,$$

"Γ-converge" (in the sense that $v_\varepsilon\to\chi_{\Omega_h}$) as $\varepsilon\to 0$ to

$$\overline{G}(u,h)=\int_{\Omega_h^+}\mathbb{C}\,e(u):e(u)\,\mathrm{d}x+\mathcal{H}^{d-1}(\partial^*\Omega_h\cap\Omega)+2\mathcal{H}^{d-1}(J_u'\cap\Omega_h^1)$$

if $u=u\chi_{\Omega_h}$, $u=u_0$ in $\omega\times(-1,0)$, $h\in BV_m(\omega;[0,M])$, $J_u':=J_u+\mathbb{R}^+e_d$