Mean value properties: Old and New

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Representative Classical Results

Blasche Theorem (1916)

An upper-semicontinuous function u is subharmonic if and only if

$$\limsup_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \left[\fint_{\partial B(x,\varepsilon)} u(y)\, d\sigma(y) - u(x) \right] \geq 0.$$

Privaloff's Theorem (1925)

An upper-semicontinuous function u is subharmonic if and only if

$$\limsup_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \left[\fint_{B(x,\varepsilon)} u(y)\, dy - u(x) \right] \geq 0.$$

Application: The sum of two subharmonic functions is subharmonic.

Modern Linear results

Case of constant coefficients

If we replace the Laplace equation $\Delta u = 0$ by a linear elliptic equation with constant coefficients $Lu = \sum_{i,j} a_{ij} u_{x_i x_j} = 0$ then mean value formulas now hold for appropriate ellipsoids instead of balls.

Linear Subelliptic case

Bonfiglioli and Lanconelli (JEMS 2012) proved extensions of Blasche and Privalov Theorems to the subelliptic case.
They use a definition of W. Kozakiewicz, *Un théorèm sur les opérateurs et son application à la théorie des Laplacians généralisés*, C.R. Soc. Varsovie 26 (1933), that is equivalent to the modern notion of viscosity solution.

A view from 30,000 ft

MVP $\Longleftrightarrow$ DPP $\Longleftrightarrow$ PDE

For an appropriate real function u we have a meta-equivalence among

1. u satisfies a Mean Value Property (in an appropriate asymptotic sense)
2. u satisfies a Dynamic Programming Principle associated to a game or control problem
3. u solves a (possibly nonlinear) PDE

Flexibility of this approach

Euclidean spaces, Riemannian manifolds, Sub-Riemannian manifolds (Heisenberg group), graphs (trees), metric-measure spaces, parabolic versions.

But **limited** to scalar $\mathbb{R}$-valued functions.

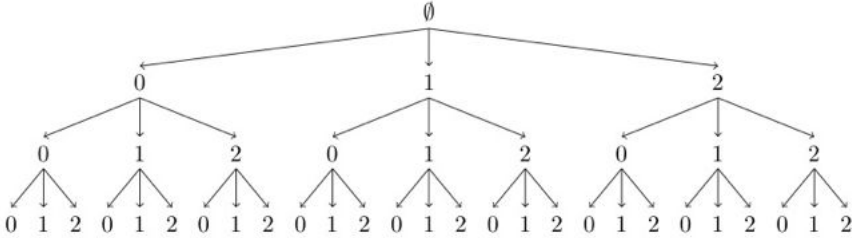
Example: Trees

A directed tree with regular 3-branching T consists of

- the empty set $\emptyset$,
- 3 sequences of length 1 with terms chosen from the set $\{0, 1, 2\}$,
- 9 sequences of length 2 with terms chosen from the set $\{0, 1, 2\}$,
- $\cdots$
- 3^r sequences of length r with terms chosen from the et $\{0, 1, 2\}$

and so on. The elements of T are called *vertices*.

Example: Trees



Calculus on Trees

Each vertex v al level r has three children (successors)

$$v_0, v_1, v_2.$$

Let $u\colon T \mapsto \mathbb{R}$ be a real valued function.

Gradient

The gradient of u at the vertex v is the vector in $\mathbb{R}^3$

$$\nabla u(v) = \{u(v_0) - u(v), u(v_1) - u(v), u(v_2) - u(v)\}.$$

Divergence

The averaging operator or *divergence* of a vector $X = (x, y, z) \in \mathbb{R}^3$ as

$$\mathrm{div}(X) = \frac{x + y + z}{3}.$$

Harmonic Functions on Trees

Harmonic functions

A function u is harmonic if satisfies the Laplace equation

$$\mathrm{div}(\nabla u) = 0.$$

The Mean Value Property

A function u is harmonic if and only if it satisfies the mean value property

$$u(v) = \frac{u(v_0) + u(v_1) + u(v_2)}{3}.$$

Thus the values of harmonic function at level r determine its values at all levels smaller than r.

The boundary of the tree

Branches and boundary

A **branch** of T is an infinite sequence of vertices, each followed by one of its immediate successors (this corresponds to the level $r = \infty$.) The collection of all branches forms the boundary of the tree T is denoted by ∂T.

The mapping $g\colon \partial T \mapsto [0,1]$ given by

$$g(b) = \sum_{r=1}^{\infty} \frac{b_r}{3^r} \quad \text{(also denoted by } b\text{)}$$

is (almost) a bijection (think of an expansion in base 3 of the numbers in [0,1]).

The Dirichlet problem

• We have a natural metric and natural measure in ∂T inherited from the interval $[0, 1]$.
• The **classical Cantor set** C is the subset of ∂T formed by branches that don't go through any vertex labeled 1.

The Dirichlet problem

Given a (continuous) function $f\colon \partial T \mapsto \mathbb{R}$ find a harmonic function $u\colon T \mapsto \mathbb{R}$ such that

$$\lim_{r\to\infty} u(b_r) = f(b)$$

for every branch $b = (b_r) \in \partial T$.

Dirichlet problem, II

Given a vertex $v \in T$ consider the subset of ∂T consisting of all branches that start at v. This is always an interval that we denote by I_v.

Solution to the Dirichlet problem, $p = 2$

The we have

$$u(v) = \frac{1}{|I_v|} \int_{I_v} f(b)\, db.$$

Note that u is a *martingale*.

We see that we can in fact solve the Dirichlet problem for $f \in L^1([0,1])$.

Where are the nonlinear PDEs?

Setting

$$\mathrm{div}_\infty(X) = \frac{1}{2}\left(\max\{x,y,z\} + \min\{x,y,z\}\right)$$

we obtain the ∞-Laplacian

$$\mathrm{div}_\infty\left(\nabla u\right) = 0$$

Choose $\alpha, \beta \in (0,1)$ such at $\alpha + \beta = 1$ and set

$$\mathrm{div}_{\alpha,\beta}(X) = \frac{\alpha}{2}\left(\max\{x,y,z\} + \min\{x,y,z\}\right) + \beta\left(\frac{x+y+z}{3}\right).$$

We obtain the (α,β)-Laplacian

$$\mathrm{div}_{\alpha,\beta}\left(\nabla u\right) = 0.$$

This operator is the homogeneous (α,β)-Laplacian.

The (homogeneous) p-Laplacian

The equations

$$\mathrm{div}_2(\nabla u) = 0, \quad \mathrm{div}_p(\nabla u) = 0, \quad \mathrm{div}_\infty(\nabla u) = 0$$

MVP

$$u(v) = \frac{\alpha}{2}\left(\max_i\{u(v_i)\} + \min_i\{u(v_i)\}\right) + \beta\left(\frac{u(v_0)+u(v_1)+u(v_2)}{3}\right).$$

In general we have $p = p(\alpha, \beta, \mathrm{geometry})$

1. The case $p = 2$ corresponds to $\alpha = 0$, $\beta = 1$.
2. The case $p = \infty$ corresponds to $\alpha = 1$, $\beta = 0$.
3. In general, there is no explicit solution formulas for $p \neq 2$

Formulas for f monotone, $p = \infty$

Suppose that f is monotonically increasing. Let us play tug-of-war in the tree with pay-off function f. Starting at the vertex v_k at level k

$$v_k = 0.b_1 b_2 \ldots b_k, \qquad b_j \in \{0, 1, 2\}$$

players always move either left (adding a 0) or right (adding a 1). In this case I_v is a Cantor-like set $I_v = \{0.b_1 b_2 \ldots b_k d_1 d_2 \ldots\}$, $d_j \in \{0, 2\}$

Formula for $p = \infty$

$$u(v) = \sup_{S_I} \inf_{S_{II}} \mathbb{E}^v_{S_I, S_{II}}[f(b)] = E^v_{S_I^\star, S_{II}^\star}[f(b)] = \int_{I_v} f(b) d\mathcal{C}_v(b)$$

Mean Value Properties in $\mathbb{R}^n$, I

We start by observing that in order to characterize continuous harmonic functions it is enough to ask that the mean value property holds in an asymptotic sense

$$u(x) = \fint_{B(x,\varepsilon)} u(y)\, dy + o(\varepsilon^2) \text{ as } \varepsilon \to 0. \tag{1}$$

In fact, even a weaker *viscosity* notion suffices.

Lemma

An upper semicontinuous function $u\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is *subharmonic* in Ω if for every $x \in \Omega$ and test function $\phi \in C^2(\Omega)$ that touches u from above at x we have that

$$\phi(x) \le \fint_{B(x,\varepsilon)} \phi(y)\, dy + o(\varepsilon^2) \text{ as } \varepsilon \to 0. \tag{2}$$

Notice that the characterization (2) gives a simple proof of Privaloff's characterization.

Mean Value Properties, II

Lemma (M-Parviainen-Rossi)

Solutions to the p-Laplace equation are characterized by

$$u(x) = \frac{p-2}{2(p+n)} \left\{ \max_{\overline{B}(x,\varepsilon)} u + \min_{\overline{B}(x,\varepsilon)} u \right\} + \frac{n+2}{n+p} \fint_{B(x,\varepsilon)} u(y)\, dy + o(\varepsilon^2) \tag{3}$$

in the viscosity sense, for p in the range $1 < p \leq \infty$.

That is, we have the analogue of Privaloff's characterization for p-subharmonic functions by replacing the regular solid average with the nonlinear average in (3) and using expansions in the viscosity sense.

Ferrari-Liu-M.

Similar characterization in the Heisenberg group for $1 < p < \infty$.

Calculus in the Heisenberg group

- $\mathcal{H}$: the Heisenberg group is just $\mathbb{R}^3$ with the group operation for $p = (x, y, z)$ and $p' = (x', y', z')$ given by

$$p \star p' = (x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y)).$$

- $X = \partial_x - (y/2)\partial_z$, $Y = \partial_y + (x/2)\partial_z$, and $Z = \partial_z$ are a basis for left-invariant vector fields.
- $[X, Y] = Z$
- For $u\colon \mathcal{H} \mapsto \mathbb{R}$, the horizontal gradient of u is the vector field

$$\nabla_H u = (Xu)X + (Yu)Y = (Xu, Yu).$$

- For $F\colon \mathcal{H} \mapsto \mathbb{R}^2$, the horizontal divergence of $F = F^1X + F^2Y + F^3Z$ is the scalar

$$\mathrm{div}_H(F) = XF^1 + YF^2.$$

Level-sets and Horizonta Mean Curvature Flow

Take $u : \mathcal{H} \times [0, \infty) \to \mathbb{R}$ such that

$$\Gamma_t = \{p \in \mathcal{H} : u(p,t) = 0\} \quad \text{for all } t \geq 0.$$

The horizontal mean curvature of Γ_t is given by

$$\kappa_H = \mathrm{div}_H \left(\frac{\nabla_H u}{|\nabla_H u|} \right).$$

The horizontal normal velocity is

$$V_H = \frac{u_t}{|\nabla_H u|} \text{ and } .$$

Horizontal mean curvature flow equation

$$V_H = \kappa_H$$

Mean Curvature Flow in the Heisenberg group

We obtain the level set equation

$$u_t = \mathrm{tr}\left[\left(I - \frac{\nabla_H u \otimes \nabla_H u}{|\nabla_H u|^2}\right)(\nabla_H^2 u)^*\right].$$

cf. [Evans-Spruck, '91] and [Chen-Giga-Goto, '91] for well-posedness in $\mathbb{R}^n$.

Wellposedness of the Level set Equation

Problem: Existence and uniqueness of viscosity solutions of

$$\text{(MCF)}\begin{cases} u_t - \operatorname{tr}\left[\left(I - \dfrac{\nabla_H u \otimes \nabla_H u}{|\nabla_H u|^2}\right)(\nabla_H^2 u)^*\right] = 0 & \text{in } \mathcal{H}\times(0,\infty) \\ u(p,0) = u_0(p) \in BC(\mathcal{H}) & \text{in } \mathcal{H}. \end{cases}$$

Main difficulty lies at the *characteristic set* (always $\neq \emptyset$):

$$\{(p,t)\in\mathcal{H}\times(0,\infty) : \nabla_H u(p,t) = 0\}.$$

Known Results:

[Capogna-Citti, '09]: Uniqueness and existence for a special class of solutions vertically separated at the boundary. [Dirr-Dragoni-Renesse, '10]: Existence via a stochastic control

Vertical Axial Symmetry

The function u is *spatially symmetry about the vertical axis* if

$$u(x, y, z, t) = u(x', y', z, t) \text{ when } (x')^2 + (y)^2 = x^2 + y^2.$$

Examples include spheres, tori and other compact surfaces.

Example (A solution initiated from the gauge sphere)

For $p = (x, y, z) \in \mathcal{H}$, $R > 0$

$$u(p, t) = (x^2 + y^2)^2 + 12t(x^2 + y^2) + 16z^2 + 12t^2 - R^4$$

is a solution of (MCF) with

$$u_0(p) = G(p) = |p|_H^4 - R^4 = (x^2 + y^2)^2 + 16z^2 - R^4.$$

The zero level set of $u(\cdot, t)$ shrinks and disappears in finite time.

Question: Is this solution unique?

Definition of Viscosity Solutions

$$F(\eta, Y) = -\operatorname{tr}\left(\left(I - \frac{\eta \otimes \eta}{|\eta|^2}\right) Y\right), \ \eta \in \mathbb{R}^2, \ Y \in \mathbb{R}^{2\times 2} \text{ symmetric}$$

Definition (Definition of subsolutions)

A bounded upper semicontinuous function u is a *subsolution* in $\mathcal{O} \subset \mathcal{H} \times (0, \infty)$ if for any smooth function ϕ and $(\hat{p}, \hat{t}) \in \mathcal{O}$ s.t.

$$\max_{\mathcal{O}}(u - \phi) = (u - \phi)(\hat{p}, \hat{t}),$$

the function ϕ satisfies at $(\hat{p}, \hat{t})$

$$\begin{cases} \phi_t + F(\nabla_H \phi, (\nabla_H^2 \phi)^*) \leq 0 & \text{if } \nabla_H \phi(\hat{p}, \hat{t}) \neq 0, \\ \phi_t \leq 0 & \text{if } \nabla_H \phi(\hat{p}, \hat{t}) = 0, \ (\nabla_H^2 \phi)^*(\hat{p}, \hat{t}) = 0. \end{cases}$$

Definition of Viscosity Solutions

$$F(\eta, Y) = -\operatorname{tr}\left(\left(I - \frac{\eta \otimes \eta}{|\eta|^2}\right) Y\right), \ \eta \in \mathbb{R}^2, \ Y \in \mathbb{R}^{2\times 2} \text{ symmetric}$$

Definition (Definition of supersolutions)

A bounded lower semicontinuous function u is a *supersolution* in $\mathcal{O} \subset \mathcal{H} \times (0, \infty)$ if for any smooth function ϕ and $(\hat{p}, \hat{t}) \in \mathcal{O}$ s.t.

$$\min_{\mathcal{O}} (u - \phi) = (u - \phi)(\hat{p}, \hat{t}),$$

the function ϕ satisfies at $(\hat{p}, \hat{t})$

$$\begin{cases} \phi_t + F(\nabla_H \phi, (\nabla_H^2 \phi)^*) \geq 0 & \text{if } \nabla_H \phi(\hat{p}, \hat{t}) \neq 0, \\ \phi_t \geq 0 & \text{if } \nabla_H \phi(\hat{p}, \hat{t}) = 0, (\nabla_H^2 \phi)^*(\hat{p}, \hat{t}) = 0. \end{cases}$$

Uniqueness for Axisymmetric Solutions

Theorem (Comparison Theorem)

Let u and v be respectively a subsolution and a supersolution of

$$u_t - \mathrm{tr}\left[\left(I - \frac{\nabla_H u \otimes \nabla_H u}{|\nabla_H u|^2}\right)(\nabla_H^2 u)^*\right] = 0$$

in $\mathcal{H} \times (0, T)$ for any $T > 0$.
$\star$ Assume that $u - a$ and $v - b$ are compactly supported for some $a, b \in \mathbb{R}$ with $a \leq b$.
$\star\star$ Assume that **one of** u or v is spatially axisymmetric about the vertical axis.
Then, we have that

$$u(p, 0) \leq v(p, 0) \text{ for all } p \in \mathcal{H} \implies u \leq v \text{ in } \mathcal{H} \times [0, T).$$

Strategy for the proof

- Double the variables for $u - v$ and take the auxiliary function:

$$u(p,t) - \left(v(q_\alpha, s_\alpha) + \alpha w(p, q_\alpha) + \alpha (t - s_\alpha)^2 + \frac{\sigma}{T - t} \right),$$

  where $\alpha > 0$ large, $\sigma > 0$ small and w is of class C^2.
- Consider the maximizer $(p_\alpha, t_\alpha, q_\alpha, s_\alpha)$ and get test functions of (p,t).
- Compare the viscosity inequalities and derive a contradiction.

It is helpful if we have w satisfying

$$(1)\ \nabla_p w(p,q) = -\nabla_q w(p,q);$$
$$(2)\ \nabla_p w(p,q) = 0 \ \Rightarrow\ \nabla_p^2 w(p,q) = 0.$$

The Euclidean choice is $w(p,q) = |p - q|^4$.
Neither $|p \cdot q^{-1}|^4$ nor $|q^{-1} \cdot p|^4$ works in the Heisenberg group.

Axial Symmetry

Lemma (Tests for axisymmetric solutions)

Let u be a axisymmetric subsolution. Suppose that there exists $(\hat{p}, \hat{t}) \in \mathcal{O} \subset \mathcal{H} \times (0, \infty)$ and $\phi \in C^2(\mathcal{O})$ such that

$$\max_{\mathcal{O}}(u - \phi) = (u - \phi)(\hat{p}, \hat{t})$$

If $\hat{p} = (\hat{x}, \hat{y}, \hat{z})$ satisfies $\hat{x}^2 + \hat{y}^2 \neq 0$, then there exists $k \in \mathbb{R}$ such that

$$\frac{\partial}{\partial x}\phi(\hat{p}, \hat{t}) = \hat{x}k \text{ and } \frac{\partial}{\partial y}\phi(\hat{p}, \hat{t}) = \hat{y}k.$$

The choice $w(p, q) = |p \cdot q^{-1}|^4$ now works for the proof.

Existence by Discrete Games

- p: *starting position*
- $\varepsilon > 0$: *step size*
- u_0: *objective function*
- t: *duration of the game*
- $N(=[t/\varepsilon^2])$: *total steps*

Player I and Player II follow the repeated rules below.

(1) Player I chooses in $\mathcal{H}$ a unit horizontal vector v, i.e., $v = (x, y, 0)$ satisfying $|v|^2 = 1$.

(2) Player II chooses $b = \pm 1$;

(3) The marker is moved from

$$p \mapsto p.e^{\sqrt{2}\varepsilon b v},$$

which is denoted simply by $p \cdot (\sqrt{2}\varepsilon b v)$.

Existence by Discrete Games, II

- The *game states* are $p = y^0, y^1, \ldots, y^N$.
- The *game value* is

$$u^\varepsilon(p,t) := \min_{v^1} \max_{b^1} \underline{\min_{v^2} \max_{b^2} \ldots \min_{v^N} \max_{b^N} u_0(y^N)}.$$

$$u^\varepsilon(p,t) = \min_{v^1} \max_{b^1} u^\varepsilon \left(p \cdot (\sqrt{2}\varepsilon b^1 v^1), t - \varepsilon^2 \right).$$

- Dynamic Programming Principle

$$\boxed{u^\varepsilon(p,t) = \min_{v \in S^1_H} \max_{b=\pm 1} u^\varepsilon \left(p \cdot (\sqrt{2}\varepsilon b v), t - \varepsilon^2 \right)}$$

Existence by Discrete Games, III

Theorem (Existence theorem by games)

Assume that u_0 is uniformly continuous function in $\mathcal{H}$ and is constant $c \in \mathbb{R}$ outside a compact set. Assume also that u_0 is spatially axisymmetric about the vertical axis. Let u^ε be the associated game value. Then u^ε converges, as $\varepsilon \to 0$, to the unique axisymmetric viscosity solution of (MCF) uniformly on compacta of $\mathcal{H} \times [0, \infty)$.

Finite-time Extinction

Theorem
Suppose that $\{\Gamma_t\}_{t\geq 0}$ denotes an axisymmetric surface evolving by the horizontal mean curvature flow. If $\Gamma_0 \subset B_r$ for some $r > 0$, then we have that $\Gamma_t = \emptyset$ for $t > r^2/\sqrt{12}$.

Here B_r is the so-called Korány ball

$$B_r = \{p = (x, y, z) \colon (x^2 + y^2)^2 + z^2 < r^4\}.$$

The proof is based on the comparison with evolution from ∂B_r.

Asymptotic Profile

Theorem
If $\Gamma_t \subset \mathcal{H}$ $(t \geq 0)$ is the horizontal mean curvature flow with initial condition a Korány sphere $\Gamma_0 = \{p \in \mathcal{H} : |p| = r\}$. Then the the normalized flow

$$\frac{\Gamma_t}{\sqrt{r^4 - 12t^2}} \to E_T$$

as $t \to T$, where E_T is given by

$$E_T := \{p \in \mathcal{H} : 12T(x^2 + y^2) + 16z^2 = 1\}.$$

and the extinction time $T = \frac{r^2}{\sqrt{12}}$.