A relaxation result in the framework of structured deformations in the BV-Setting

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Structured deformations from a region Ω

(Del Piero & Owen)

- $(g, G, \mathcal{K})$

– g macroscopic deformation

– G deformation without disarrangements

$$\nabla g = (\nabla g - G) + G$$

– $\mathcal{K}$ disarrangement site

- limit of simple deformations $(f_n, \mathcal{K}_n)$

$$\text{“}f_n \to g,\ \nabla f_n \to G,\ \mathcal{K}_n \to \mathcal{K}''$$

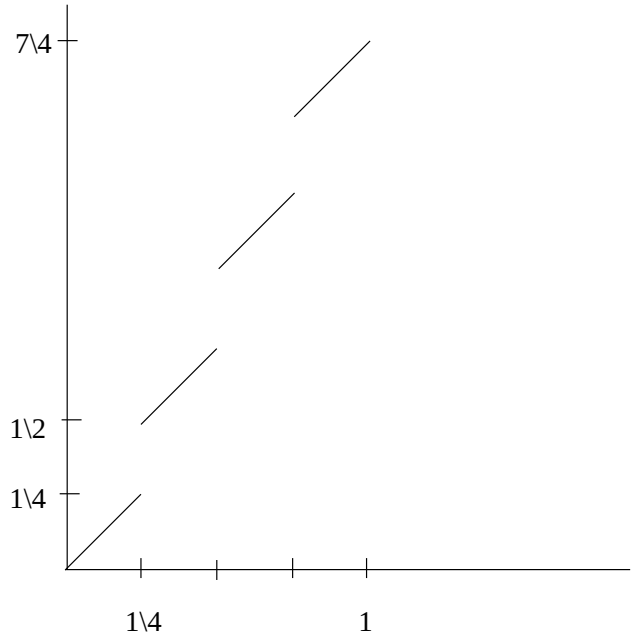
Simple deformations - Examples

a) Broken ramp sequence $(f_n, \mathcal{K}_n)$

- $N = 1$, $\Omega = (0, 1)$

- $f_n(x) = x + \frac{k}{n}, \quad \frac{k}{n} \leqslant x < \frac{k+1}{n}, \quad k = 0, ..., n-1$

- $\mathcal{K}_n = \{\frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}\}$



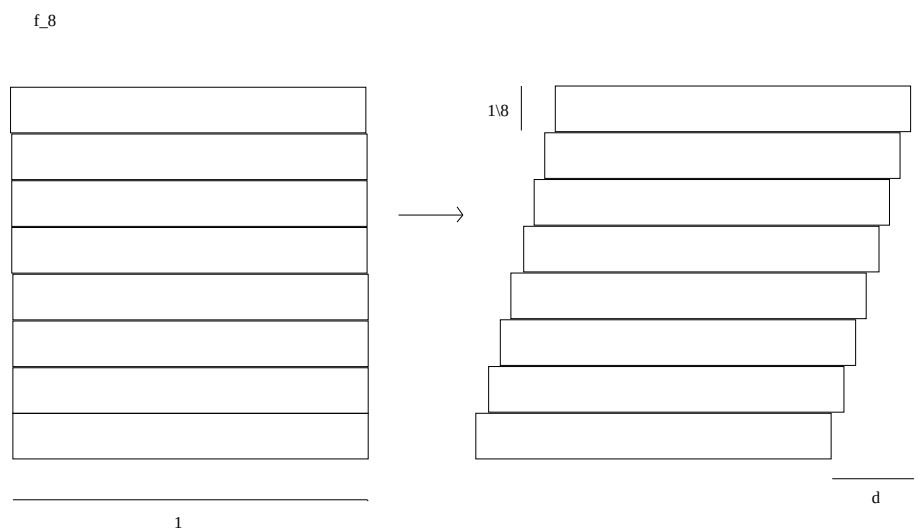
f_4

b) The deck of cards $(f_n, \mathcal{K}_n)$

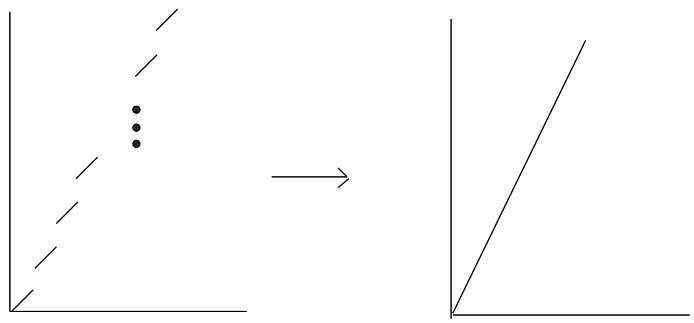
- $N = 2$, $\Omega = (0,1)^2$

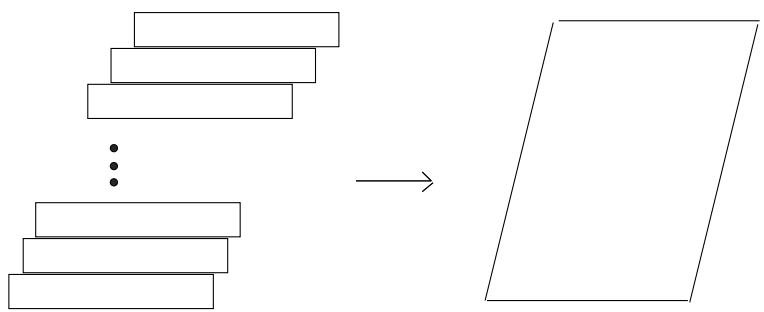
- $f_n(x,y) = (x + \frac{k}{n}d, y), \quad \frac{k}{n} \leqslant x < \frac{k+1}{n}, \quad k = 0, ..., n-1$

- $\mathcal{K}_n = (0,1) \times \frac{k}{n}$, $k = 1, ..., n-1$



In the limit...





That is, we get:

a) $(\emptyset, g, G)$

- $g(x) = 2x$

- $G(x) = 1$

b) $(\emptyset, g, G)$

- $g(x, y) = (x + yd, y)$ (simple shear)

- $G(x, y) = \mathbb{I}$

Classical problem in elastostatics

To minimize:

$$\int_\Omega W(\nabla g)\, dx$$

Energy to be assigned to $(g, G, \mathcal{K})$? (Choksi & Fonseca)

Step 1. To assign an energy (bulk + interfacial contribution) to each simple deformation

Step 2. To define the energy for $(g, G, \mathcal{K})$ as the limit of approximating simple deformation energies

Step 3. To get an integral representation of the limit energy in terms of appropriate bulk and interfacial limit densities...

Setting : SBV functions

Recall $BV(\Omega;\mathbb{R}^d)$ & $SBV(\Omega;\mathbb{R}^d)$

1) $BV(\Omega;\mathbb{R}^d)$

- $u \in L^1(\Omega;\mathbb{R}^d)$

- $D_j u_i \in \mathcal{M}(\Omega)$

Properties:

- $Du = \nabla u \mathcal{L}^N \lfloor \Omega + D^s u$

- $Du = \nabla u \mathcal{L}^N \lfloor \Omega + [u] \otimes \nu_u \mathcal{H}^{N-1} \lfloor S_u + C_u$

where:

- S_u ... jump set of u ...

- $[u] := u^+ - u^-$

- $\nu \in S^{N-1}$ ... normal to S_u ...

- Ω_u ... points of approximate continuity of u ...

- $C_u = D^s u \lfloor (\Omega \setminus \Omega_u)$ ... Cantor part of Du ...

2) $SBV(\Omega; \mathbb{R}^d)$ (De Giorgi & Ambrosio)

$u \in BV(\Omega; \mathbb{R}^d)$, $C_u = 0$

Choksi & Fonseca's previous results:

$(g, G) \in SBV(\Omega; \mathbb{R}^d) \times L^1(\Omega; \mathbb{R}^{d\times N})$

$$\mathcal{I}(g, G) := \inf_{\{u_n\}\subset SBV(\Omega;\mathbb{R}^d)} \left\{ \liminf_{n\to\infty} E(u_n), \;\; u_n \xrightarrow{L^1} g, \;\; \nabla u_n \overset{\mathcal{M}(\Omega)}{\rightharpoonup} G \right\}$$

$$E(u) = \int_\Omega W(\nabla u)\, dx + \int_{S_u} \psi([u], \nu(u))\, d\mathcal{H}^{N-1}$$

In this work...

$SBV^2(\Omega;\mathbb{R}^d) = \{u \in SBV(\Omega;\mathbb{R}^d),\ \nabla u \in SBV(\Omega;\mathbb{R}^{d\times N})\}$

(Carriero, Leaci & Tomarelli)

Notation:

$\nabla^2 u = \nabla(\nabla u)$

(absolutely continuous part of $D(\nabla u)$ w.r.t $\mathcal{L}^N$)

Energy:

$$E(u) = \int_\Omega W(\nabla u, \nabla^2 u)\, dx + \int_{S_u} \Psi_1([u], \nu(u))\, d\mathcal{H}^{N-1}(x)$$
$$+ \int_{S_{\nabla u}} \Psi_2([\nabla u], \nu(\nabla u))\, d\mathcal{H}^{N-1}(x)$$

... under appropriate hypotheses ...

Main problem.

$(g, G) \in BV^2(\Omega; \mathbb{R}^d) \times BV(\Omega; \mathbb{R}^{d\times N})$

$BV^2(\Omega; \mathbb{R}^d) = \{v \in BV(\Omega; \mathbb{R}^d),\ \nabla v \in BV(\Omega; \mathbb{R}^{d\times N})\}$

To characterize:

$$I(g, G) = \inf_{\{u_n\}\subset SBV^2(\Omega;\mathbb{R}^d)} \left\{ \liminf_{n\to\infty} E(u_n),\ u_n \xrightarrow{L^1} g,\ \nabla u_n \xrightarrow{L^1} G \right\}$$

Exist such u_n?

Alberti's Lusin Type Theorem For Gradients: Given $f \in L^1(\Omega; \mathbb{R}^{d\times N})$, there exists $u \in SBV(\Omega; \mathbb{R}^d)$ & $g : \Omega \to \mathbb{R}^{d\times N}$ (Borel function) s.t

$$Du = f\mathcal{L}^N + g\mathcal{H}^{N-1}\lfloor S_u,$$

$$\int_{S_u} |g| \, d\mathcal{H}^{N-1} \leqslant C||f||_{L^1(\Omega;\mathbb{R}^{d\times N})},$$

$$||u||_{L^1(\Omega)} \leqslant C||f||_{L^1(\Omega;\mathbb{R}^{d\times N})}$$

Main Result.

$$
\begin{aligned}
I(g,G) \;=& \int_\Omega \Big(W_1(G-\nabla g) + W_2(G,\nabla G)\Big)\, dx + \int_{S_g} \gamma_1([g],\nu(g)) d\mathcal{H}^{N-1} \\
&+ \int_{S_G} \gamma_2(G^+,G^-,\nu(G)) d\mathcal{H}^{N-1} + \int_\Omega W_1\left(-\frac{dD^c g}{d|D^c g|}\right) d|D^c g| \\
&+ \int_\Omega W_2^\infty\left(G, \frac{dD^c G}{d|D^c G|}\right) d|D^c G|
\end{aligned}
$$

First Step:

To obtain an integral representation in the SBV-setting...

Namely, in the SBV-setting we obtain that:

$$I(g,G) = \int_\Omega \Big(W_1(G - \nabla g) + W_2(G, \nabla G) \Big) dx + \int_{S_g} \gamma_1([g], \nu(g)) d\mathcal{H}^{N-1}$$
$$+ \int_{S_G} \gamma_2(G^+, G^-, \nu(G))\, d\mathcal{H}^{N-1}$$

$(g,G) \in SBV^2(\Omega; \mathbb{R}^d) \times SBV(\Omega; \mathbb{R}^{d\times N})$

Auxiliary results for the proof...

Reshetnyak's Theorem: Let μ, μ_n be finite $\mathbb{R}^d$–valued Radon measures in Ω s.t

- $\mu_n \overset{*}{\rightharpoonup} \mu$

- $|\mu_n|(\Omega) \to |\mu|(\Omega)$

Then

$$\int_\Omega f\left(x, \frac{\mu_n}{|\mu_n|}(x)\right) d|\mu_n| \to \int_\Omega f\left(x, \frac{\mu}{|\mu|}(x)\right) d|\mu|$$

$\forall\ f : \Omega \times S^{d-1} \to \mathbb{R}$ continuous & bounded

Alberti's Lusin Type Theorem For Gradients (recall from before...)

Alberti's Rank One Theorem: Let $u \in BV(\Omega; \mathbb{R}^d)$ with $D^s u = M|D^s u|$ (M: polar decomposition of $D^s u$). Then $M(x)$ has rank one, i.e,

$$M(x) = \eta(x) \otimes \nu(x), \quad |\eta(x)| = |\nu(x)| = 1,$$

for $|D^s u|$-a.e $x \in \Omega$.

Matias' Lemma: Given $A \in \mathbb{R}^{d \times N}$, there exists $u \in SBV(\Omega; \mathbb{R}^d)$ s.t

- $u_{\lfloor \partial Q} = 0$

- $\nabla u = A$ a.e in Ω

- $|D^s u|(\Omega) \leqslant C(N)|A|\,|\Omega|$.

Sketch of the proof for the SBV-setting.

Step 1. $$I(g,G) = I_1(g,G) + I_2(G)$$

$$I_1(g,G) = \inf \left\{ \liminf_{n\to\infty} \int_{S_{u_n}} \Psi_1([u_n],\nu(u_n))\, d\mathcal{H}^{N-1}, \right.$$
$$\left. \{u_n\} \subset SBV^2(\Omega;\mathbb{R}^d),\ u_n \xrightarrow{L^1} g,\ \nabla u_n \xrightarrow{L^1} G \right\}$$

$$I_2(G) = \inf \left\{ \liminf_{n\to\infty} \int_{\Omega} W(v_n,\nabla v_n)\, dx + \int_{S_{v_n}} \Psi_2([v_n],\nu(v_n))\, d\mathcal{H}^{N-1}, \right.$$
$$\left. \{v_n\} \subset SBV(\Omega;\mathbb{R}^{d\times N}),\ v_n \xrightarrow{L^1} G \right\}$$

I_2?

Bouchitté, Fonseca & Mascarenhas

$$\Downarrow$$

$$I_2(G) \;=\; \int_\Omega W_2(G, \nabla G)\, dx + \int_{S_G} \gamma_2(G^+, G^-, \nu(G))\, d\mathcal{H}^{N-1}$$

I_1?

To show:

$$I_1(g,G) = \int_\Omega W_1(G - \nabla g)\, dx + \int_{S_g} \gamma_1([g], \nu(g)) d\mathcal{H}^{N-1}$$

Step 2. Localize & show that $I_1(g, G, \cdot)\lfloor \mathcal{A}(\Omega)$ is a Radon measure absolutely continuous w.r.t

$$\mathcal{L}^N + \mathcal{H}^{N-1}\lfloor S_g$$

Step 3. (Lower Bound)

$$I_1(g,G,A) \geqslant \int_A W_1\left(G(x) - \nabla g(x)\right)\, dx + \int_{A\cap S_g} \gamma_1\left([g],\nu(g)\right)\, d\mathcal{H}^{N-1}$$

- $u_n \in SBV^2(\Omega;\mathbb{R}^N)$, $u_n \xrightarrow{L^1} g$, $\nabla u_n \xrightarrow{L^1} G$

$$\lim_{n\to\infty} \int_{S_{u_n}} \Psi_1([u_n],\nu(u_n))\, d\mathcal{H}^{N-1} < \infty.$$

- $\mu_n(B) = \int_{B\cap S_{u_n}} \Psi_1([u_n],\nu(u_n)) d\mathcal{H}^{N-1}$

- (up to a subsequence) $\mu_n \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega;\mathbb{R}^d)$

- $\mathcal{L}^N$-almost every $x_0 \in \Omega$

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geqslant W_1(G(x_0) - \nabla g(x_0))$$

- $\mathcal{H}^{N-1} \lfloor S_g$-almost every $x_0 \in \Omega$

$$\frac{d\mu}{d\mathcal{H}^{N-1} \lfloor S_g}(x_0) \geqslant \gamma_1([g](x_0), \nu(g)(x_0)).$$

Then:

$$\begin{aligned}\liminf_{n\to\infty} \mu_n(A) &\geqslant \mu(A) \\ &\geqslant \int_A \mu_a(x)dx + \int_{A\cap S_g} \mu_g^s(x) d\mathcal{H}^{N-1}(x) \\ &\geqslant \int_A W_1\left(G(x) - \nabla g(x)\right) dx + \int_{A\cap S_g} \gamma_1\left([g], \nu(g)\right) d\mathcal{H}^{N-1}\end{aligned}$$

• μ_a absolutely continuous part of μ w.r.t $\mathcal{L}^N$, μ_g^s absolutely continuous part of μ w.r.t $\mathcal{H}^{N-1}\lfloor S_g$

Finally take the infimum over all such $\{u_n\}$ ...

Step 4. (Upper bound)

$$I_1(g,G,A) \leqslant \int_A W_1\left(G(x) - \nabla g(x)\right)\, dx + \int_{A\cap S(g)} \gamma_1\left([g],\nu(g)\right)\, d\mathcal{H}^{N-1}$$

To show:

- $\dfrac{dI_1(g,G,\cdot)}{d\mathcal{L}^N}(x_0) \leqslant W_1(G(x_0) - \nabla g(x_0)), \quad \mathcal{L}^N$-a.e $x_0 \in \Omega$

- $\dfrac{dI_1(g,G,\cdot)}{d\mathcal{H}^{N-1}\lfloor S_g}(x_0) \leqslant \gamma_1([g](x_0),\nu(g)(x_0)), \quad \mathcal{H}^{N-1}$ -a.e $x_0 \in S_g$

Idea: (argument of Ambrosio, Mortolla & Tortorelli)

Enough to consider $g = \lambda\chi_E$, $\lambda \in \mathbb{R}$, E set of finite perimeter

Two cases:

- E is a polyhedron

- General case. Two results are also needed:

Proposition: There exist constants $C_1, C_2 > 0$ such that

$$|\gamma_1(\lambda, \nu) - \gamma_1(\lambda', \nu)| \leqslant C_1 |\lambda - \lambda'|, \quad \forall \lambda, \lambda' \in \mathbb{R}^d,$$

$$|W_1(A) - W_1(B)| \leqslant C_2 |A - B|, \quad \forall A, B \in \mathbb{R}^{d \times N}.$$

Moreover γ_1 is upper semicontinuous with respect to ν.

Proposition: Let $(g_n, G_n) \in SD(\Omega; \mathbb{R}^d)$ be such that

$$g_n \xrightarrow[n \to \infty]{L^1} g \quad \text{and} \quad G_n \xrightarrow[n \to \infty]{L^1} G.$$

Then

$$I_1(g, G) \leqslant \liminf_{n \to \infty} I_1(g_n, G_n).$$

Sketch of the proof in the BV-setting.

Step 1. (as before)

$$I(g,G) = I_1(g,G) + I_2(G)$$

I_2?

$$\begin{aligned} I_2(G) \;&=\; \int_\Omega W_2(G,\nabla G)\,dx + \int_{S_G} \gamma_2(G^+,G^-,\nu(G))\,d\mathcal{H}^{N-1} \\ &\quad + \int_\Omega W_2^\infty\left(G, \frac{dD^cG}{d|D^cG|}\right)|D^cG| \end{aligned}$$

(Bouchitté, Fonseca & Mascarenhas)

I_1?

To show:

$$I_1(g,G) = \int_\Omega W_1(G-\nabla g)\,dx + \int_{S_g} \gamma_1([g],\nu(g))d\mathcal{H}^{N-1} + \int_\Omega W_1\left(-\frac{dD^c g}{d|D^c g|}\right) d|D^c g|$$

Auxiliary Lemma: Let $(g,G) \in GSD(\Omega;\mathbb{R}^d)$ and $A \in \mathcal{A}(\Omega)$. Then

$$I_1(g,G,A) = \inf_{\substack{\{g_n\}\subset SBV^2 \\ \{G_n\}\subset SBV}} \left\{ \liminf_{n\to\infty} I_1(g_n,G_n,A),\ g_n \xrightarrow[n\to\infty]{L^1} g,\ G_n \xrightarrow[n\to\infty]{L^1} G \right\}$$

Step 2. (as before) Localize & show that $I_1(g, G, \cdot)\lfloor\mathcal{A}(\Omega)$ is a Radon measure absolutely continuous w.r.t

$$\mathcal{L}^N + \mathcal{H}^{N-1}\lfloor S_g + |D^c g|$$

Step 3. (Upper bound)

Enough to see that

$$\frac{dI_1(g, G, .)}{d|D^c g|}(x_0) \leqslant W_1\left(-\frac{dD^c g}{d|D^c g|}(x_0)\right), \ |D^c g| - a.e \ x_0 \in \Omega$$

Step 4. (Lower Bound)

Enough to see that

$$\frac{dI_1(g,G,.)}{d|D^c g|}(x_0) \geqslant W_1\left(-\frac{dD^c g}{d|D^c g|}(x_0)\right),\ |D^c g| - a.e\ x_0 \in \Omega$$