Quantitative results in stochastic homogenization for uniformly elliptic equations

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Overview: Random homogenization

- We are interested in linear, uniformly elliptic equations like:

$$-\operatorname{div}\left(A(\tfrac{x}{\varepsilon})Du^\varepsilon\right) = f(x) \quad \text{and} \quad -\operatorname{tr}\left(A(\tfrac{x}{\varepsilon})D^2u^\varepsilon\right) = f(x)$$

  or, more generally, nonlinear analogues, such as

$$-\operatorname{div}(D_pL(Du^\varepsilon, \tfrac{x}{\varepsilon})) = f(x) \quad \text{and} \quad F(D^2u^\varepsilon, \tfrac{x}{\varepsilon}) = f(x)$$

- In each case we assume that the coefficients are sampled by an underlying (given) probability measure which is *stationary* and satisfies a *finite range of dependence.*
- Our goal is to describe the solutions u^ε for $0 < \varepsilon \ll 1$.
- We demand that our results be *quantitative.*

Qualitative results

Under more general assumptions (ergodic rather than independent coefficients), we have the following classical qualitative results:

- Papanicolaou-Varadhan('81), Kozlov ('81), Yurinskii ('82):

$$\begin{array}{c} -\operatorname{div}\left(A(\tfrac{x}{\varepsilon})Du^\varepsilon\right) = f(x) \\ -\operatorname{tr}\left(A(\tfrac{x}{\varepsilon})D^2u^\varepsilon\right) = f(x) \end{array} \underset{\varepsilon\to 0}{\Longrightarrow} \begin{array}{c} -\operatorname{div}\left(\overline{A}Du\right) = f(x) \\ -\operatorname{tr}\left(\overline{A}D^2u\right) = f(x) \end{array}$$

- Dal Maso-Modica ('86):

$$-\operatorname{div}(D_pL(Du^\varepsilon, \tfrac{x}{\varepsilon})) = f(x) \underset{\varepsilon\to 0}{\Longrightarrow} -\operatorname{div}\left(D\overline{L}(Du)\right) = f(x)$$

- Caffarelli-Souganidis-Wang ('05):

$$F(D^2u^\varepsilon, \tfrac{x}{\varepsilon}) = f(x) \underset{\varepsilon\to 0}{\Longrightarrow} \overline{F}(D^2u) = f(x).$$

- In each case, an abstract ergodic theorem is used to obtain almost sure homogenization. Typically, this kind of soft argument is difficult to make quantitative.

Quantitative results for $-\operatorname{div}(A(x/\varepsilon)Du^\varepsilon) = f$

For linear, divergence form equations, the first results are due to **Yurinskii ('86)**. He proved in $d \geq 3$ a suboptimal algebraic rate of convergence for the second stochastic moment of the L^∞ norm: i.e., for some $\alpha > 0$,

$$\mathbb{E}\left[\|u^\varepsilon - u_{\text{hom}}\|_{L^\infty}^2\right] \leq C\varepsilon^\alpha.$$

This result was unsurpassed for more than 20 years until the recent explosion of results, beginning with **Gloria-Otto ('11),** who proved optimal error estimates in all dimensions and with better stochastic integrability. They used a critical idea from an earlier paper of **Naddaf-Spencer ('98)**. We also mention **Mourrat ('11), Gloria-Neukamm-Otto ('13), Gloria-Mourrat ('12), Nolen ('13), Marahrens-Otto ('13), Lamacz-Neukamm-Otto ('13), Gloria-Marahrens ('14), Mourrat-Otto ('14), . . .**

Quantitative results for $-\operatorname{tr}(A(x/\varepsilon)D^2u^\varepsilon) = f$

For linear, nondivergence form equations, the best previous results are due to **Yurinskii**, in a series of papers in the 1980s. These papers were completely forgotten and/or ignored. (Until this year, his best paper on the topic had *zero* citations according to both mathscinet and google scholar.)

He proved:

- Suppose $d \geq 5$. Then there exists $\alpha(d, \Lambda) > 0$ such that

$$\mathbb{P}\left[\, \|u^\varepsilon - u_{\mathrm{hom}}\|_{L^\infty} \geq \varepsilon^\alpha \,\right] \lesssim \varepsilon^\alpha.$$

- Suppose that $d \in \{3, 4\}$. Then there exists $\delta > 0$ such that, if $\Lambda - 1 \leq \delta$, where Λ is the ellipticity ratio, then the above conclusion still holds.
- In dimension $d = 2$, Yurinskii gets a slower, logarithmic rate.

Quantitative results for $-\operatorname{div}(D_pL(Du^\varepsilon, \frac{x}{\varepsilon})) = f$

No previous work exists to our knowledge.

Note that all the methods developed to date for the linear case (both probabilistic and pde) rely heavily on linearity. For example, at the core of the method of Gloria-Otto are estimates for the heterogeneous Green's functions that are transferred to the solutions of the equation by the formula.

Quantitative results for $F(D^2u^\varepsilon, \frac{x}{\varepsilon})) = f$

Caffarelli-Souganidis ('10) obtained:

$$\mathbb{P}\left[\|u^\varepsilon(x) - u_{\text{hom}}(x)\|_{L^\infty} \geq \exp\left(-\alpha\sqrt{|\log\varepsilon|}\right)\right] \lesssim \underbrace{\exp\left(-\alpha\sqrt{|\log\varepsilon|}\right)}_{=\varepsilon^{c|\log\varepsilon|^{-1/2}}}$$

Note that this convergence rate is *sub-algebraic*, which limits its usefulness in applications. Since their arguments seemed to optimally quantify the best known proofs of qualitative homogenization, it raised the question of whether an algebraic rate of convergence is even correct.

New strategy for attacking these problems

We propose a new method for attacking these problems. It consists roughly of the following steps:

1. Find the correct subadditive and superadditive quantities that control the solutions. Using independence and decomposing large cubes into smaller cubes and iterating, show get an algebraic rate of convergence for these quantities.
2. Use an oscillating test function argument to get an algebraic rate of convergence in L^∞ for the Dirichlet problem.
3. Get higher regularity by perturbing off the limiting equation. (Like **Avellaneda-Lin ('89)** in the periodic case.)
4. Use the higher regularity to get optimal error estimates by plugging into appropriate concentration inequalities.

Results in nondivergence form case

Steps 1 and 2. For the equation $F(D^2 u^\varepsilon, \frac{x}{\varepsilon}) = f$, in joint work with Charles Smart, we proved the estimate: for every $p \in (0, d)$, there exists $\alpha(p, d, \Lambda) > 0$ such that

$$\mathbb{P}\left[\|u^\varepsilon(x) - u_{\mathrm{hom}}(x)\|_{L^\infty} \geq C\varepsilon^\alpha\right] \lesssim \exp\left(-c\varepsilon^{-p}\right).$$

Note that this improves not only Caffarelli-Souganidis but also Yurinskii in the linear case.

Step 3: By doing an iteration in dyadic balls (a quantitative/random version of Avellaneda-Lin's idea in the periodic setting) and using the previous estimate, we get a *pointwise* $C^{1,1}$ *estimate:*

for all $p < d/2$, there exists $C(p, d, \Lambda) > 0$ such that for all $t > 0$,

$$\mathbb{P}\left[|D^2 u^\varepsilon(0)| > Ct\right] \leq \exp(-ct^p).$$

Results in nondivergence form case

Steps 4. Using the $C^{1,1}$ estimate and specializing to the linear case (to use Green's function formulas) we obtain the following *optimal* error estimates in all dimensions:
For every $p < \infty$, and fixed $x_0 \in U$,

$$\mathbb{E}\left[\, |u^\varepsilon(x_0) - u_{\mathrm{hom}}(x_0)|^p \right]^{1/p} \lesssim \mathrm{Err}_d(\varepsilon),$$

where the characteristic size of the error in dimension d is

$$\mathrm{Err}_d(\varepsilon) := \begin{cases} \varepsilon \, |\log \varepsilon| & \text{in } d = 2, \\ \varepsilon^{3/2} & \text{in } d = 3, \\ \varepsilon^2 \, |\log \varepsilon|^{1/2} & \text{in } d = 4, \\ \varepsilon^2 & \text{in } d \geq 5. \end{cases}$$

Subadditive quantity μ

The hardest part of the program is Step 1, identify and analyze the subadditive quantities. Given $F \in \Omega$ and $U \subseteq \mathbb{R}^d$, we define:

$$\mu(U,F) := \sup \left\{ \frac{|\partial \Gamma_u(U)|}{|U|} \, : \, u \in C(\overline{U}) \text{ and } F(D^2 u, x) \geq 0 \text{ in } U \right\}.$$

where

$$\Gamma_u := \text{ convex envelope of } u,$$

and

$$\begin{aligned} \partial \Gamma_u(U) &:= \text{image of } U \text{ under the subdifferential of } \Gamma_u \\ &= \text{set of slopes of planes we can touch } u \text{ by from below in } U. \end{aligned}$$

Subadditive quantity μ

- $\mu(U,F)$ which measures how big of a "bump" the graph of a solution $u \in C(\overline{U})$ of $F(D^2u,x) = 0$ in U may have.
- Subadditivity: If $U_1,\ldots,U_k$ are disjoint, then

$$\mu(U_1 \cup \cdots \cup U_k, F) \leq \sum_{j=1}^{k} \frac{|U_j|}{|U|} \mu(U_j, F).$$

- By the subadditive ergodic theorem, there exists $\overline{\mu}$ such that $\mathbb{P}\left[\lim_{R\to\infty} \mu(B_R,F) = \overline{\mu}\right] = 1.$
- To get a rate for this limit: we adapt the regularity theory for the Monge-Ampère equation to show that, if $\overline{\mu} > 0$, then solutions with the biggest bump (i.e., maximizers for μ) are close to parabolas.

Higher regularity

To get the $C^{1,1}$ estimate, we use an idea of Avellaneda-Lin. Given a solution u of

$$-\operatorname{tr}\left(A(x)D^2u\right) = 0 \quad \text{in } B_s$$

for $s \gg 1$, we consider, for each $1 \leq R < s/2$, the quantity

$$k_R := \frac{1}{R^2} \inf_{p \in \mathbb{R}^d} \operatorname*{osc}_{x \in B_R} \left(u(x) - p \cdot x\right).$$

By comparing u to the solution of the homogenized equation with same Dirichlet boundary condition on B_R, using the error estimates, get:

$$k_R \leq (1 + CR^{-\alpha})k_{\theta R}, \quad \theta > 1 \text{ universal.}$$

(This holds with overwhelming probability for $R \geq O(1)$.)
Since $k_1 \approx |D^2u(0)|$, this gives the conclusion.

Setup for divergence form equations

We consider the optimization problem

$$\begin{cases} \text{minimize} \quad I_U[w] := \displaystyle\int_U L\left(Dw(x), \frac{x}{\varepsilon}\right) dx \quad \text{over} \quad w \in H^1(U) \\ \qquad \text{subject to some suitable boundary condition on } \partial U. \end{cases}$$

where

$L(p,x)$ is uniformly convex and growing quadratically in p

L is sampled from an underlying probability measure $\mathbb{P}$

$\mathbb{P}$ is stationary and satisfies a finite range of dependence in x

More precise assumptions

We fix $\Lambda > 1$ and consider L's satisfying:

(L1) $L : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a *Carathéodory function*, that is, $L(p,x)$ is measurable in x and continuous in p.

(L2) L is uniformly convex in p: for every $p_1, p_2, x \in \mathbb{R}^d$,

$$\frac{1}{4}|p_1 - p_2|^2 \leq \frac{1}{2}L(p_1,x) + \frac{1}{2}L(p_2,x) - L\left(\frac{1}{2}p_1 + \frac{1}{2}p_2, x\right) \leq \frac{\Lambda}{4}|p_1 - p_2|^2.$$

We define $\Omega := \{\text{all such } L\}$.

Assumptions II

We encode dependence on the environment by endowing Ω with the following collection of σ–algebras: for every Borel $U \subseteq \mathbb{R}^d$,

$$\mathcal{F}(U) := \text{the } \sigma\text{–algebra generated by the maps } L \mapsto \int_U L(p,x)\phi(x)\,dx,$$
$$\text{where } p \in \mathbb{R}^d \text{ and } \phi \in C_c^\infty(\mathbb{R}^d).$$

The largest of these we denote by $\mathcal{F} := \mathcal{F}(\mathbb{R}^d)$.
$\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$.

Assumptions III

Here are the assumptions on $\mathbb{P}$:

(P1) $\mathbb{P}$ is stationary with respect to $\mathbb{Z}^d$–translations: for every $z \in \mathbb{Z}^d$ and $F \in \mathcal{F}$,

$$\mathbb{P}\left[F\right] = \mathbb{P}\left[T_z F\right],$$

where $T_z : \Omega \to \Omega$ defined for each $z \in \mathbb{Z}^d$ by $T_z L(p,x) := L(p, x+z)$ and extended to $\mathcal{F}$ by $T_z F := \{T_z L \,:\, L \in F\}$.

(P2) $\mathbb{P}$ has a unit range of dependence: for all Borel subsets $U, V \subseteq \mathbb{R}^d$ such that $\operatorname{dist}(U,V) \geq 1$,

$$\mathcal{F}(U) \text{ and } \mathcal{F}(V) \text{ are } \mathbb{P}\text{–independent.}$$

(P3) The constant C in (L2) is uniformly bounded on the support of $\mathbb{P}$, that is, there exists $K_0 \geq 0$ such that

$$\mathbb{P}\left[\forall p, x \in \mathbb{R}^d,\ |p|^2 - K_0(1+|p|) \leq L(p,x) \leq \Lambda |p|^2 + K_0(1+|p|)\right] = 1.$$

Previous work

Main previous contribution is due to **Dal Maso and Modica ('86)**. They proved (under much more general assumptions) a *qualitative* homogenization result, which states that, $\mathbb{P}$-almost surely, the heterogeneous energy functional Γ-converges as $\varepsilon \to 0$ to a functional with a deterministic integrand $\overline{L}(p)$.

Their method centers on studying the following quantity:

$$\nu(U,L,p) := \min \left\{ \fint_U L(p + Dw(x), x)\, dx \,:\, w \in H^1_0(U) \right\}.$$

ν represents the energy of the minimizer of the Dirichlet problem with planar boundary conditions $p \cdot x$.

This is a replacement for the cell problem. Key step is to show that the minimizers for $\nu(U,L,p)$ stay close to the plane $p \cdot x$ when U is large. This is accomplished by showing that the energy spreads.

Quantitative results

In joint work with Charlie Smart (to appear soon), we prove quantitative versions of Dal Maso-Modica's result. Here is a sample theorem:

Theorem (A.-Smart (in prep.))

Fix a smooth bounded domain $U \subseteq \mathbb{R}^d$ *and* $g \in C^2(\partial U)$. *Let* u^ε *be the minimizer for* $L(p, \frac{x}{\varepsilon})$ *and* u *be the minimizer of* $\overline{L}$, *subject to the Dirichlet condition* g. *Then for every* $p < d$ *there exists* $\alpha(p, d, \Lambda) > 0$ *such that*

$$\mathbb{P}\left[\sup_U |u^\varepsilon - u| > \varepsilon^\alpha\right] \leq C \exp\left(-c\varepsilon^{-p}\right).$$

Moreover, every $0 < \beta < 1$ *and* $p < (1-\beta)d$, *there exists* $\alpha(\beta, p, d, \Lambda) > 0$ *such that*

$$\mathbb{P}\left[\sup_{y \in U}\left|\overline{L}(Du(y)) - \fint_{B_{\varepsilon^\beta}(y)} L(Du^\varepsilon, \tfrac{x}{\varepsilon})\, dx\right| > \varepsilon^\alpha\right] \leq C \exp\left(-c\varepsilon^{-p}\right).$$

Qualitative proof of Dal Maso-Modica

Without the normalization (the slash in the integral), the quantity ν is *subadditive*. This means that if U is a disjoint union of $U_1, \ldots, U_k$, then

$$|U|\nu(U, L, p) \leq \sum_{i=1}^{k} |U_i|\nu(U_i, L, p).$$

This is true because a minimizer candidate on the large domain U can be assembled by stitching together the minimizers on the smaller domains U_i.

Now apply the *subadditive ergodic theorem*: there exists $\overline{\nu}(p)$ such that, for any open set U,

$$\mathbb{P}\left[\lim_{t\to\infty} \nu(tU, L, p) = \overline{\nu}(p)\right] = 1.$$

Define $\overline{L}(p) := \overline{\nu}(p)$.

Qualitative proof of Dal Maso-Modica II

Once ν is shown to have a deterministic, almost sure limit, it follows that, in the limit, the subadditivity of ν must be close to additivity. This means that the value of the ν in a large cube is close to the average value of ν in smaller subcubes.

By convexity, this means that stitching together the subcube minimizers is a very good approximation to the minimizer in the big cube. This means that the big cube minimizer is close to being a plane.

Once we can homogenize planar boundary conditions, obtaining a very general homogenization result is relatively easy.

The rest of the argument consists in showing that, since we can homogenize the Dirichlet problem with planar boundary conditions, then due to the $C^{1,\alpha}$ regularity of the limiting deterministic problem (which is more than we need), we can homogenize more general boundary conditions by approximation.

Quantitative strategy

To obtain quantitative results, the hard part is to quantify the almost sure limit

$$\mathbb{P}\left[\lim_{t\to\infty} \nu(tU, L, p) = \overline{\nu}(p)\right] = 1.$$

There are two parts ($Q_n := (3^{-n}, 3^n)^d$ are triadic cubes):

(i) Quantify the rate at which the monotone sequence $\mathbb{E}\left[\nu(Q_n, p)\right]$ converges to $\overline{\nu}(p)$.

(ii) Quantify the rate at which the variance of $\nu(Q_n, p)$ decays to 0.

It turns out that (i) is hard, but (ii) is easy if we have (i).

Proof of (ii) assuming (i)

Using the finite range of dependence, it can be easily shown that at least one of the following holds:

- $\mathbb{E}[\nu(Q_n, p)] - \mathbb{E}[\nu(Q_{n+1}, p)]$ is large compared to $\operatorname{var}[\nu(Q_n, p)]$, or
- $\operatorname{var}[\nu(Q_{n+1}, p)]$ is much smaller than $\operatorname{var}[\nu(Q_n, P)]$.

Proof:

In the additive regime, we can use the finite range of dependence and run the argument that proves that the variance of the average of iid random variables must contract. (The variance of a sum is the sum of the variances.)

Why (i) is hard

The reason is that (i) is hard is that all we know about $\mathbb{E}[\nu(Q_n, p)]$ is that it is a monotone sequence in n, by subadditivity. But a monotone sequence can obviously converge at any rate.

Our nightmare situation is the one in which $\mathbb{E}[\nu(Q_n, p)]$ seems to be converging rapidly to $a > \overline{\nu}$ for $n = 1, \ldots, 10^{10}$. The variance is low, and all is well...

Fixing this issue does not come from a finer analysis of ν itself, but by considering an entirely new quantity.

The dual quantity

Rather than μ, it is better to consider the "dual" quantity:

$$\mu(U, L, q) := \min \left\{ \fint_U \left(L(Dw(x), x) - q \cdot Dw(x)\right) dx \,:\, w \in H^1(U) \right\}.$$

When $q = 0$, this looks similar to ν, but it is much different because the boundary condition is free. We are minimizing over *all* H^1 functions.

Notice that while ν was *sub*additive, the new quantity μ is *super*additive, by restriction. If we have a large scale minimizer, we can restrict it to each subdomain to get a candidate for smaller scale energy minimizers.

Why is μ dual to ν?

To see the duality (in a convex analysis sense) in the simplest situation, consider the case when $L(p,x) = L(p)$ is already constant in x. Then we get

$$\nu(U, L, p) = L(p)$$

and

$$\mu(U, L, q) = \min_{p \in \mathbb{R}^d} \{L(p) - q \cdot p\} = -L^*(q)$$

where L^* is the Legendre-Fenchel transform of L.

Identifying $\overline{L}$ with μ

The previous slide suggests that if we could show that

$$\mu(Q_n, L, q) \to \overline{\mu}(q) \quad \text{as } n \to \infty$$

then $\overline{\mu}(q) = -\overline{L}^*(q)$ and so we could identify $\overline{L}$ by duality:

$$\overline{L}(p) = \sup_{q \in \mathbb{R}^d} \left(p \cdot q - \overline{L}^*(q) \right) = \sup_{q \in \mathbb{R}^d} \left(p \cdot q + \overline{\mu}(q) \right).$$

This works, and gives a new proof of Dal Maso and Modica's qualitative result. Moreover, this proof strategy gives quantitative results more easily, because the limit $\mu(Q_n, L, q) \to \overline{\mu}(q)$ can be quantified.

Quantitative strategy

We need to quantify the limit

$$\lim_{n\to\infty} \mu(Q_n) = \overline{\mu}.$$

(Again, $Q_n := (3^{-n}, 3^n)^d$ are triadic cubes and we take $q = 0$ WLOG.)

The quantitative strategy is the same as it was for ν :

(i) Quantify the rate at which the monotone sequence $\mathbb{E}\left[\mu(Q_n)\right]$ converges to $\overline{\mu}$.

(ii) Quantify the rate at which the variance of $\mu(Q_n)$ decays to 0.

For the same reasons as before, (i) is hard, but (ii) is easy if we have (i). So let's focus on (i).

Quantifying the limit $\mathbb{E}[\mu(Q_n)] \to \overline{\mu}$

We want to show that if $\mathbb{E}[\mu(Q_{n+1})] - \mathbb{E}[\mu(Q_n)]$ is small, then both have to be close to $\overline{\mu}$. This rules out the "nightmare scenario" from earlier.

Here is the idea: If $\mathbb{E}[\mu(Q_{n+1})] - \mathbb{E}[\mu(Q_n)]$ is small, then we can find a deterministic p_n such that

$$\mathbb{E}\left[\nu(Q_{2n}, p_n)\right] - \mathbb{E}\left[\mu(Q_{n+1})\right] \leq C\left(\mathbb{E}[\mu(Q_{n+1})] - \mathbb{E}[\mu(Q_n)]\right).$$

But since $\mu(U) \leq \nu(U,p)$ for any p, we deduce that

$$\overline{\mu} - \mathbb{E}\left[\mu(Q_{n+1})\right] \leq C\left(\mathbb{E}[\mu(Q_{n+1})] - \mathbb{E}[\mu(Q_n)]\right).$$

Be a standard iteration, we get, for some $\theta < 1$,

$$\overline{\mu} - \mathbb{E}\left[\mu(Q_n)\right] \leq C\theta^n = C3^{-\alpha n}.$$

Then we are done, at least with the hard part (i).

Flatness of large scale minimizers

The key step was to show that

$$\mathbb{E}\left[\nu(Q_{2n}, p_n)\right] - \mathbb{E}\left[\mu(Q_{n+1})\right] \leq C\left(\mathbb{E}[\mu(Q_{n+1})] - \mathbb{E}[\mu(Q_n)]\right).$$

To prove this, we just need to show that large scale minimizers for μ are approximately flat, that is, close to a plane. Then we can slightly perturb this plane to have planar boundary conditions, so that it is a candidate energy minimizer for $\nu(\cdot, p)$.

Here is the idea: if $\mathbb{E}[\mu(Q_{n+1})] - \mathbb{E}[\mu(Q_n)]$ is small, then we are in the "additive regime". This implies by convexity that the restriction of the minimizer on Q_{n+1} to the 3^d smaller subcubes of size Q_n is actually a very good approximation in H^1. In particular, the gradient of the minimizer should be almost independent in these subcubes.

Flatness of large scale minimizers II

We define $p_n := \mathbb{E}\left[\fint_{Q_n} Du_n(x)\, dx \right]$ where u_n is the (unique, up to a constant) energy minimizer for μ in Q_n.

Making the idea in the last slide rigorous and quantitative leads to:

$$\mathbb{E}\left[\left(\fint_{Q_n} Du_{n+1}(x) - p_n \right)^2 \right] \leq C \left(\mathbb{E}[\mu(Q_{n+1}] - \mathbb{E}[\mu(Q_n]) \right).$$

Formulating and proving this estimate is the trickiest part of the argument.

Flatness of large scale minimizers III

Using Poincaré inequality and the estimate in the previous slide, we can build candidate minimizers v_{2n} for the energy on the cube Q_{2n} such that, in each of the 3^{dn} subcube Q_n^i of size Q_n,

$$\fint_{Q_n^i} v_{2n} - p_n \cdot x = 0$$

and

$$\mathbb{E}\left[\fint_{Q_{2n}} L(Dv_{2n}(x), x)\, dx\right] \leq \mathbb{E}\left[\mu(Q_{n+1})\right] + C\left(\mathbb{E}[\mu(Q_{n+1})] - \mathbb{E}[\mu(Q_n)]\right).$$

The first line and Poincaré says that v_{2n} is *really close* to $p_n \cdot x$ when viewed from the much larger scale cube Q_{2n}. This permits us to modify v_{2n} to give it planar boundary conditions, without perturbing the (normalized) energy very much.

Convergence of $\mu(Q_n)$

As a result, we get the estimate

$$E[\nu(Q_{2n}, p_n)] - \mathbb{E}[\mu(Q_{n+1})] \le C\left(\mathbb{E}[\mu(Q_{n+1})] - \mathbb{E}[\mu(Q_n)]\right).$$

Hence

$$\overline{\mu} - \mathbb{E}[\mu(Q_{n+1})] \le C\left(\mathbb{E}[\mu(Q_{n+1})] - \mathbb{E}[\mu(Q_n)]\right).$$

By a standard iteration,

$$\overline{\mu} - \mathbb{E}[\mu(Q_n)] \le C3^{-\alpha n}.$$

$$\begin{aligned} E[\nu(Q_{2n}, p_n)] - \mathbb{E}[\mu(Q_{2n})] &\le E[\nu(Q_{2n})] - \mathbb{E}[\mu(Q_{n+1})] \\ &\le C\left(\mathbb{E}[\mu(Q_{n+1})] - \mathbb{E}[\mu(Q_n)]\right) \\ &\le C3^{-\alpha n}. \end{aligned}$$

This implies $|p_n - p| \le C3^{-\alpha n}$ for some limiting slope p.

Concentration I

Now we can handle the fluctuations using a very strong concentration argument.

We have a nonnegative subadditive quantity $\nu(\cdot, p) - \mu$ which is converging to zero in expectation at an algebraic rate. In this situation, the most elementary of all concentration arguments yields the bound

$$\mathbb{P}\left[\nu(Q_n, p) - \mu(Q_n) > 3^{-\alpha(d-p)n}\right] \leq C \exp\left(-c3^{pn}\right), \quad p < d.$$

We just need to consider the exponential moment generating function

$$\mathbb{E}\left[\exp\left(t(\nu(Q_n, p) - \mu(Q_n))\right)\right].$$

Concentration II

Compute:

$$
\begin{aligned}
&\log \mathbb{E}\left[\exp\left(t3^{dm}(\overline{\nu}(Q_{n+m}) - \mu(Q_{n+m}))\right)\right] \\
&\quad \leq \log \mathbb{E}\left[\prod_{i=1}^{3^{dm}} \exp\left(t(\nu(Q_n^i) - \mu(Q_n^i))\right)\right] && \text{(by subadditivity)} \\
&\quad = \sum_{i=1}^{3^{dm}} \log \mathbb{E}\left[\exp\left(t(\nu(Q_n^i) - \mu(Q_n^i))\right)\right] && \text{(by independence)} \\
&\quad = 3^{dm} \log \mathbb{E}\left[\exp\left(t(\nu(Q_n) - \mu(Q_n))\right)\right] && \text{(by stationarity).}
\end{aligned}
$$

Concentration III

From previous slide:

$$\log \mathbb{E}\left[\exp\left(t3^{dm}(\nu(Q_{n+m}) - \mu(Q_{n+m}))\right)\right] \\ \leq 3^{dm} \log \mathbb{E}\left[\exp\left(t(\nu(Q_n) - \mu(Q_n))\right)\right].$$

Now we use that $\nu - \mu \leq |\nu| + |\mu| \leq K$ for some $K \leq C$ and apply the elementary inequalities:

$$\begin{cases} \exp(s) \leq 1 + 2s & \text{for all } 0 \leq s \leq 1, \\ \log(1+s) \leq s & \text{for all } s \geq 0, \end{cases}$$

So take $t := 1/K$ to get

$$\log \mathbb{E}\left[\exp\left(t(\nu(Q_n) - \mu(Q_n))\right)\right] \leq 2\mathbb{E}\left[t(\nu(Q_n) - \mu(Q_n))\right] \\ \leq Ct3^{-\alpha n} \leq C3^{-\alpha n}.$$

Concentration IV

Combining stuff on previous slide:

$$\log \mathbb{E}\left[\exp\left(c3^{dm}(\nu(Q_{n+m}) - \mu(Q_{n+m}))\right)\right] \leq C3^{dm-\alpha n}$$

Chebyshev:

$$\mathbb{P}\left[\nu(Q_{n+m}) - \mu(Q_{n+m}) \geq K\exp(-3^{dm})\right] \leq CK^{-1}3^{dm-\alpha n}.$$

Optimize parameters. Choosing $K = \exp(3^{q(m+n)})$ for $p < q < d$ and then taking m to be a very large multiple of n gives

$$\mathbb{P}\left[\nu(Q_n) - \mu(Q_n) \geq C\exp(-3^{\alpha n})\right] \leq C\exp\left(-3^{pn}\right).$$

This is optimal in terms of stochastic integrability: we can never have an estimate like this with $\exp(-3^{dn})$ *on the right side.*

Higher regularity

The estimate is new in the linear case, in which L is a quadratic form:

$$L(p, x) = p \cdot A(x)p.$$

We can now recover a stochastic *a priori* $C^{0,1}$ estimate by comparing to the Laplacian. In particular, we can recover an estimate that says that any solution of

$$-\operatorname{div}(A(x)Du) = f(x) \quad \text{in } B_R, \quad R > 10,$$

with $f \in C^\beta(B_R)$ satisfies

$$\operatorname*{osc}_{B_1} u \leq \mathcal{C}\left(R \sup_{B_R} |f| + R^{1+\beta}[f]_{C^\beta(B_R)} + R^{-1} \operatorname*{osc}_{B_R} u \right),$$

where $\mathcal{C}$ is a random variable satisfying

$$\mathbb{E}\left[\exp\left(\mathcal{C}^p\right)\right] < +\infty \quad \text{for every } p < d.$$

Specializing to linear equations

A version of the previous "annealed $C^{0,1}$ estimate" was previously proved (in the discrete setting, with L^p stochastic integrability) for $C^{0,\alpha}$ for $\alpha < 1$ by **Marahrens-Otto ('13)** using very different methods based on the Log Sobolev inequality. This was extended to the continuum setting by **Gloria-Marahrens (to appear)**.

We can apply the $C^{0,1}$ estimate to the modified correctors and the Green's functions to get optimal estimates for the gradients of both. At this point, we can, following **Gloria-Otto ('11)** and **Gloria-Neukamm-Otto ('13)**, plug into special concentration inequalities designed for optimally estimating the variance (i.e., the Spectral Gap inequality or Log Sobolev inequality) and recover optimal error estimates error in homogenization.

Thank you for your attention!